A Nonlinear Control Law for Two-Wheeled Self-Balanced Vehicles

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***Abstract*—This paper presents the design of a nonlinear control law for two-wheeled self-balanced vehicles. The design is based on forwarding and gives a Lyapunov function that allows us to obtain an estimation of the domain of attraction for the resultant law.**

I. Introduction

Self-balanced systems are undergoing a considerable development nowadays. They have been made popular by the vehicle called Segway. Its control around the vertical position can be studied by applying linear methods and does not present special problems. However, when problems associated to the increase of the domain of attraction are to be tackle, the use of nonlinear methods become inevitable and the problem gains a remarkable difficulty.

To study these problems, inspiration can be sought in the results shown by the studies about the inverted pendulum, device that shares outstanding features with the self-balanced system, although it differs to a great extent from it. The inverted pendulum, either on a cart or the rotatory Furuta pendulum, has been the subject of many studies in the last decades. Some well-known, pendulum related equipment are Furuta pendulum [1], acrobot [2], pendubot [3], the reaction wheel pendulum [4], and other pendulum based vehicles like the one that can be found in [5].

The difference between the self-balanced vehicle and the inverted pendulum lies in the fact that, in the first one the axle of the motors is at the same time the pivot point of the pendulum, whereas in the second one the pendulum goes freely around the pivot point. Therefore, this paper proposes working on a dynamic model different from the model of the inverted pendulum. The difference between the two models makes impossible the mere application of the results obtained for the inverted pendulum to the case of the self-balanced system. Moreover, in this paper, only the movement of the vehicle following a straight line is considered.

In spite of the fact that the linear control laws applied to these vehicles have good practical features, the domain of attraction that can be estimated analytically for these laws is extremely small as it is proved in [6]. In this paper, a new nonlinear control law is proposed, which has been designed by means of a procedure similar to the forwarding method for the control of nonlinear systems, since, as it will be discussed below, the self-balanced system presents the appropriate structure. An analysis of the domain of attraction for the new controller is also exposed and it is compared with the estimate of the domain of attraction of a LQR law through its Lyapunov function by using the method proposed in [7]. The advantages of the method proposed here are demonstrated.

An unmanned self-balanced vehicle with the features analyzed in this paper is being constructed as a benchmark for the designed controllers. Some simulation results are shown, which have been obtained by applying the control law described and the particular parameters for the design of this vehicle.

The paper is organized as follows. In Section II, the model of movement proposed is presented. In Section III, the design of a linear and a nonlinear control law for the system is shown. Subsequently, in Section IV, the domain of attraction for the two designed control laws are estimated. Finally, in Section V some simulation results are presented for a particular vehicle design.

II. Vehicle Modeling

The two-wheeled vehicle is an inverted pendulum where the pivot point matches the axle of the motors. Thus, the external torque applied by the motors produces effects of the same value on wheels and pendulum but with opposite direction.

The system constituted by the vehicle consists of two parts or subsystems. On the one hand, the two motors, the electronic control devices and other auxiliary devices are fixed to the frame to compose the pendulum. On the other hand, the wheels are fixed to the axle of the motors, constituting the second subsystem.

Let us define the system variables θ, the inclination angle or deviation between the pendulum and vertical line; $\dot{\theta}$, the angular rate of the pendulum; and $\dot{\varphi}$, the angular rate of the axle of the motors. These variables are shown in Fig. 1.

In order to simplify the model of the vehicle, we can assume the mass of the entire pendulum set (frame, motors and other elements) to be a punctual mass located on the center of gravity of the physical pendulum. Thus, the pendulum has

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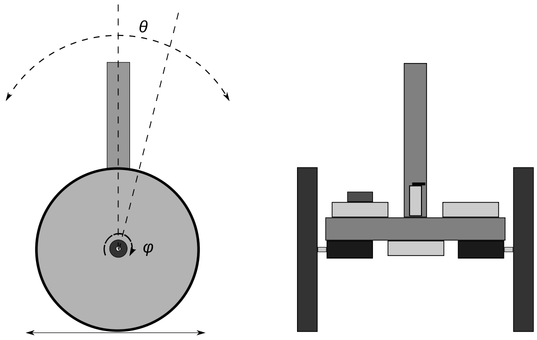

Fig. 1. Diagram of the two-wheeled vehicle

a mass m separated a distance l from the axle, where there are two wheels fixed with radius r and mass M_r.

By using the Lagrangian of the dynamical system and the Euler-Lagrange approach for non-conservative forces, which have been omitted here in order not to extend the explanation unnecessarily, the movement equations for the system (1) and (2) can be obtained.

$$2ml^2\ddot{\theta} + mlr\ddot{\varphi}\cos\theta - mgl\sin\theta = -\tau + k\dot{\varphi} \quad (1)$$

$$\begin{aligned} \left(\tfrac{3}{2}M_r + m\right) r^2\ddot{\varphi} + mlr\ddot{\theta}\cos\theta & \\ -mlr\dot{\theta}^2\sin\theta &= \tau - k\dot{\varphi} \end{aligned} \quad (2)$$

where g is the gravitational constant and k is a constant that represents the static friction of the motor.

We define the following constants

$$\begin{aligned} \alpha &= 2ml^2 \\ \beta &= mlr \\ \gamma &= mgl \\ \eta &= \left(\frac{3}{2}M_r + m\right) r^2 \,. \end{aligned}$$

Let us consider the state variables $x_1 = \theta$, $x_2 = \dot{\theta}$ and $x_3 = \dot{\varphi}$. If we regroup the terms in (1) and (2), we obtain the state equations for the system shown in (3), (4) and (5).

$$\dot{x}_1 = x_2 \quad (3)$$

$$\begin{aligned} \dot{x}_2 = \frac{1}{\Delta}\Big[&\gamma\eta\sin x_1 - \beta^2 x_2^2 \sin x_1 \cos x_1 \\ &+ (\eta + \beta\cos x_1)(kx_3 - u)\Big] \end{aligned} \quad (4)$$

$$\begin{aligned} \dot{x}_3 = \frac{1}{\Delta}\Big[&-\beta\gamma\sin x_1\cos x_1 + \alpha\beta x_2^2\sin x_1 \\ &- (\alpha + \beta\cos x_1)(kx_3 - u)\Big] \end{aligned} \quad (5)$$

where $\Delta = \alpha\eta - \beta^2\cos^2 x_1$.

By means of the partial linearization [8] defined as

$$\begin{aligned} u = \frac{\beta\gamma\sin x_1\cos x_1}{\alpha + \beta\cos x_1} - \frac{\alpha\beta x_2^2\sin x_1}{\alpha + \beta\cos x_1} & \\ + kx_3 + \frac{\left(\alpha\eta - \beta^2\cos^2 x_1\right)}{(\alpha + \beta\cos x_1)} v & \end{aligned} \quad (6)$$

the following state space representation of the system is obtained

$$\dot{x}_1 = x_2 \quad (7)$$

$$\begin{aligned} \dot{x}_2 = & \frac{\gamma\sin x_1}{\alpha + \beta\cos x_1} + \frac{\beta x_2^2\sin x_1}{\alpha + \beta\cos x_1} \\ & - \frac{(\eta + \beta\cos x_1)}{\alpha + \beta\cos x_1} v \end{aligned} \quad (8)$$

$$\dot{x}_3 = v\,. \quad (9)$$

Since this is a prototype designed to work like a benchmark for control laws, the designer has some freedom to choose the dimensions of the elements that compound the vehicle. As a result of this freedom we can force $\eta = \alpha$ modifying the set of parameters and, thus, the equations that describe the system in (7), (8) and (9) can be simplified as shown in (10), (11) and (12). In this way, the design process of the control law is expected to be easier.

$$\dot{x}_1 = x_2 \quad (10)$$

$$\dot{x}_2 = \frac{\gamma\sin x_1}{\alpha + \beta\cos x_1} + \frac{\beta x_2^2\sin x_1}{\alpha + \beta\cos x_1} - v \quad (11)$$

$$\dot{x}_3 = v\,. \quad (12)$$

III. Control Laws Design

The main objective for the control of this system is to stabilize the vehicle (the pendulum) in the upper vertical position. To achieve this aim, different control strategies can be used. Two different control laws are going to be established in this article, a linear one based on LQR (Linear-Quadratic Regulator) and a nonlinear one based on *forwarding* as proposed in [9].

A. Linear LQR Law

The LQR method consists in a minimization of a cost function

$$J = \int_0^\infty \left(x^\top Qx + u^\top Ru\right) dt \quad (13)$$

where Q and R penalize the error in the state variables and the control signal, respectively.

The LQR controller works with the linear model representation of the system $\dot{x} = Ax + Bu$. The system shown in (10), (11) and (12) can be linearized around the origin $x = (0, 0, 0)$, that is the desired equilibrium point, to obtain

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma}{\alpha+\beta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} u\,. \quad (14)$$

The control law K that minimizes the cost function is

$$u = -Kx\,. \quad (15)$$

B. Nonlinear Law

As it has been mentioned previously, the system has an upper triangular structure, so that techniques similar to *forwarding* can be used because the system matches the structure

$$\begin{aligned} \dot{z} &= f(z) + \Psi(z,\xi) + g(z,\xi)v & (16)\\ \dot{\xi} &= a(\xi) + b(\xi)v\,. & (17) \end{aligned}$$

By analyzing the equations for the system, it is possible to identify the lower subsystem (17) with equations (10) and (11), whereas the upper subsystem (16) can be identified with (12).

The lower subsystem can be proved to be unstable. These techniques require $\dot{\xi} = a(\xi)$ to be stable and, in order to achieve this, it is neccesary to use a previous control law

$$v = \frac{2\left(\gamma + \beta x_2^2\right)\sin x_1}{\alpha + \beta\cos x_1} + u_d \qquad (18)$$

where a new control variable u_d is introduced in order to stabilize the upper subsystem.

When the aforementioned control law is applied, the resulting system is

$$\begin{aligned} \dot{x}_1 &= x_2 & (19)\\ \dot{x}_2 &= -\frac{\left(\gamma + \beta x_2^2\right)\sin x_1}{\alpha + \beta\cos x_1} - u_d & (20)\\ \dot{x}_3 &= \frac{2\left(\gamma + \beta x_2^2\right)\sin x_1}{\alpha + \beta\cos x_1} + u_d & (21) \end{aligned}$$

where the lower subsystem (17) can be identified with equations (19) and (20), and the upper subsystem is identified with equation (21). In this case, the lower subsystem with $u_d = 0$ is stable. The system $\dot{z} = f(z)$, that corresponds to $\dot{x}_3 = 0$, is globaly stable and, therefore, forwarding techniques can be applied.

The subsystem composed by (19) and (20), with $u_d = 0$, admits an invariant function

$$\nu_1 = \frac{\gamma + \beta x_2^2}{\beta\left(\alpha + \beta\cos x_1\right)^2} \qquad (22)$$

that can be considered a kind of energy function.

Let us consider the sum of the invariant function (22) for the lower subsystem plus a quadratic term ν as a Lyapunov function candidate. Then

$$V = \frac{\gamma + \beta x_2^2}{\beta\left(\alpha + \beta\cos x_1\right)^2} - \frac{\gamma}{\beta\left(\alpha + \beta\right)^2} + \frac{1}{2}\rho\nu^2 \qquad (23)$$

where $\rho > 0$ is a tuning parameter.

The control law u_d can be obtained by forcing the Lyapunov function derivative for the whole system to be negative definite.

By differentiating the Lyapunov function (23), it results

$$\begin{aligned} \dot{V} &= \tfrac{2x_2^2}{(\alpha+\beta\cos x_1)^2}\dot{x}_2 + \tfrac{2\left(\gamma+\beta x_2^2\right)\sin x_1}{(\alpha+\beta\cos x_1)^3}\dot{x}_1 + \rho\nu\nabla\nu\dot{x}\\ &= \rho\nu\left(x_2\tfrac{\partial\nu}{\partial x_1} + \tfrac{\left(\gamma+\beta x_2^2\right)\sin x_1}{(\alpha+\beta\cos x_1)}\left(-\tfrac{\partial\nu}{\partial x_2} + 2\tfrac{\partial\nu}{\partial x_3}\right)\right) \qquad (24)\\ &+ u_d\left(-\tfrac{2x_2}{(\alpha+\beta\cos x_1)^2} + \rho\nu\left(-\tfrac{\partial\nu}{\partial x_2} + \tfrac{\partial\nu}{\partial x_3}\right)\right)\,. \end{aligned}$$

In order to ensure $\dot{V}$ is negative, the first term in (24) can be forced to zero, yielding the following Partial Differential Equation (PDE)

$$x_2\frac{\partial\nu}{\partial x_1} + \frac{\left(\gamma + \beta x_2^2\right)\sin x_1}{(\alpha + \beta\cos x_1)}\left(-\frac{\partial\nu}{\partial x_2} + 2\frac{\partial\nu}{\partial x_3}\right) = 0\,. \qquad (25)$$

The solution for the PDE provides a new invariant function

$$\nu_2 = 2x_2 + x_3\,. \qquad (26)$$

Finally, the control law u_d can be calculated, ensuring $\dot{V} \leq 0$, by means of

$$u_d = -\left(-\frac{2x_2}{\left(\alpha + \beta\cos x_1\right)^2} + \rho\nu\left(-\frac{\partial\nu}{\partial x_2} + \frac{\partial\nu}{\partial x_3}\right)\right)\,. \qquad (27)$$

The result is

$$u_d = \frac{2x_2}{\left(\alpha + \beta\cos x_1\right)^2} + \rho\left(2x_2 + x_3\right)\,. \qquad (28)$$

The complete control law for the system (10), (11) and (12) is

$$v = \frac{2\left(\gamma + \beta x_2^2\right)\sin x_1}{\alpha + \beta\cos x_1} + \frac{2x_2}{\left(\alpha + \beta\cos x_1\right)^2} + \rho\left(2x_2 + x_3\right)\,. \qquad (29)$$

IV. Stability Analysis

In order to compare the designed control laws, a stability analysis is done by studying the domain of attraction for each of them. Studying the linear case, it is not possible to precisely measure the domain of attraction, so that it has to be estimated. The domain of attraction estimate for each control law can be compared.

A. Domain of Attraction for the LQR Law

When the LQR based control law is applied to a strongly nonlinear system like this vehicle, it is not possible to know exactly the domain of attraction. For this reason, the domain must be estimated. To get an estimation we must find a bound for the Lyapunov function of the system that guarantees stability, and the domain of attraction will be stated as the region contained into the maximum level surface of V inside the bound. The following result is based on [7].

Let us consider (14) the linearization of the system determined by (10), (11) and (12), around the desired equilibrium point. The linearized system can be stabilized by using a LQR control law. The law, $u = -Kx$, asyntotically stabilizes the system (14) and locally stabilizes the system (10), (11) and (12). Considering the local stabilization of the nonlinear system, a local region of attraction, U, can be determined for the closed-loop system, according to the procedure that will be described subsequently.

$$U = \{(x_1, x_2, x_3)^\top : |x_1| \le a, |x_2| \le b, |x_3| \le c\} \quad (30)$$

with a, b and c, positive constants.

By rewriting the stabilized system as the addition of the linear part plus the nonlinear part, it can be obtained

$$\dot{x} = (A - BK)\, x + \left(0, \tfrac{(\gamma+\beta x_2^2)\sin x_1}{\alpha+\beta\cos x_1} - \tfrac{\gamma x_1}{\alpha+\beta}, 0\right)^\top . \quad (31)$$

Since the sistem $A-BK$ is stable, the Lyapunov's equation

$$(A - BK)^\top P + P\,(A - BK) = -I \quad (32)$$

has a solution, P, that is positive definite.

Choosing $V = x^\top P x$ as Lyapunov function for the system, the derivative of V through the trajectory of the system (31) is

$$\begin{aligned} \dot{V} = \quad & x^\top \left((A-BK)^\top P + P\,(A-BK)\right) x \\ & +2x^\top P \left(0, \tfrac{(\gamma+\beta x_2^2)\sin x_1}{\alpha+\beta\cos x_1} - \tfrac{\gamma x_1}{\alpha+\beta}, 0\right)^\top \\ \le \quad & -\|x\|^2 + 2\,\|x\|\,\|P\| \left|\tfrac{(\gamma+\beta x_2^2)\sin x_1}{\alpha+\beta\cos x_1} - \tfrac{\gamma x_1}{\alpha+\beta}\right| . \end{aligned} \quad (33)$$

Focusing on the expression for the absolute value of the second term of the inequality, it can be separated into new terms through the next inequality

$$\begin{aligned} & \left|\frac{(\gamma+\beta x_2^2)\sin x_1}{\alpha+\beta\cos x_1} - \frac{\gamma x_1}{\alpha+\beta}\right| \\ \le & \left|\tfrac{\gamma \sin x_1}{\alpha+\beta\cos x_1} - \tfrac{\gamma\sin x_1}{\alpha+\beta}\right| + \left|\tfrac{\gamma\sin x_1}{\alpha+\beta} - \tfrac{\gamma x_1}{\alpha+\beta}\right| + \left|\tfrac{\beta x_2^2 \sin x_1}{\alpha+\beta\cos x_1}\right| \\ \le & \frac{\beta\gamma}{\alpha^2-\beta^2}\frac{|x_1|^3}{2} + \frac{\gamma}{\alpha+\beta}\frac{|x_1|^3}{6} + \frac{\beta}{\alpha-\beta}\,|x_2|^2\,|x_1| \ , \end{aligned} \quad (34)$$

where the next inequalities have been used

$$|\sin x_1| \le |x_1|, \quad |\sin x_1 - x_1| \le \tfrac{|x_1|^3}{2},$$
$$|1-\cos x_1| \le \tfrac{|x_1|^2}{2}, \quad \alpha+\beta\cos x_1 \ge \alpha-\beta .$$

By means of (34), and taking into account that $|x_i| \le \|x\|$, the expression can be bounded using the inequality

$$\left|\tfrac{(\gamma+\beta x_2^2)\sin x_1}{\alpha+\beta\cos x_1} - \tfrac{\gamma x_1}{\alpha+\beta}\right| \le \|x\|^3 \left(\tfrac{\beta\gamma}{2(\alpha^2-\beta^2)} + \tfrac{\gamma}{6(\alpha+\beta)} + \tfrac{\beta}{\alpha-\beta}\right) . \quad (35)$$

Consequently, by introducing (35) into (33) and rearranging the terms, we can obtain the next bound.

$$\dot{V} \le \|x\|^2 \left[2\,\|x\|^2\,\|P\| \left(\tfrac{\beta\gamma}{2(\alpha^2-\beta^2)} + \tfrac{\gamma}{6(\alpha+\beta)} + \tfrac{\beta}{\alpha-\beta}\right) - 1\right] . \quad (36)$$

Let us take a positive constant parameter that satisfies

$$\bar{x} < \left[2\,\|P\| \left(\tfrac{\beta\gamma}{2(\alpha^2-\beta^2)} + \tfrac{\gamma}{6(\alpha+\beta)} + \tfrac{\beta}{\alpha-\beta}\right)\right]^{-1/2} . \quad (37)$$

The inequality $\dot{V} < 0$ can be ensured inside the region

$$U_0 = \left\{x = (x_1, x_2, x_3)^\top : \|x\| < \bar{x}, x \ne (0,0,0)^\top\right\} . \quad (38)$$

Let us define $r = \bar{x}\sqrt{\lambda_{min}/\lambda_{max}}$, where λ_{min} and λ_{max} are the minimum and maximum eigenvalue of the matrix P, respectively. The region (30) with positive constants a, b and c satisfying

$$a^2 + b^2 + c^2 \le r^2$$

can be ensured to be an estimate for the domain of attraction of the system shown in (10), (11) and (12) by using the linear control law (15).

For any $x_0 \in U$, the level surface $C_{x_0} = \{x : x^\top P x = x_0^\top P x_0\}$ of V at x_0 is entirely contained in U_0, since for any $x_0 \in U$

$$\begin{aligned} \lambda_{min}\,\|x\|^2 \le x^\top P x \\ = x_0^\top P x_0 \le \lambda_{max}\,\|x_0\|^2 \le \lambda_{max} r^2 \end{aligned} \quad (39)$$

which results in

$$\|x\|^2 \le \frac{\lambda_{max}}{\lambda_{min}} r^2 = \bar{x}^2 . \quad (40)$$

And thus, this expression implies that $\|x\| \le \bar{x}$.

B. Domain of Attraction for the Nonlinear Law

In order to study the stability of the system (10), (11) and (12) with the nonlinear control law (29), we focus on the following Lyapunov function

$$V = \frac{\gamma+\beta x_2^2}{\beta\,(\alpha+\beta\cos x_1)^2} - \frac{\gamma}{\beta\,(\alpha+\beta)^2} + \frac{1}{2}\rho\,(2x_2+x_3)^2$$

whose derivative through the trajectories of the system is

$$\dot{V} = -\left(\frac{2x_2}{(\alpha+\beta\cos x_1)^2} + \rho\,(2x_2+x_3)\right)^2 . \quad (41)$$

By using this Lyapunov function, it is possible to prove the stability of the system since the function satisfies the next conditions

$$V(0) = 0 \quad \text{y} \quad V(x) > 0 \quad \forall x \ne 0$$
$$\dot{V}(x) \le 0 \quad \forall x .$$

Lyapunov stability can be proved because $\dot{V}(x)$ is negative semi-definite, but it is not possible to prove the asymptotical stability. It is necessary to use the LaSalle's Invariance Principle to prove that the maximum invariant set such that $\dot{V}(x) = 0$ is the origin.

Forced $\dot{V}(x) \equiv 0$, what, through (24), is equivalent to $u_d = 0$, the residual dynamics of the system results

$$\dot{x}_1 = x_2 \quad (42)$$

$$\dot{x}_2 = -\frac{(\gamma+\beta x_2^2)\sin x_1}{\alpha+\beta\cos x_1} \quad (43)$$

$$\dot{x}_3 = \frac{2\,(\gamma+\beta x_2^2)\sin x_1}{\alpha+\beta\cos x_1} \quad (44)$$

with

$$F \triangleq \frac{2x_2}{\left(\alpha+\beta\cos x_1\right)^2} + \rho\left(2x_2+x_3\right) \equiv 0\,. \qquad (45)$$

By studying the dynamics of F through the trajectories of (42), (43) and (44), it can be obtained

$$\dot{F} = \frac{\partial F}{\partial x_1}\dot{x}_1 + \frac{\partial F}{\partial x_2}\dot{x}_2 + \frac{\partial F}{\partial x_3}\dot{x}_3 \qquad (46)$$

$$= \frac{2\sin x_1\left(\beta x_2^2-\gamma\right)}{\left(\alpha+\beta\cos x_1\right)^3}\,. \qquad (47)$$

Function $\dot{F}$ equals 0 for $x_1 = 0$ and $x_1 = \pm n\pi$ with $n \in \mathbb{N}$, $n \geq 1$, and for the values $x_2 = \pm\sqrt{\frac{\gamma}{\beta}}$. The points that contain these values represent the candidates to be members of invariant sets. Since this condition demands x_2 to be constant, we obtain $\dot{x}_2 = 0$ and from (43) it can be concluded that $\gamma + \beta x_2^2 = 0$, which implies a contradiction. Therefore, the points with $x_2 = \pm\sqrt{\frac{\gamma}{\beta}}$ do not belong to the invariant set.

Points with $x_1 = 0$ guarantee $\dot{x}_2 = 0$ and $\dot{x}_3 = 0$. The requirement of $\dot{x}_1$ to equal 0 implies that $x_2 = 0$ and through (45) it is guaranteed that $x_3 = 0$. Thus, the origin is a member of the invariant set.

The case with $x_1 = \pm n\pi$ sets the bounds for the local region where the system is asymptotically stable. In particular, the bounds are $x_1 = \pm\pi$, that represent physically the same point, and this point is unachievable according to the characteristics of the vehicle.

Therefore, the maximum invariant set for this system (in the feasible work zone) matches up with the equilibrium point $x = (0,0,0)^\top$, and we can conclude that the system is asymptotically stable, being the domain of attraction bounded by the maximum level surface of (41) that satisfies $x_1 < |\pi|$.

V. Simulation Results

This study is being developed jointly with the construction of a two-wheeled self-balanced vehicle that will be used as a benchmark to prove the results obtained. For the simulation of the system behaviour, we use the model of the vehicle shown in Section II, whose parameters have been experimentally identified from the designed vehicle.

Fig. 1 describes the vehicle and shows an outline of the hardware. The vehicle is composed of an aluminium framework in the shape of an inverted T, with two motors fixed on its lower section, whose axles are at the same time the axles for the two wheels. Two boxes are shown, where the electronics and sensors needed to implement the control of the system (microcontroller board, motor controller, wireless transmitter, batteries and Inertial Measurement Unit) are placed to be properly protected. A preliminary version of the vehicle is shown in Fig. 2.

The designed control law can be programmed into the memory of the embedded microcontroller. Thus, experimental data can be reported to a PC via a bluetooth-serial connection.

Electronics and auxiliary elements for the vehicle are located near the axle and, in that way, the effective center of mass is lowered.

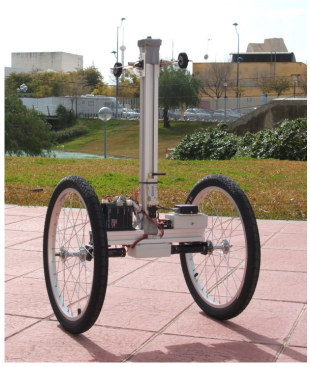

Fig. 2. Preliminary version of the two-wheeled self-balanced vehicle

The pendulum mass is 3 kg and its equivalent center of mass is located at a distance of 13 cm from the axle. The system has been designed to fulfill $\eta = \alpha$, what implies that the wheels have a radium of 15 cm and a mass of 0.5 kg each one.

The main parameters that characterize the model result in the next values: $\alpha = 0.1014$, $\beta = 0.0585$ and $\gamma = 3.8259$.

Considering these values for the parameters, the estimate of the domain of attraction of the system with the LQR law results in $r = 6.38 \cdot 10^{-4}$, that is an extremely small estimate of the domain of attraction due to the high conservatism necessary to justify mathematically the stability. On the other hand, the estimate of the domain of attraction of the system with the nonlinear law can be extended to the larger Lyapunov level surface such that the control law is defined inside it.

In order to compare the domain of attraction of the system using both control laws, the volume of the regions can be calculated to have quantitative values that can be compared.

According to (39), the ellipsoid $x^\top P x = \lambda_{max} r^2$ bounds the estimate of the domain of attraction of the system with the LQR law. The volume inside the ellipsoid can be calculated in the following way

$$\text{Volume}_{ellipsoid} = \frac{4\pi}{3}\sqrt{\frac{\left(\lambda_{max} r^2\right)^3}{|P|}}\,, \qquad (48)$$

and, using the values for the parameters of the system, the volume associated to the LQR law is $\text{Volume}_{LQR} = 7.0956 \cdot 10^{-7}$.

The maximum level surface of (41) that satisfies $x_1 < |\pi|$, for the system with the nonlinear law and the parameters that have been presented, is

$$\frac{\gamma+\beta x_2^2}{\beta\left(\alpha+\beta\cos x_1\right)^2} - \frac{\gamma}{\beta\left(\alpha+\beta\right)^2} + \frac{1}{2}\rho\left(2x_2+x_3\right)^2 = C\,, \qquad (49)$$

with $C = 32978$.

This level surface is shown in Fig. 3 and the volume contained in it can be calculated through integration and results $\text{Volume}_{NL} = 78544.98$. The practical operating region of the system is included in this volume.

Despite the fact that LQR controller shows a good behaviour stabilizing the system in the practice, the domain of attraction

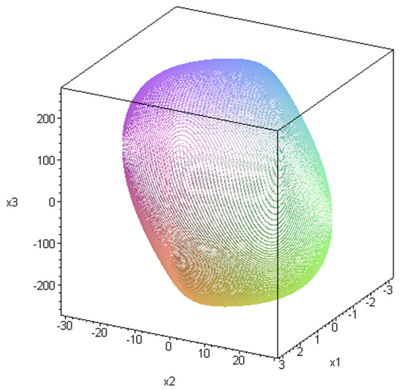

Fig. 3. Domain of attraction of the system with the nonlinear law

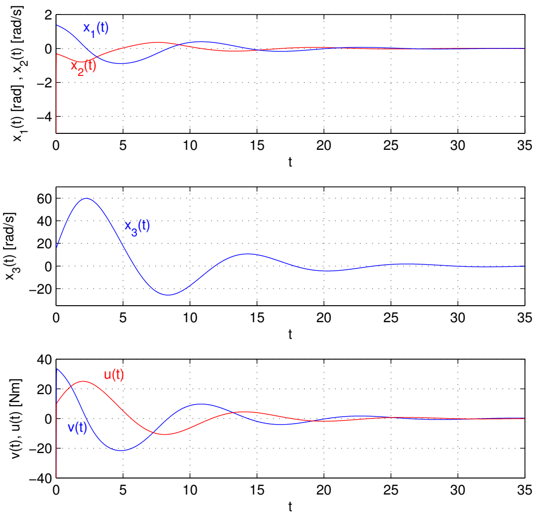

Fig. 4. Simulation results for the nonlinear law

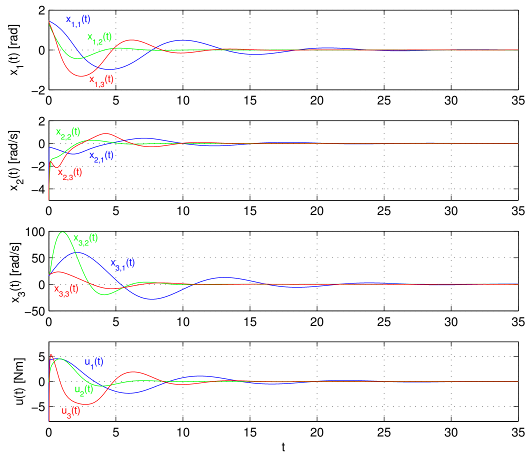

Fig. 5. Robustness to parameter uncertainty

that can be mathematically proved for the LQR law is much smaller than the theoretical one proved for the nonlinear law.

Fig. 4 presents the simulation data for the system taking the initial conditions $x = (1.42 \text{ rad}, -5 \text{ rad/s}, 20 \text{ rad/s})^\top$ that are considerably separated from the origin. The figure shows the state variables of the system, $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$ and $x_3(t) = \dot{\varphi}(t)$, the control signal $v(t)$ for the partial-linearized system and the control signal $u(t)$ for the complete system. The evolution of the state variables shows that the system is stabilized. Another advantage of the nonlinear law is that its control signal is smaller than the linear one. This is due to the fact that, according to x_1, with $x_2 = 0$ and $x_3 = 0$, the nonlinear control signal is always under the linear one. This implies that, when the initial x_1 is far from the origin, the inicial peak in the nonlinear control signal is smaller than in the linear one.

As a preliminary study, the robustness of the control law to parameter uncertainty can be checked by using different sets of parameter associated with the real system in our laboratory in the simulation. Fig. 5 shows a comparison of the state variables $x_{i,j}(t)$ and the control signal $u_j(t)$, where $j = 1$ refers to the case $\eta = \alpha = 0.1014$; $j = 2$ to $\eta = 0.0084$, $\alpha = 0.1014$; and $j = 3$ to $\eta = 0.4484$, $\alpha = 0.2782$. The simulation shows the good behavior of the control law in the three cases.

In conclusion, for this two-wheeled self-balanced vehicle, a new nonlinear control law has been designed which allows to prove the possibility of asymptotic stabilization of the system with a domain of attraction including the whole operation region.

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