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# A new swing-up law for the Furuta pendulum

F. GORDILLO†*, J. A. ACOSTA† and J. ARACIL†

In this paper the swing-up problem for the Furuta pendulum is solved applying Fradkov's speed-gradient (SG) method to a dimension 4 model of the system. The new law is compared with the conventional Åström–Furuta strategy, based on a dimension 2 model. A comparative analysis, including simulations and experiments, whereby the advantages and effectiveness of the new law for swinging the pendulum up are shown, is included.

## 1. Introduction

The inverted pendulum is a very simple device that displays very interesting behaviour modes, which have attracted the attention of many control researchers. In this paper, the interest is focused on the rotating type of inverted pendula. A schematic representation of the system is shown in figure 1. As displayed in this figure, $\theta$ denotes the angle of the pendulum with the upright vertical and $\varphi$ denotes the angle of the rotor arm. The rotating pendulum is also known as the Furuta pendulum, and has been studied by many authors such as Åström and Furuta (2000), and others (Wiklund *et al.* 1993, Bloch *et al.* 1999, Pagano 1999). The inverted pendulum gives rise to many interesting control problems. It is a non-linear underactuated mechanical system that is unstable at the desired position. Furthermore, the actuator limitations produce very complex and interesting behaviours that deserve careful analysis (Aracil *et al.* 1998). As a matter of fact, it shows two different and very interesting control problems. One is swinging the pendulum up from the hanging position to the upright one. To deal with this problem an energy control strategy is usually adopted. When the pendulum is close to the desired upright position with low enough speed, a stabilization or balancing strategy is applied. These two problems are quite interesting. The first one, in particular, a truly non-linear control problem, displays many difficulties. The simplest and best known solution to the swing-up problem and that which is easiest to implement is the one proposed by Åström and Furuta (2000) and Wiklund *et al.* (1993). It is based on neglecting the reaction torques from the pendulum to the arm, so that the energy control of the pendulum can be studied without considering the position and the velocity of the arm. This allows us to greatly simplify the model, which is reduced to a second-order one. With this reduced model, and with the help of Fradkov's speed-gradient (SG) method (Andrievskii *et al.* 1996, Fradkov and Pogromsky 1998), the desired energy injection can be easily computed and a very successful control law for the swing-up problem is obtained. However, it is based on simplifying assumptions, as mentioned above. Furthermore, the arm speed must be low at the switching time but this speed is not considered in the dimension 2 law. Fortunately, for usual pendula parameters and for the usual initial conditions (the lower position with no velocity) the dimension 2 law behaves well.

In this paper a new control strategy is proposed that does not neglect the reaction torques from the pendulum to the arm, and therefore considers the velocity of the arm. Thus, an objective function for Fradkov's speed-gradient method which includes not only the energy, but also the arm momentum, is proposed. With this objective function, Fradkov's method leads to a control law which prevents some of the problems found with the Åström and Furuta one. If the reaction torques from the pendulum to the arm cannot be neglected, for example, due to the mass of the pendulum or to the low friction on the arm, the new law solves the swing-up problem for cases where Aström and Furuta's method fails. Furthermore, the arm speed is also controlled during the swing-up process in such a way that the arm speed is low when the pendulum approaches the upright position. Concrete examples are given below.

On the other hand, the new law keeps the nice properties of Åström–Furuta's law when compared with other swing-up laws based on the dimension 4 model but obtained with other techniques: the new swing-up law is able to accomplish the goal with very small control signal magnitude (unlike Olfati-Saber (2000)) and the control signal converges to zero as the homoclinic orbit is reached (unlike Fantoni and Lozano (2002)).

The paper is organized as follows. Section 2 is devoted to recall Fradkov's speed-gradient method, with emphasis on the pseudogradient method. Then in § 3, the Hamiltonian formulation is used to obtain models for the Furuta pendulum. A dimension 4 model and an approximate dimension 2 model are derived. Control laws for these models are derived by the SG method in § 4. For the dimension 4 model an objective function

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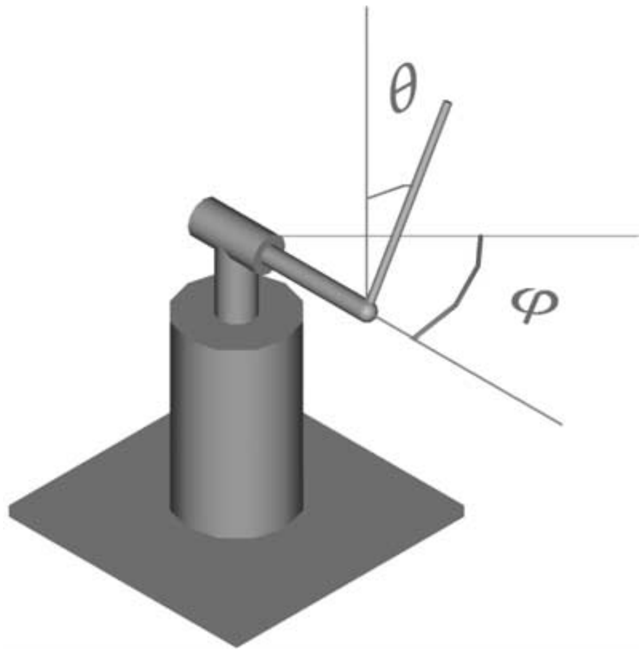


Figure 1. The Furuta pendulum.

with two terms—the energy and the arm momentum—is proposed, while for the dimension 2 model the objective function depends only on the energy resulting in the well-known Åström–Furuta's law. Both control laws have been checked on a concrete experimental pendulum, where the friction effects are compensated with a LuGre friction model (Canudas de Wit *et al.* 1995). The results of these experiments are reported in § 5. In § 6, some conclusions are given.

## 2. Speed pseudogradient algorithm

In this section the speed pseudogradient algorithm (Fradkov and Pogromsky 1998) in finite form is recalled. Consider the time-invariant, affine-in-control system given by

$$\dot{x} = f(x) + g(x)u \tag{1}$$

$$y = h(x) \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^l$ is the output and $u(t) \in \mathbb{R}^m$ is the input. Also consider the control objective $y(t) \to 0$ when $t \to \infty$; this control objective can be written with the objective function

$$Q(x) = \tfrac{1}{2}|h(x)|^2 \tag{3}$$

and defining the goal as $\lim_{t\to\infty} Q(x(t)) = 0$.

For this objective function the speed pseudogradient algorithm in finite form, can be written as

$$u = -\lambda (L_g h(x))^{\mathrm{T}} h(x) \tag{4}$$

Stability properties of the algorithm (4) are described in Theorem 2.21 in Fradkov and Pogromsky (1998, p. 101). For the sake of completeness this theorem is reproduced here.

**Theorem 1** (Fradkov and Pogromsky 1998)**:** *Consider the system* (1), (2) *and* (4) *under the following assumptions:*

- **A1.** *The functions* $f$, $g$, $h$ *are smooth and bounded together with their second partial derivatives in the region* $\Omega_0 = \{x \in \mathbb{R}^n : Q(x) \le Q_0\}$ *for some* $Q_0$.
- **A2.** *For all* $x \in \Omega_0$ *it follows that* $h(x)^{\mathrm{T}} L_f h(x) \le 0$.
- **A3.** *There exists a positive number* $\epsilon > 0$ *such that any connected subset of the set*

$$D_\epsilon = \Omega_0 \cap \{x \in \mathbb{R}^n : \det((L_g h(x))^{\mathrm{T}} L_g h(x)) \le \epsilon\}$$

  *is compact.*
- **A4.** *The matrix* $L_g h(x)$ *has rank* $l$ *for any* $x \in \Omega_0$ *such that* $Q(x) \ne 0$.

*Then in system* (1), (2) *and* (4) *for any initials conditions* $x(0) \in \Omega_0$ *the goal* $y(t) \to 0$ *when* $t \to \infty$ *is achieved.*

**Remark 1:** If assumption A4 is violated on some set $D_0$ then it can be shown that all trajectories of the closed loop system tend to a maximal invariant subset $M_0$ of $D_0$. Particularly if set $M_0$ is countable and consists of isolated points such that in all these points the matrix $\nabla_x f$ has a least one eigenvalue with the positive real part, then the statement of Theorem 1 remains true for almost all initial conditions from $\Omega_0$.

**Remark 2** (Fradkov *et al.* 1997)**:** Condition A4 severely restricts the class of controlled plant models. Indeed, it can be fulfilled only if $m \ge l$, i.e. if the number of controlling inputs is not less than the number of controlled outputs. It was shown in Fradkov *et al.* (1997) that A4 can be weakened at the cost of strengthening A2. Namely, let A2 be complemented by the conservativity-like condition

**A2.** $h(x)^{\mathrm{T}} L_f h(x) \le 0$ for all $x \in \Omega_0$ and $h(x)^{\mathrm{T}} L_f h(x) = 0$ if $x \in \Omega_0$ and $L_g h(x) = 0$.

Let A4 be replaced by Shiriaev's rank condition

$$\dim \mathcal{S}(x) \ge l \qquad \forall x \in \Omega_0$$

where $\mathcal{S}(x) = \mathrm{span}\{L_f^k(L_g h(x)), k = 0, 1, \ldots\}$, then the theorem statement remains true.

The essence of this theorem is that the positive definite objective function $Q(x)$ is a Lyapunov function and, therefore, the goal will be achieved. Some applications of the SG method are given in Fradkov *et al.* (1995), Fradkov (1996), Konjukhov *et al.* (1996) and Fradkov and Pogromsky (1998).

## 3. Models of the Furuta pendulum

### 3.1. *Dimension 4 model*

Consider the pendulum shown in figure 1. The rotor arm (corresponding to angle $\varphi$) is subjected to a torque, while no torque is applied directly to the pendulum shaft (angle $\theta$). Therefore, it is an underactuated system. The system parameters are: the mass of the pendulum $m$, the pendulum length $2l$, the arm radius $r$, moment of inertia of the pendulum $\mathcal{J}$, the moment of inertia of the motor, $\mathcal{J}_m$, the moment of inertia of the motor and the arm $\mathcal{J}_a$ and the torque constant $K$ (the control torque $\mathcal{F}$ is equal to $Ku$). The following parameters are introduced

$$\omega_0 = \sqrt{\frac{mgl}{\mathcal{J}}}; \quad \alpha = \frac{mrl}{\mathcal{J}}; \quad \beta = \frac{\mathcal{J}_a + mr^2}{\mathcal{J}}; \quad \gamma = \frac{K}{mgl}$$

The generalized coordinates are $[q_1, q_2] = [\theta, \varphi]$ and their conjugate momenta are defined as

$$p_1 = \mathcal{J}(\dot{q}_1 + \alpha \dot{q}_2 \cos q_1) \tag{5}$$

$$p_2 = \mathcal{J}(\alpha \dot{q}_1 \cos q_1 + (\beta + \sin^2 q_1)\dot{q}_2) \tag{6}$$

The Hamiltonian that represents the energy of the unforced system is

$$\mathcal{H}_{4D} = \frac{1}{2\Delta}\{(\beta + \sin^2 q_1)p_1^2 + p_2^2 - 2\alpha p_1 p_2 \cos q_1\} + \mathcal{J}\omega_0^2(\cos q_1 - 1) \tag{7}$$

with $\Delta = \mathcal{J}[\beta + \sin^2 q_1 - \alpha^2 \cos^2 q_1]$. Thus, the Hamilton equations are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \partial\mathcal{H}_{4D}/\partial q_1 \\ \partial\mathcal{H}_{4D}/\partial q_2 \\ \partial\mathcal{H}_{4D}/\partial p_1 \\ \partial\mathcal{H}_{4D}/\partial p_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \mathcal{J}\gamma\omega_0^2 \end{bmatrix} u \tag{8}$$

or

$$\dot{q}_1 = \frac{1}{\Delta}\left[(\beta + \sin^2 q_1)p_1 - \alpha p_2 \cos q_1\right] \tag{9}$$

$$\dot{q}_2 = \frac{1}{\Delta}[p_2 - \alpha p_1 \cos q_1] \tag{10}$$

$$\dot{p}_1 = \frac{1}{2\Delta^2}\mathcal{J}\left[(\beta + \sin^2 q_1)p_1^2 + p_2^2 - 2\alpha p_1 p_2 \cos q_1\right] \times \left[(1 + \alpha^2)\sin 2q_1\right] + \mathcal{J}\omega_0^2 \sin q_1 - \frac{1}{2\Delta}\left[p_1^2 \sin 2q_1 + 2\alpha p_1 p_2 \sin q_1\right] \tag{11}$$

$$\dot{p}_2 = \mathcal{J}\gamma\omega_0^2 u \tag{12}$$

### 3.2. *Dimension 2 model*

Åström and Furuta's control law was derived using a model of a pendulum for which the linear acceleration of the pivot is the control action. In this case the energy of the uncontrolled pendulum can be approximated by

$$\mathcal{H}_{2D} = \frac{1}{2\mathcal{J}}p_1^2 + \mathcal{J}\omega_0^2(\cos q_1 - 1) = \tfrac{1}{2}\mathcal{J}\dot{q}_1^2 + \mathcal{J}\omega_0^2(\cos q_1 - 1) \tag{13}$$

where $p_1 = \mathcal{J}\dot{q}_1$ is the conjugate momentum for variable $q_1$.

The Hamilton equations corresponding to this Hamiltonian function are

$$\begin{bmatrix} \dot{q}_1 \\ \dot{p}_1 \end{bmatrix} = \begin{bmatrix} p_1/J_p \\ \mathcal{J}\omega_0^2 \sin q_1 \end{bmatrix} + \begin{bmatrix} 0 \\ -Kl\cos q_1 \end{bmatrix} u \tag{14}$$

The dimension 2 energy (13) is an approximation to the energy (7). In order to study the validity of the approximation, expression (7) may be written as a function of $(q_1, q_2, \dot{q}_1, p_2)$

$$\mathcal{H}_{4D} = \frac{1}{2}\left(\frac{\Delta}{\beta + \sin^2 q_1}\right)\dot{q}_1^2 + \frac{1}{2}\frac{p_2^2}{\mathcal{J}(\beta + \sin^2 q_1)} + \mathcal{J}\omega_0^2(\cos q_1 - 1) \tag{15}$$

Note that if $r \to \infty$ and $\mathcal{J}_a/mr^2 \to \infty$

$$\beta \triangleq \frac{\mathcal{J}_a + mr^2}{\mathcal{J}} \to \infty \tag{16}$$

and therefore

$$\mathcal{H}_{4D} \to \tfrac{1}{2}\left(\frac{\Delta}{\beta + \sin^2 q_1}\right)\dot{q}_1^2 + \mathcal{J}\omega_0^2(\cos q_1 - 1)$$

and

$$\Delta \triangleq \mathcal{J}(\beta + \sin^2 q_1 - \alpha^2 \cos^2 q_1) \to \mathcal{J}(\beta - \alpha^2 \cos^2 q_1)$$

Besides

$$\frac{\alpha^2}{\beta} = \frac{l^2}{\mathcal{J}((\mathcal{J}_a/mr^2) + 1)}$$

and, as $\mathcal{J}_a/mr^2 \to \infty$, then $\alpha^2/\beta \to 0$. Therefore, $\Delta \to \mathcal{J}\beta$ and, thus, $\Delta/(\beta + \sin^2 q_1) \to \mathcal{J}$.

Therefore, if $r \to \infty$ and $\mathcal{J}_a/mr^2 \to \infty$, the dimension 4 energy (15) approaches the dimension 2 approximation (13). This fact suggests that, under this assumption, the laws obtained from the dimension 2 model may be valid for the actual system. As a matter of fact, the well-known control law proposed in Åström and Furuta (2000), is successful for most of the experiments carried out with the current laboratory pendula. The reason for this success is that the conventional design of pendula leads to parameter values that fulfill the assumptions above. However, some practical counter-examples will be seen in § 5.

## 4. Speed pseudogradient control laws

The swing-up problem consists of swinging the pendulum to the upright position. This can be accomplished driving the energy towards the energy of the upright position. Two strategies for swinging the pendulum up are presented; both are based in the SG method. First, the law presented by Åström and Furuta (2000) is recalled. This law is based on the dimension 2 model (14), which is an approximation to model (9)–(12). Then, a dimension 4 model is used and a new law that takes into account the velocity of the arm is proposed. However, in practice, variable $\varphi$ does not affect the obtained control law. Therefore, a dimension 3 control law is actually obtained.

### 4.1. *Law obtained with a dimension 2 model*

The control law proposed in Wiklund *et al.* (1993) and Åström and Furuta (2000) can be derived by the SG method when only the pendulum is considered, and the arm is neglected. Then, the dimension 2 model (14) is the one taken into account.

In order to apply the SG method consider the objective function $Q = \frac{1}{2}(\mathcal{H}_{2D} - \mathcal{H}^*_{2D})^2$, where $\mathcal{H}^*_{2D}$ is the desired energy of the upright position, which is zero when the origin for the potential energy is the one chosen here. Therefore, the affine-in-control system is given by equation (14) and the objective output is

$$y = h(x) = \mathcal{H}_{2D} - \mathcal{H}^*_{2D} \tag{17}$$

where $x = (q, p)$ and where $q = q_1$ and $p$ is the conjugate momentum for variable $q$. Clearly, if the SG method is applied, the control law is given by

$$\begin{aligned} u &= -\lambda \frac{Kl}{J_p}(\mathcal{H}_{2D} - \mathcal{H}^*_{2D}) p \cos q \\ &= -\lambda K l (\mathcal{H}_{2D} - \mathcal{H}^*_{2D}) \dot{q} \cos q \end{aligned} \tag{18}$$

where $\lambda$ is a positive control gain. In Åström and Furuta (2000) some modifications are proposed in order to make it more efficient, e.g. $u = \text{sat}\{\overline{\lambda}(\mathcal{H}_{2D} - \mathcal{H}^*_{2D})\} \times \text{sign}(\dot{q} \cos q)$, with $\overline{\lambda} = \lambda k l$. This variant of (18) is more efficient and also makes $\dot{Q} \leq 0$. Some other modifications serve as attempts to decrease the arm velocity trying to correct the lack of consideration of $\dot{q}_2$ in the simplified model. In these cases, $Q$ is no longer a Lyapunov function.

**Proposition 4.1:** *When control law* (18) *is applied to system* (14), $Q \to 0$ *for almost all initial conditions from* $\Omega_0 = \{x \in \mathbb{R}^n : Q(x) \leq Q_0\}$ *and, thus, the pendulum will tend to the homoclinic orbit* $\mathcal{H}_{2D} = 0$.

**Proof:** Let us check the conditions of Theorem 1. Assumption A1 is fulfilled because all functions $f(x)$, $g(x)$ and $h(x)$, in model (14)–(17), and their second partial derivatives are bounded in a region $\Omega_0$. As $Q(x)$ is conservative when $u = 0$, $L_f h(x) = 0$ and Assumption A2 is valid. Assumption A3 is also valid, because any connected subset between sets of $D_\epsilon = \Omega_0 \cap \mho_0$ is compact where $\mho_0 = \{q, p: p = 0\} \cup \{q, p: q = k(\pi/2), k = \pm 1, \pm 2, \ldots\}$. The last assumption A4 is not fulfilled. Matrix $L_g h(x)^{\mathrm{T}} = (k/J_p) p \cos q$, has a rank $l = 0 \neq 1$ for $D_0 = \{p = 0\} \cup \{q = k(\pi/2), k = \pm 1, \pm 2, \ldots\}$. As assumption A4 is violated for the set $D_0$, the statement of the proposition is derived using Remark 1. □

### 4.2. *Law obtained with a dimension 4 model*

In order to take into account the arm velocity in the derivation of the control law, the dimension 4 model is adopted. The affine-in-control system is now given by (9)–(12). If the objective function is still the square of the divergence of the total system energy $Q = Q_1 = \frac{1}{2}\zeta_1^2(\mathcal{H} - \mathcal{H}^*)^2$, the mechanical system will tend towards the surface $\mathcal{H} = 0$ in the space $(q_1, q_2, \dot{q}_2) = (q_1, \dot{q}_1, \dot{q}_2)$ (see figure 2(*a*)). This aim does not guarantee that the system will pass near the origin. If now the objective function is modified, so $Q = Q_1 + Q_2$ with $Q_2$ a positive semidefinite function, the system will tend towards the curve $(Q_1 = 0, Q_2 = 0)$. If $Q_2$ is chosen correctly, the origin of the state space will belong to this curve. One possibility is to choose both $Q_1$ and $Q_2$ conservative for the unforced system. The Hamiltonian structure of the system model can help to find function $Q_2$. It can be seen that, as the system is symmetrical with respect to angle $\varphi$, the conjugate momentum $p_2$ is conservative for $u = 0$. This fact is obvious in (12). Thus, it is reasonable to choose $Q_2 = \frac{1}{2}\zeta_2^2(p_2 - p_2^*)^2$. Note that the objective curve is a trajectory for the open loop system ($u = 0$) corresponding to the homoclinic orbit (figure 2(*b*)).

Therefore, the objective of the SG controller will be to bring the system to this homoclinic orbit and then, with $u = 0$, the system will evolve towards the desired position and, once close to it, the control strategy can be commuted to a local stabilizing controller. Thus the objective function is

$$Q = \underbrace{\tfrac{1}{2}\zeta_1^2(\mathcal{H} - \mathcal{H}^*)^2}_{Q_1} + \underbrace{\tfrac{1}{2}\zeta_2^2(p_2 - p_2^*)^2}_{Q_2}$$

and the output of system (9)–(12) is

$$y = h(x) = \begin{bmatrix} \zeta_1(\mathcal{H} - \mathcal{H}^*) \\ \zeta_2(p_2 - p_2^*) \end{bmatrix} \tag{19}$$

where $x = (q_1, q_2, p_1, p_2)$ and $\zeta_1$, $\zeta_2$ are arbitrary positive constants.

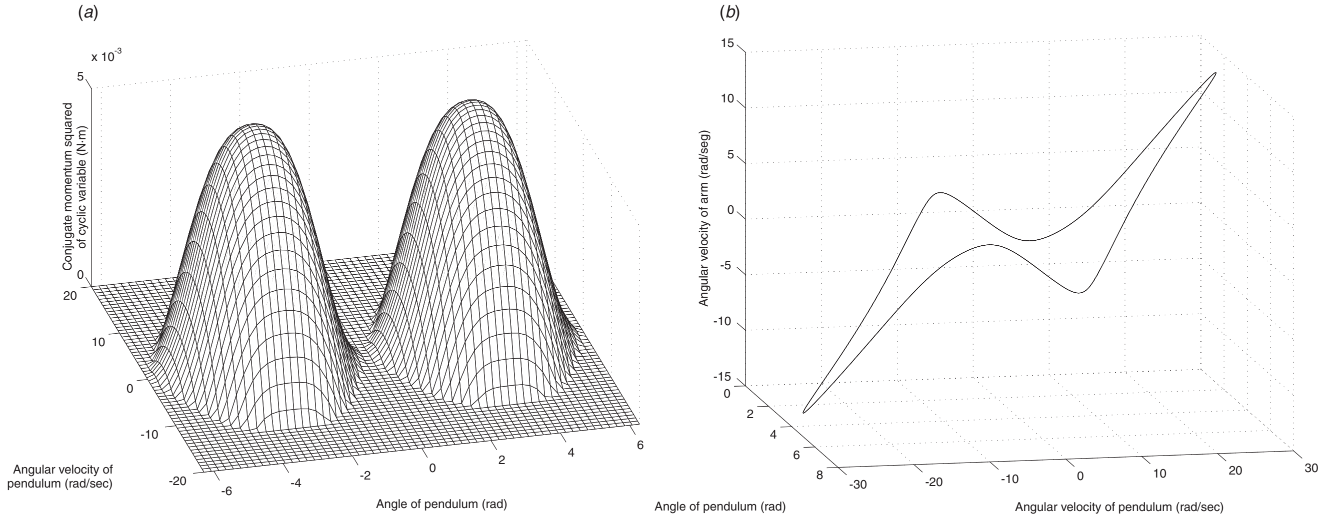


Figure 2. (*a*) Surfaces $Q_1 = 0$ and $Q_2 = 0$ in the space $(\theta, \dot{\theta}, p_2)$. (*b*) Homoclinic orbit in the space $(\theta, \dot{\theta}, \dot{\varphi})$.

The SG algorithm (4) yields

$$u = -\bar{\lambda}(\zeta_1^2(\mathcal{H} - \mathcal{H}^*)\dot{q}_2 + \zeta_2^2(p_2 - p_2^*)) \qquad (20)$$

where $\bar{\lambda} = \lambda \mathcal{J}\gamma\omega_0^2$.

The effectiveness of this control law depends on a careful selection of constants $\zeta_1$ and $\zeta_2$. However, for any value of $\zeta_1$ and $\zeta_2$ the system works well, in the sense that the objective is reached. It should be noted that the control law depends only on the variables $[q_1, \dot{q}_1, \dot{q}_2]$, so it can be considered a dimension 3 control problem.

**Proposition 4.2:** *When control law* (20) *is applied to system* (9)–(12), $Q \to 0$ *for almost all initial conditions from* $\Omega_0$ *and, thus, the pendulum will tend to the homoclinic orbit* $\{\mathcal{H} = 0\} \bigcap \{p_2 = 0\}$.

**Proof:** Let us check the conditions of Theorem 1. Assumption A1 is fulfilled because all functions $f(x)$, $g(x)$ and $h(x)$, in model (9)–(12) and (19), and their second partial derivatives are bounded in a region $\Omega_0 = \{x \in \mathbb{R}^n : Q(x) \leq Q_0\}$. Since $Q(x)$ is conservative for $u = 0$ then $L_f h(x) = 0$ and Assumption A2 is fulfilled. Assumption A3 is also true, since $D_\epsilon = \Omega_0 \cap \mho_0 = \Omega_0$, because $\mho_0 = \{x \in \mathbb{R}^n : \det((L_g h(x))^{\mathrm{T}} L_g h(x))\}$ and

$$|L_g h(x)^{\mathrm{T}} L_g h(x)| = (\mathcal{J}\gamma\omega_0^2)^2 \begin{bmatrix} \zeta_1 \dot{q}_2 \\ \zeta_2 \end{bmatrix} [\zeta_1 \dot{q}_2 \quad \zeta_2] = 0 \quad (21)$$

As in the dimension 2 case, Assumption A4 is not fulfilled. Furthermore, in this case, Remark 1 is not applicable. Matrix $L_g h(x)^{\mathrm{T}} \propto [\zeta_1 \dot{q}_2, \zeta_2]^{\mathrm{T}}$, has rank $1 < l = 2$ for all points in the state space. On the other hand, this assumption can only be valid if $m \geq l$ (number of the inputs $\geq$ number of outputs), and in our case the system is underactuated. Nevertheless, using Remark 2 the statement of the proposition is proven since

$$\mathcal{S}(x) = \mathrm{span}\left\{ \begin{bmatrix} \zeta_1 \dot{q}_2 \\ \zeta_2 \end{bmatrix}, (\mathcal{J}\gamma\omega_0^2) \begin{bmatrix} \zeta_1/\Delta \\ 0 \end{bmatrix} \right\} \qquad (22)$$

and $\dim \mathcal{S}(x) = l = 2 \ \forall x \in \Omega_0$. □

**Remark 3:** Notice from (20) that $u \to 0$ as the system approaches the homoclinic orbit $\{\mathcal{H} = \mathcal{H}^*\} \bigcap \{p_2 = p_2^*\}$.

## 5. Benchmark: simulations and experiments

This section will show simulations and experimental results on the Furuta pendulum depicted in figure 3 (see the Appendix). In order to approach a Hamiltonian system, a LuGre model (Canudas de Wit *et al.* 1995) has been used in order to partially compensate the friction of the pendulum arm. The saturation limits in all experiments are $|u| \leq 0.25$. In order to compare both control laws (18) and (20) some experiments are included.

Control law (18) based on the approximate dimension 2 model does not work for every value of $\lambda$ (even in the simulations). If the assumptions made in the previous section are not valid, the swing-up is only accomplished for some particular values of parameter $\lambda$. In other words, parameter $\lambda$ must be tuned. In the benchmark Furuta pendulum used here (see figure 3), $\beta \approx 6.4$ and $\alpha^2/\beta \approx 0.25$. We have found that for $\lambda \leq 0.01$ the task is not achieved. In figure 4 the results of a simulation and an experiment for the case $\lambda = 0.02$ in which the objective is achieved can be seen. On the other hand, control law (20) is easier to be tuned. See in figure 5 the results of a real experiment with law (20) with a response that it is faster than the one of figure 4 obtained with law (18).

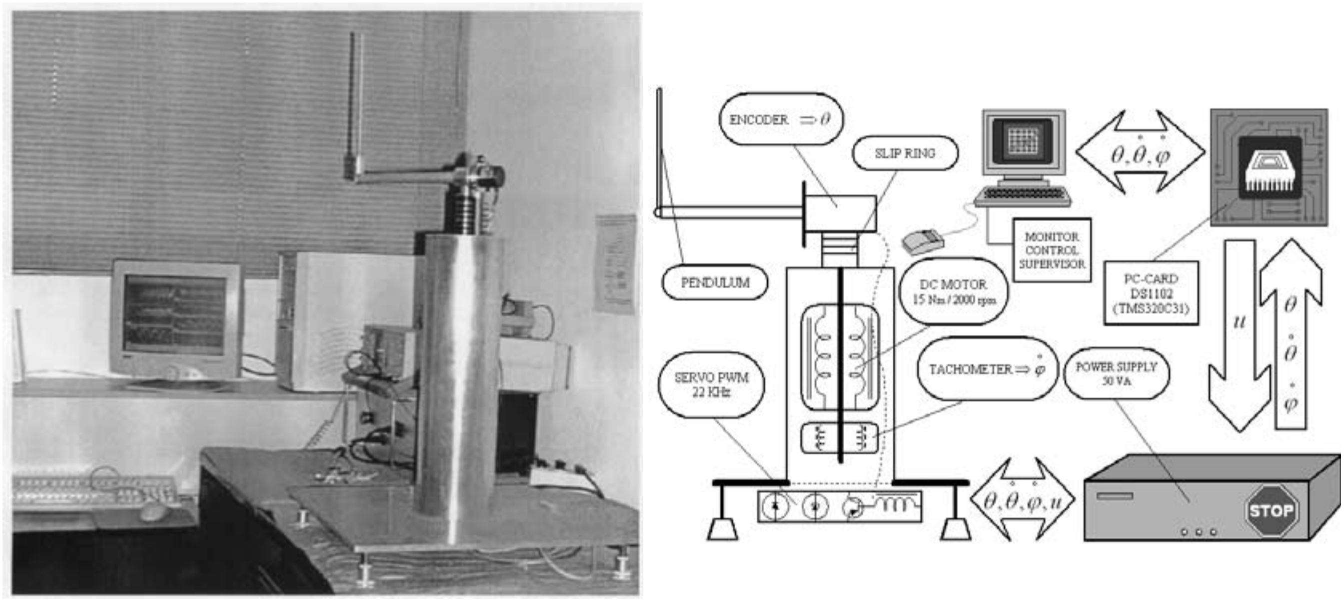


Figure 3. Experimental Furuta pendulum system (*a*) and control system (*b*).

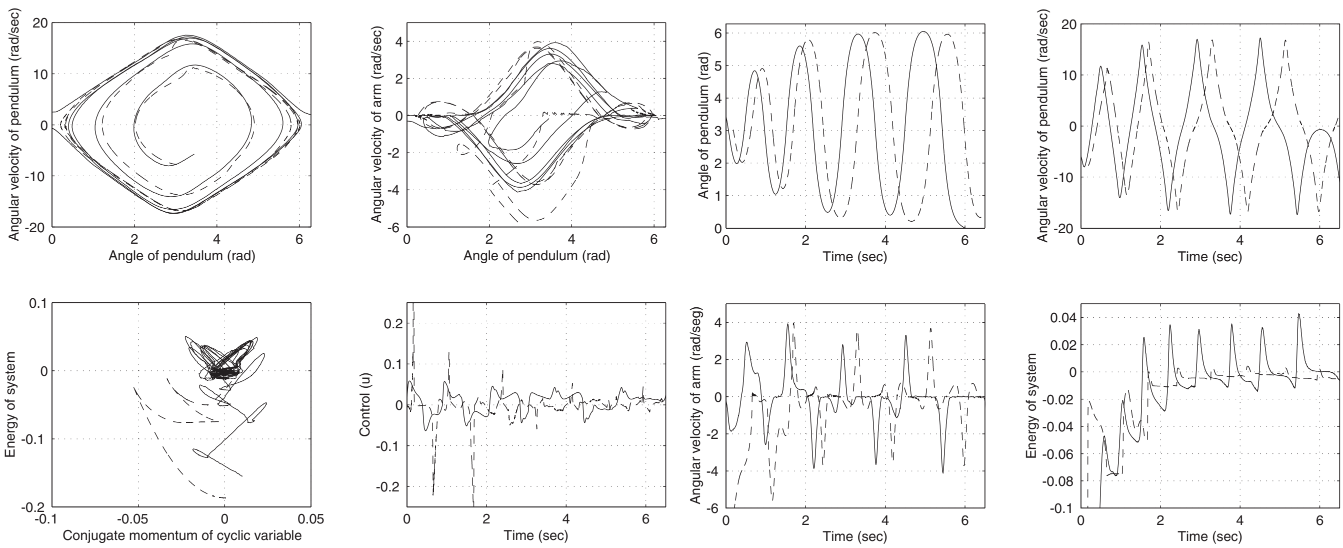


Figure 4. Simulated (dashed) and experimental (solid) results for control law (18).

In order to make evident the advantages of the new law, some physical changes in our laboratory pendulum have been performed in order to violate some of the conditions above. We have chosen to increase the mass of the pendulum putting an additional weight of 300 g at the centre of mass of the pendulum. With this change, control law (18) fails for every value of $\lambda$. Figure 6 shows an example with $\lambda = 0.1$. It can be seen that the state of the system does not pass close to the origin and, therefore, if the hybrid controller were used, the local law would not have the opportunity to work. On the other hand, control law (20) is successful. See, e.g., figure 7 where $\lambda = 0.5$, $\zeta_1 = 0.3$ and $\zeta_2 = 1$.

A final set of experiments have been performed in order to show the behaviour of the pendulum when the hybrid controller with both stages (swing-up and stabilization) is implemented. The stabilization problem is solved with an LQR controller. Experiments comparing control laws (18) and (20) for this whole problem have been performed with the initial configuration of the pendulum. Figure 8 shows experiments with both hybrid control laws. In both experiments when the pendulum is close to the upright position,

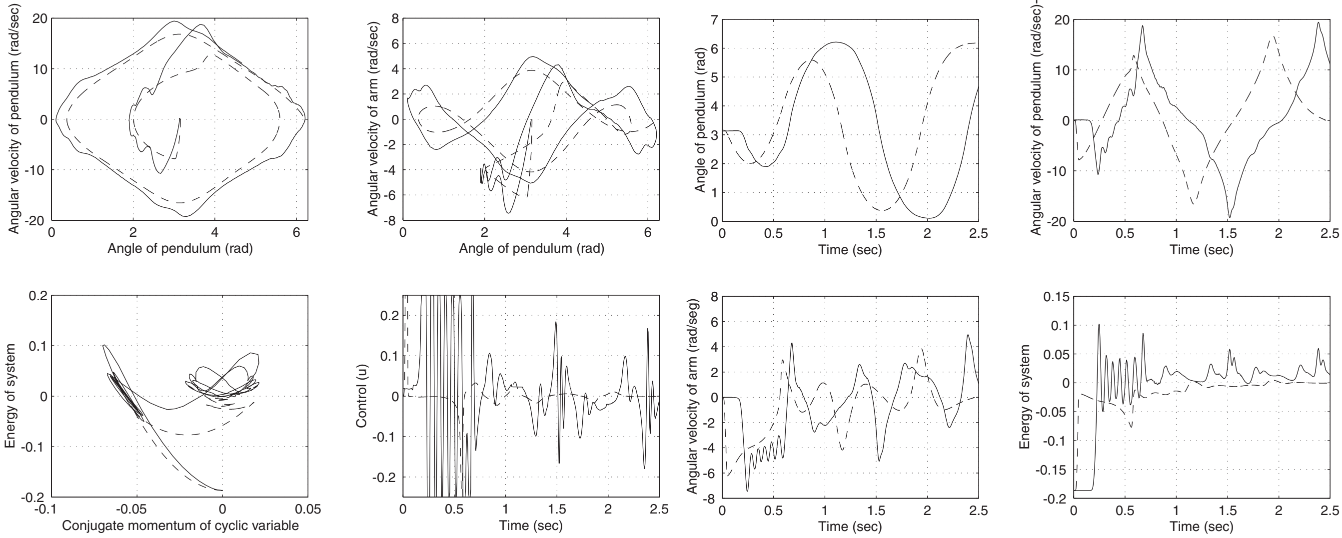


Figure 5. Simulated (dashed) and experimental (solid) results for control law (20).

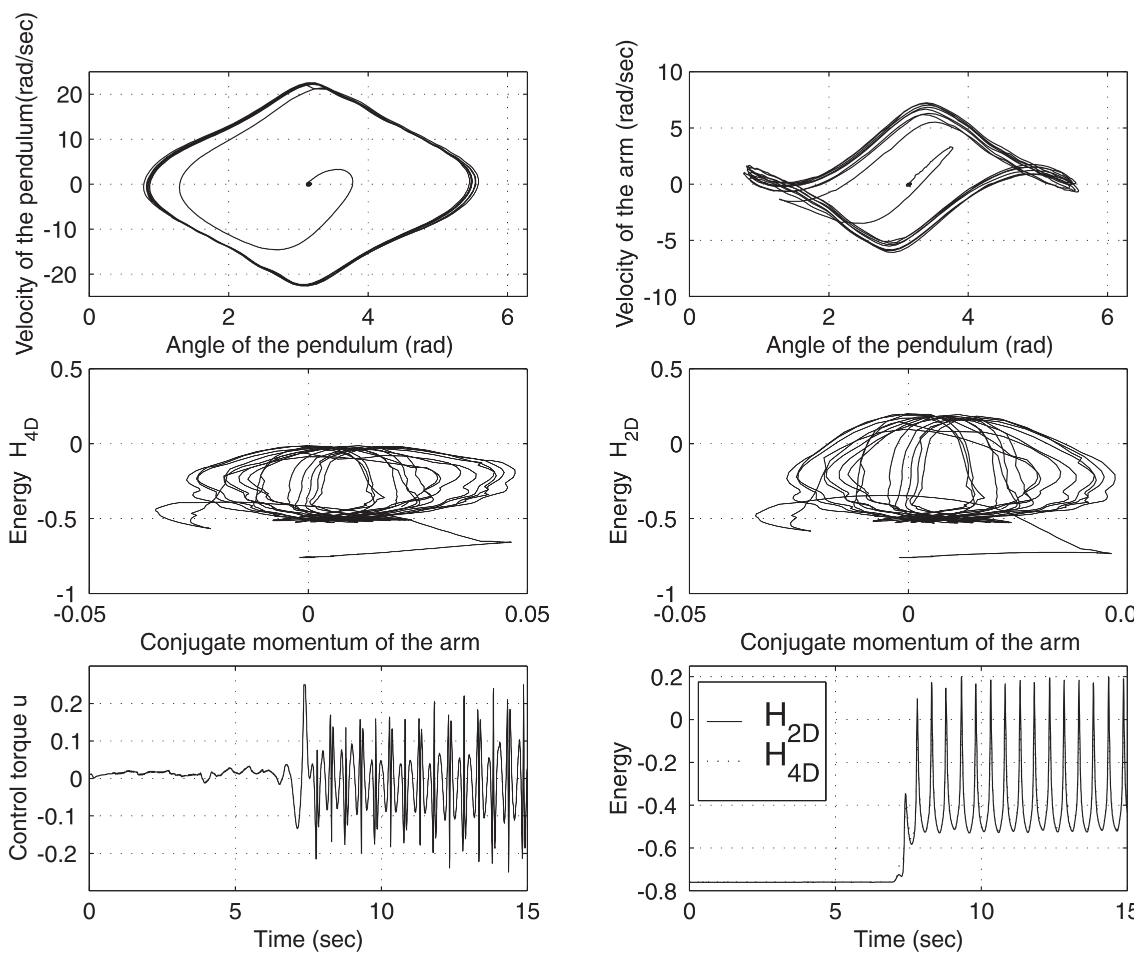


Figure 6. Pathological behaviour with control law (18) where a weight has been added to the pendulum.

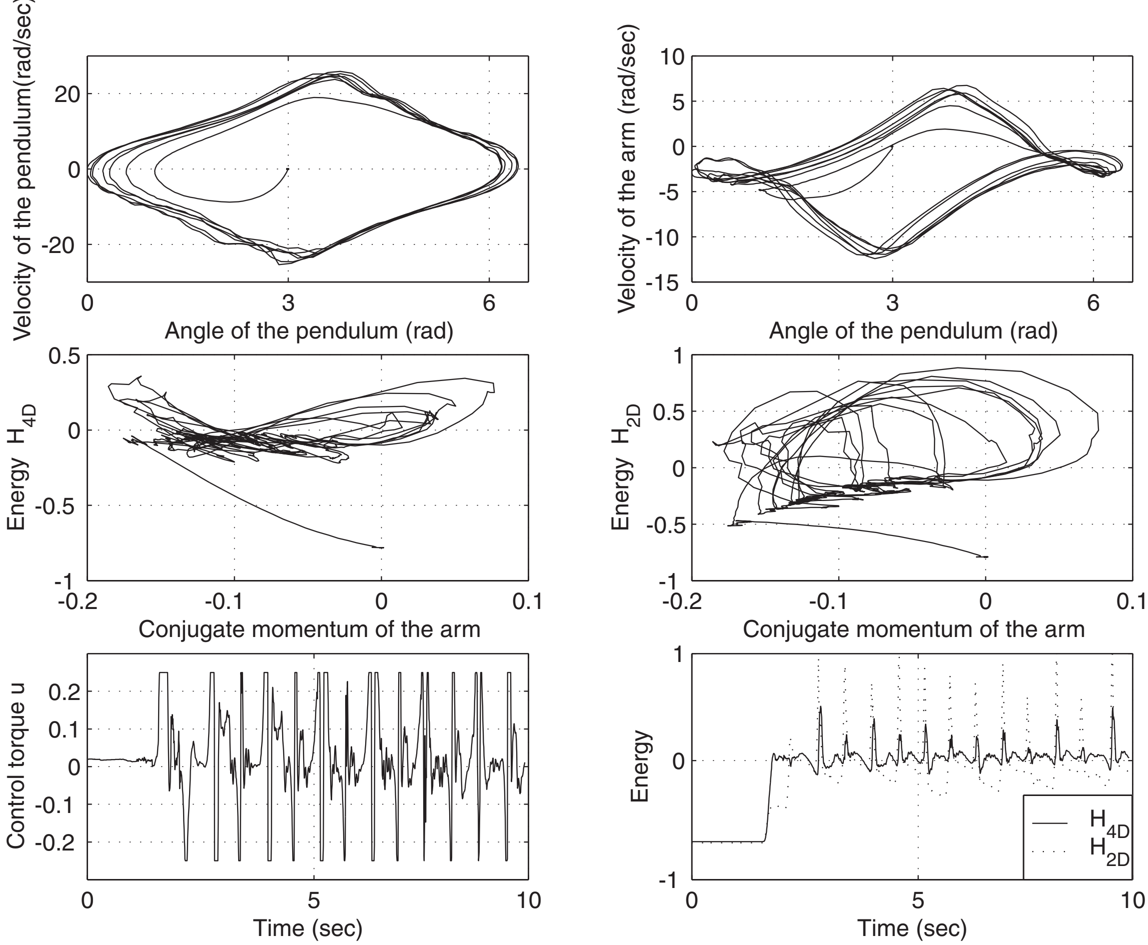


Figure 7. Success of the control law (20) where a weight has been added to the pendulum.

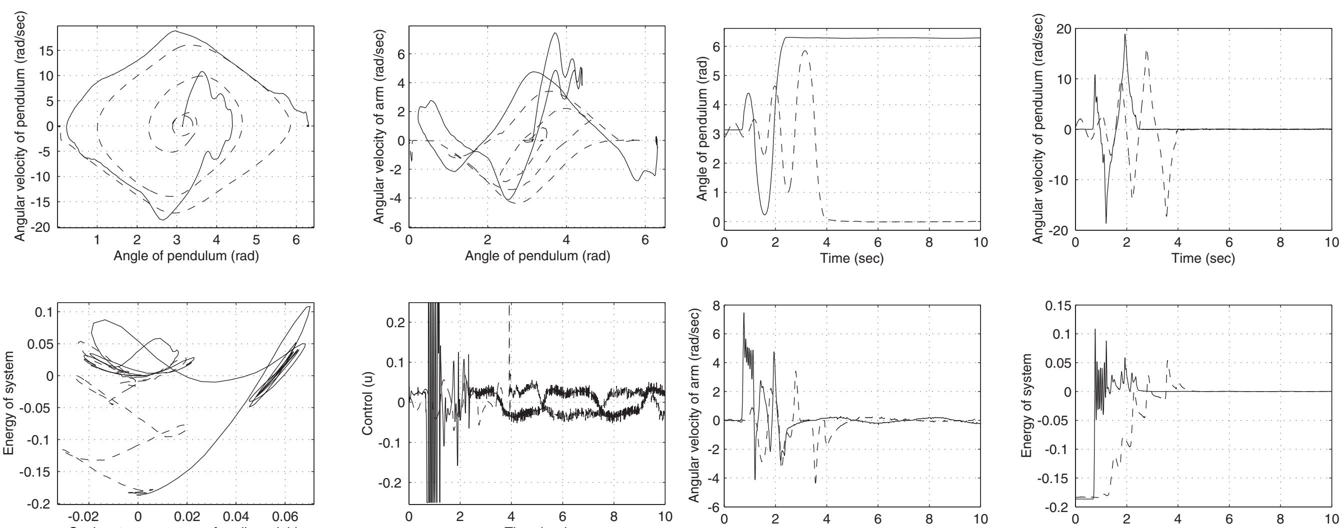


Figure 8. Hybrid control by switching to a linear local controller. Phase portraits and variables versus time, for law (18) (dashed) and law (20) (solid). Experimental framework.

the control law is switched to a linear controller, in order to stabilize the pendulum. Both experiments correspond to the same initial conditions $(\theta, \dot{\theta}, \dot{\varphi}) = (3.14, 0, 0)$. It can be seen that both control laws are successful.

## 6. Conclusions

A new control law for swinging the Furuta pendulum up has been obtained applying Fradkov's speed-gradient method to a dimension 4 model. The associated Lyapunov function is the sum of the squares of the errors in the energy—which is conservative for the unforced system since it is Hamiltonian—and in the conjugate momentum associated to the cyclic variable—which is also a conservative quantity for the unforced system. Global stability (except for a zero-measure initial set) has been proved.

The new law outperforms the well-known Åström and Furuta law as has been shown by simulations and experiments due to the fact that this law is based on an approximate dimension 2 model. Nevertheless, in order to show examples where the dimension 2 law fails, the pendulum has been brought to extreme conditions. Thus, in the experimental example the mass of the pendulum has been sensibly increased. This fact suggests that the dimension 2 law can be successful in normal operation of most laboratory Furuta pendula. In any case, the new law presented in this paper is formally more correct, has a wider range of operation conditions and is easier to be tuned.

Finally, it must be pointed out that this law can be directly extended to other underactuated systems with two degrees of freedom with one cyclic variable. This last property makes the associated conjugate momentum conservative for the unforced system and, therefore, the same procedure can be applied. Note that, if the well-known partial linearization is applied in advance, this nice property will be broken and the same procedure is no longer applicable in spite of the partial linearization simplifying the system equations. Partial linearization is very useful for many problems but in some cases, as the one presented here, it does not help since it destroys the Hamiltonian structure of the system.

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