Problem-specific theoretical results for "Assembly flowshop scheduling problem: Speed-up procedure and computational evaluation".

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In this appendix, we provide an equivalent proof of the results presented in Sections 3 using solely concepts from the specific scheduling problem. Firstly, using the forward and backward codifications, two definitions for the forward and backward critical path are introduced (see Definitions 1 and 2, respectively). Then, by using these definitions we are able to prove that at least one critical path exists in the problem, and how it changes when a new job is inserted in the sequence.

Before starting with some required definitions, we show an example of the forward and backward codifications. Thereby, in Figures 1 and 2 we represent a sequence $(2, 3, 1, 4)$ in an instance composed by three pre-assembly machines (machines $1, 2$ and 3) and three assembly machines (machines $4, 5$ and 6). From these figures, we can see (using both codifications) that there are operations, denoted as critical, which cannot be moved without worsen the objective function of the problem (see e.g. operation $O_{6,1}$ in Figure 1). In this regard, we introduce two new definitions, forward critical path and backward critical path (see Definitions 1 and 2).

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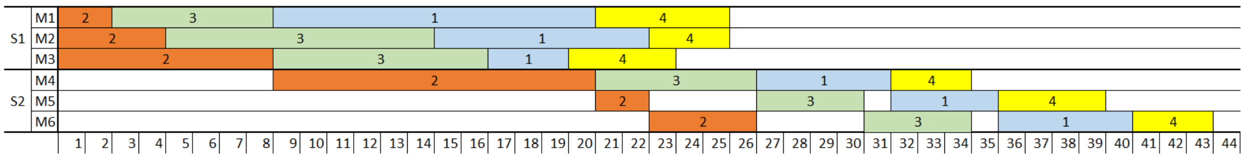

Figure 1: Example of forward codification with four jobs and six machines

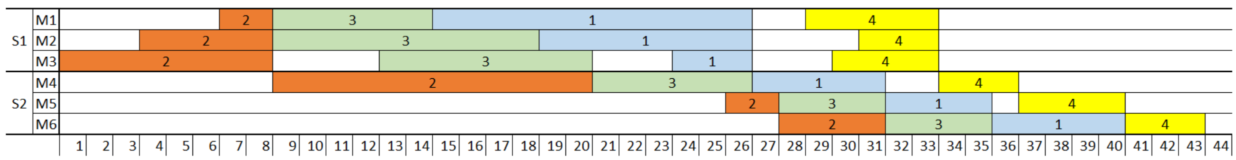

Figure 2: Example of backward codification with four jobs and six machines

Definition 1. ***(Fordward critical path)*** *Let $\mathcal{I}$ be an instance of the $DPm \rightarrow Fm||C_{max}$ problem and $\Pi = (\pi_1, \ldots, \pi_n)$ a solution for this problem. Then, we define the forward critical path of Π, $\mathcal{P}(\Pi)$, as a path to go from some operation O_{i,π_1} in the pre-assembly phase (with $i \in \{1, \ldots, m_1\}$) to operation O_{m,π_n}, without adding any idle or waiting times, and moving always either to a posterior job or stage. Equivalently, we define the forward critical path of operation $O_{i,j}$, denoted as $\mathcal{P}(O_{i,j})$, as the path to go from the first operation to $O_{i,j}$ without adding idle or waiting times.* □

Definition 2. ***(Backward critical path)*** *Let $\mathcal{I}$ be an instance of the $DPm \rightarrow Fm||C_{max}$ problem and $\overline{\Pi}$ the reversed sequence with respect to $\Pi = (\pi_1, \ldots, \pi_n)$, i.e. $\overline{\Pi} = (\overline{\pi}_1, \ldots, \overline{\pi}_n) = (\pi_n, \ldots, \pi_1)$. Then, we define backward critical path of Π, $\overline{\mathcal{P}}(\Pi)$, as a path to go from some operation O_{m,π_n} in the assembly phase to the latest operation O_{i^*,π_1} in the shop (with $i^* \in \{1 \ldots m_1\}$), without adding any idle or waiting times, and moving always either in a decreasing direction of jobs or stages. Equivalently, we define the backward critical path of operation $O_{i,j}$, denoted as $\overline{\mathcal{P}}(O_{i,j})$, as the path to go from the last operation to $O_{i,j}$ without adding idle or waiting times.* □

Note that in both cases, the sum of all processing times in the corresponding critical path must be equal to the makespan. In Figures 3 and 4, we show the forward and backward critical paths for the previous example, respectively. Next, in Theorem 1, we prove that the forward and backward critical path must exist in the problem under considera-

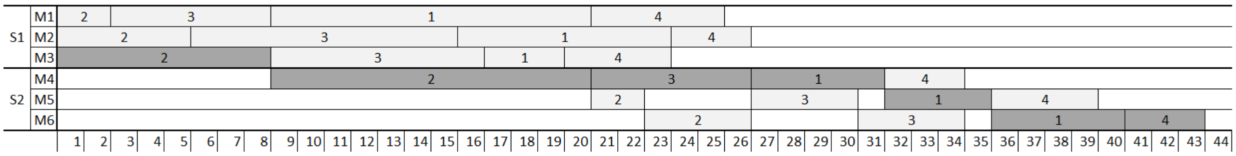

Figure 3: Example of forward critical path

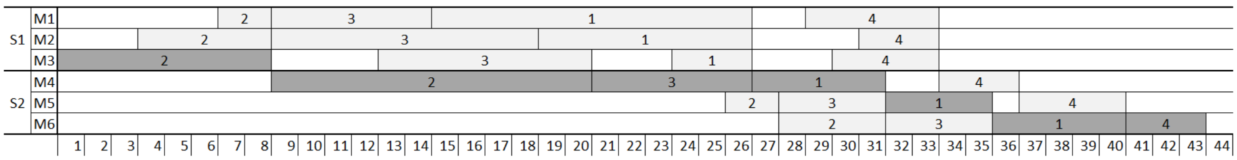

Figure 4: Example of backward critical path

tion. Finally, the Theorem 2 is introduced to indicate an efficient equation to calculate the objective function when a job is inserted in the best position of a partial sequence.

Theorem 1. *Let $\mathcal{I}$ be an instance of the $DPm \to Fm||C_{max}$ problem and $\Pi = (\pi_1, \dots, \pi_n)$ a solution for this problem. Then, there is at least one forward critical path, $\mathcal{P}(\Pi)$, using the forward codification.*

Proof. Without lost of generality, let us consider last scheduled operation, O_{m,π_n}, in the shop. Then, according to Equation 4 and in case that $m_2 > 1$ and $n > 1$, we can move either to operation O_{m-1,π_n} or operation $O_{m,\pi_{n-1}}$ without adding any idle time, since either $C_{m,\pi_n} = C_{m-1,\pi_n} + p_{m,\pi_n}$, or $C_{m,\pi_n} = C_{m,\pi_{n-1}} + p_{m,\pi_n}$, respectively. This move can be repeated until either $k = 1$ or $i = m_1 + 1$, which are analysed below:

Case $k = 1$: According to Equation 3, we have $C_{i,\pi_1} = C_{i-1,\pi_1} + p_{i,\pi_1}, i \in \{m_1 + 2, \dots, m\}$. In other words, we are moving from operation O_{i,π_1} to operation O_{i-1,π_1} without any idle time between them. Similarly, considering $k = 1$ and $i = m_1 + 1$, Equation 2 establishes $C_{m_1+1,\pi_1} = \max_{i \le m_1}\{C_{i,\pi_1}\} + p_{m_1+1,\pi_1}$. Denoting by i^* the machine with maximum completion time for job π_1 in the pre-assembly phase (i.e. $i^* = \arg\max_{i \le m_1} C_{i,\pi_1}$), the equation can be rewriting as $C_{m_1+1,\pi_1} = C_{i^*,\pi_1} + p_{m_1+1,\pi_1}$, and operations O_{m_1+1,π_1} and O_{i^*,π_1} join without idle time. Finally, for $k = 1$ and machine $i^* \le m_1$, Equation 1 establishes that $C_{i\pi_1} = p_{i,\pi_k}$, which is the first operation in the shop.

Case $i = m_1 + 1$*:* According to Equation 2, $C_{m_1+1,\pi_k} = \max\{\max_{i \leq m_1}\{C_{i,\pi_k}\}, C_{m_1+1,\pi_{k-1}}\} + p_{m_1+1,\pi_k}, k \in \{1,\ldots,n\}$, i.e. either operation O_{m_1+1,π_k} moves to operation $O_{m_1+1,\pi_{k-1}}$ (if $C_{m_1+1,\pi_k} = C_{m_1+1,\pi_{k-1}} + p_{m_1+1,\pi_k}$) or to the machine i^* which satisfies $C_{m_1+1,\pi_k} = C_{i^*,\pi_k} + p_{m_1+1,\pi_k}$, without adding idle time in both cases. We have two possibilities, either the critical path move to operation $k = 1$ (analysed above) or to machine i^*. For this subcase i^*, according to Equation 1 $C_{i^*\pi_k} = C_{i^*,\pi_{k-1}} + p_{i,\pi_k}, k \in \{1,\ldots,n\}$, i.e. operation O_{i^*,π_k} moves to operation $O_{i^*,\pi_k - 1}$.

So, it is demonstrated that a path to go from the first to the last operation in the shop must exist.

□

From this theorem, the following corollaries can be derived.

Corollary 1. *Let $\mathcal{I}$ be an instance of the $DPm \rightarrow Fm||C_{max}$ problem, $\Pi = (\pi_1,\ldots,\pi_n)$ a solution for this problem, and $\mathcal{P}(\Pi)$ one critical path. Then, $\mathcal{P}(\Pi)$ is also a critical path applying the backward codification.*

Proof. The proof is obvious according to Definition 1. Since there is no idle time between the operations in the critical path, it is clear that the path to go from the first to the last operation without adding idle times is the same that to go from the last to the first operation, and $\mathcal{P}(\Pi)$ is also a critical path for the backward codification. □

Corollary 2. *Let $\mathcal{I}$ be an instance of the $DPm \rightarrow Fm||C_{max}$ problem and $\Pi = (\pi_1,\ldots,\pi_n)$ a solution for this problem. Then, there must exist at least one forward and one backward critical path $\mathcal{P}(O_{i,j})$ and $\overline{\mathcal{P}}(O_{i,j})$.*

Proof. The proof is obvious using the same reasoning as in Theorem 1. □

Finally, based on these previous results, we can demonstrate the following theorem which describes how to obtain the makespan when a new job σ is inserted in any position of a partial sequence.

Theorem 2. *Let $\mathcal{I}$ be an instance of the $DPm \rightarrow Fm||C_{max}$ problem and $\Pi = (\pi_1, \ldots, \pi_{k-1})$ a solution for this problem with $k-1$ jobs. Then, the makespan obtained after inserting job σ in a position l (with $l \in \{1, \ldots, k\}$) of Π is defined by:*

$$C_{max} = \max_{i \in \{1,\ldots,m_1+m_2\}} \{C^{\sigma}_{il} + \overline{C}_{i\pi_l}\} \tag{1}$$

where C^{σ}_{il} is the completion time on machine i of job σ when is inserted in position l of sequence Π (using the forward codification), and $\overline{C}_{i\pi_l}$ is the completion time of job Π_l using backward codification.

Proof. The theorem can be proved by contradiction. Let us assume that C_{max} is either lower or greater than $\max_{i=1,\ldots,m_1+m_2}\{C^{\sigma}_{il} + \overline{C}_{i\pi_l}\}$.

Let us first start with the first assumption, i.e. $C_{max} < \max_{i\in\{1,\ldots,m_1+m_2\}}\{C^{\sigma}_{il} + \overline{C}_{i\pi_l}\}$. Let $i^{'}$ denote the machine with maximum value of both completion times, i.e. $i^{'} = \arg\max_{i\in\{1,\ldots,m_1+m_2\}}\{C^{\sigma}_{il} + \overline{C}_{i\pi_l}\}$. On the one hand, by Equations 1, 2, 3, and 4, the completion time of operation $O_{i'\sigma}$ cannot be reduced considering forward codification, without reducing their processing times (or some previous one). On the other hand, by Equations 5, and 6, the completion time of operation $\overline{O}_{i'\pi_l}$ cannot be reduced considering backward codification (analogously without reducing any processing time), i.e. the time between the starting time of operation $\overline{O}_{i'\pi_l}$ and the final operation in the shop cannot be compressed more than $\overline{C}_{i\pi_l}$. Then, as operation $O_{i'\sigma}$ is scheduled before operation $\overline{O}_{i'\pi_l}$, and the completion times (C^{σ}_{il} and $\overline{C}_{i\pi_l}$) cannot be compressed more, it is clear than $C_{max} \geq \max_{i\in\{1,\ldots,m_1+m_2\}}\{C^{\sigma}_{il} + \overline{C}_{i\pi_l}\}$.

Secondly, let us suppose that $C_{max} > \max_{i\in\{1,\ldots,m_1+m_2\}}\{C^{\sigma}_{i\pi_l} + \overline{C}_{i\pi_l}\}$. Regarding a generic i, $C_{max} > \max_{i\in\{1,\ldots,m_1+m_2\}}\{C^{\sigma}_{il} + \overline{C}_{i\pi_l}\}$ can be written as $C_{max} > C^{\sigma}_{il} + \overline{C}_{i\pi_l}$ ($i \in \{1, \ldots, m_1 + m_2\}$). Since it has been proved that C^{σ}_{il} and $\overline{C}_{i\pi_l}$ cannot be compressed, this expression can be equivalently formulated by $C_{max} = IT_i + C^{\sigma}_{il} + \overline{C}_{i\pi_l}$, where IT_i ($IT_i > 0$, $i \in \{1, m_1 + m_2\}$) is the idle time between job σ and job π_l on machine i, when the sequence is scheduled forward until σ and backward from π_l. Finally, as idle time IT_i is greater than 0 for every machine, the schedule is not semi-active and there is a contradiction. $\square$

Let us continuous with the previous example supposing that job 5 is going to be inserted in the second position of the actual sequence (2,3,1,4). Then, according to Theorem 2, the makespan after the insertion can be calculated using $C_{max} = \max_{i \in \{1,\ldots,m_1+m_2\}}\{C^{\sigma}_{il} + \overline{C}_{i\pi_l}\}$. In Figure 5, we indicate the values for $C^{\sigma}_{il} + \overline{C}_{i\pi_l}$ for each machine $i \in \{1, \ldots, 6\}$. Clearly, the makespan is obtained, as the sum of both variables, in the second machine.

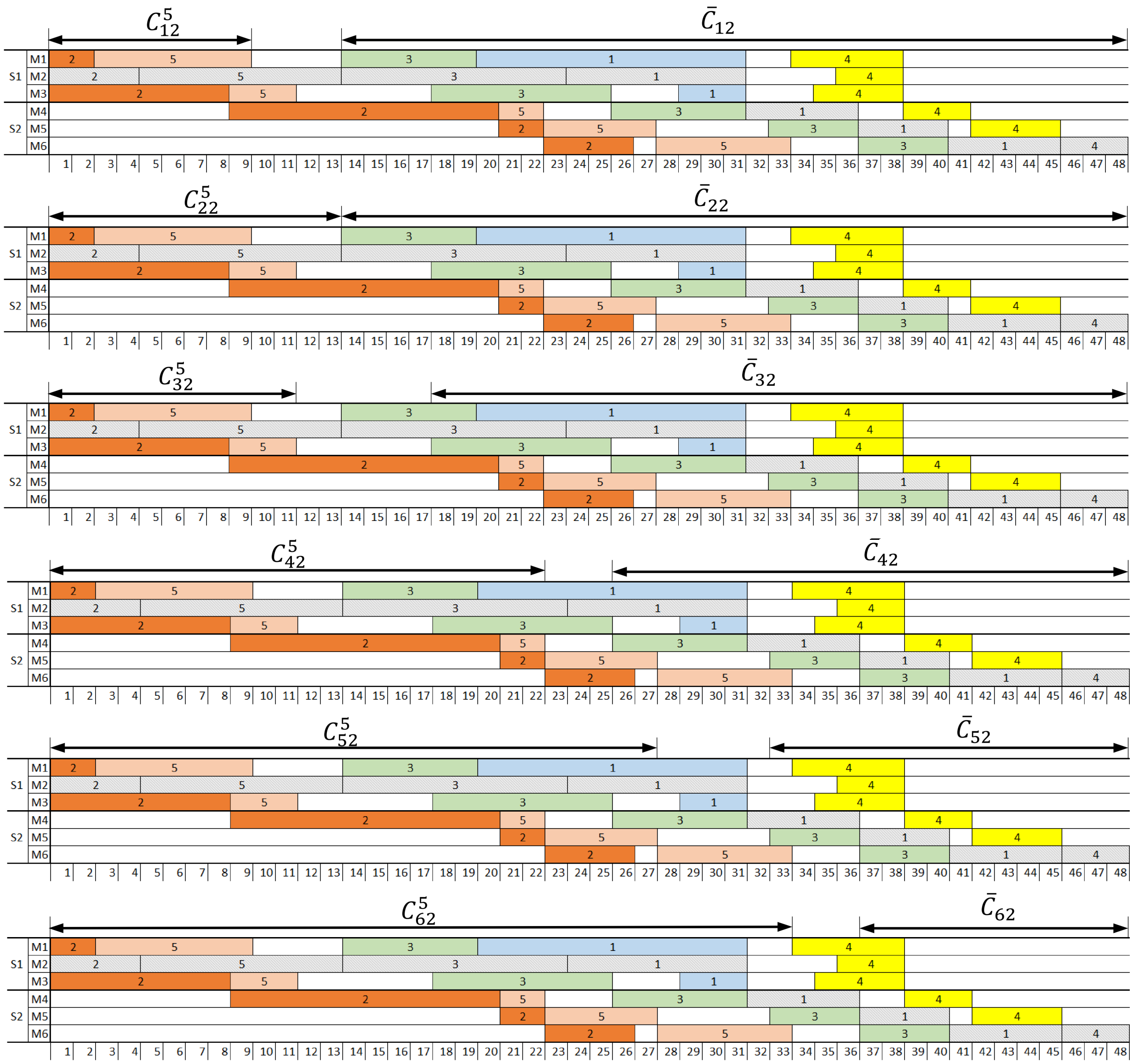

Figure 5: Example for Theorem 2