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Short rational generating functions for solving some families of fuzzy integer programming problems

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Abstract

In this paper we present new complexity results for solving single and multiobjective fuzzy integer programs with soft constraints and fuzzy objective function coefficients. Our method is based on the use of Barvinok's short rational generating functions of convenient transformations of the fuzzy problems to crisp ones. This analysis allows us to provide new insights on the structure of these families of fuzzy programs.
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Keywords: Fuzzy mathematical programming; Integer programming; Theoretical complexity; Generating functions

1. Introduction

Integer linear programming (IP) deals with the problem of minimizing or maximizing a linear function over the solutions of a diophantine system of inequalities. IP is a very important area of activity because it is a natural tool to model and handle many real life situations, as for instance management and efficient use of resources: distribution of goods, production scheduling, machine sequencing, capital budgeting, etc. Furthermore, from a mathematical point of view many interesting combinatorial and geometrical problems in Graph Theory and Logic can be seen as integer programs. For these reasons, integer programming has attracted many researchers from different areas and there is nowadays an extensive body of literature (textbooks and publications) exclusively devoted to the analysis and resolution of these problems, both from a theoretical and a practical viewpoint. The interested reader is referred to the classical textbooks by Schrijver [1], Nemhauser and Wolsey [2] or the more recent by Sierksma [3] among many others, for further details. Different methodologies have been designed to solve integer programming problems, among them, we mention the widely used algorithms based on branch-and-bound, cutting planes, branch-and-cut methods or

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dynamic programming. However, it is well-known that solving an IP is generally NP-hard when the dimension is part of the input, although polynomial-time solvable in fixed dimension (see [1,4]).

Decision-makers usually find vague information when trying to model a real-world situation as a mathematical programming problem. This imprecision, which causes difficulties in modeling, can be considered by assuming that some of the elements in the formulation are fuzzy entities (see [5,46]). In [6, Chapter 4] the authors present a detailed list of options to "fuzzify" a mathematical programming problem. When some of the elements in an IP are fuzzy, the problem falls into a different class that it is usually called Fuzzy Integer Programming (FIP). This latter class of problems has been widely studied in the literature (see for instance, [7–15] among many others). However, its theoretical complexity has not been yet stated. It is clear that solving a FIP is, in general, much harder than solving an IP, since a standard integer program is a very particular case of FIP. In particular, FIP is also NP-hard. Nevertheless, although it is well-known that IP can be solved in polynomial-time when one fixes the dimension of the space of variables, there are not known results concerning the complexity of FIP when the dimension is fixed.

In this paper we provide new exact algorithms for solving some important models of FIP problems (namely those described in [12,16] and [17]) and we analyze their theoretical complexity. Both the results and the tools are new in this field. Furthermore, although some of the proofs presented in this paper are based on previous results by Barvinok [18], they are not trivial extensions since Barvinok's theory states the polynomial-time complexity of (crisp) IP, but not of its fuzzy versions. For each of the models that we consider, we need to perform convenient transformations that allow us to apply short rational generating functions results (not only Barvinok's results [18] but also some others related to the complexity of multiobjective integer programming [19]). In particular, we deal with three different fuzzifications of integer programming problems. One is based on using fuzzy sets to represent the vagueness of the inequality relations in the constraints of the integer program, usually called *flexible programming* (see [20,21] and the references therein); the second one, on the consideration of fuzzy coefficients in the linear objective function (see Section 4.4 in [22] or [23,24]), i.e., when the imprecision concerns the costs associated to the variables in the model; and finally, single and multiobjective IP problems where both the constraints and the coefficients of the objective function are fuzzy elements. We give polynomiality results for FIP similar to those proved by Lenstra [25] about the polynomiality of (crisp) IP in fixed dimension. In our approach, we perform some transformations of the considered fuzzy problems to multiobjective integer programs. These transformations allow us to apply new tools borrowed from the theory of short rational generating functions to prove the polynomial complexity of several models of fuzzy integer programming. The use of short rational generating functions for solving fuzzy optimization problems is instrumental and new; and it is the milestone that allows us to prove the new complexity results.

The paper is organized as follows. Section 2 is devoted to recall some previous notions and results about short rational generating functions (SRGF) of rational polytopes, in particular, the main results concerning the complexity of computing them and its application for solving multiobjective integer programming problems. We present, in Section 3, complexity results for fuzzy integer programs where the fuzziness is induced by considering soft constraints. In Section 4 we analyze integer programs with fuzzy coefficients in the objective functions. At the end of that section we state similar complexity results for single and multiobjective fuzzy integer programs where both the coefficients of the objective functions and the constraints are fuzzy. Finally, in Section 5 we draw some conclusions about the results presented in this paper.

2. Preliminaries

2.1. Short rational generating functions

Short rational generating functions (SRGF) were first used by Barvinok [18] for counting integer points inside rational bounded polyhedra (polytopes), based on the previous geometrical paper by Brion [26]. The main idea is encoding those integral points in a rational function with as many variables as the dimension of the space where the body lives. Let $P \subset \mathbb{R}^d_+$ be a given convex polyhedron, the integral points may be expressed in a formal sum $f(P,z) = \sum_\alpha z^\alpha$ with $\alpha = (\alpha_1, \dots, \alpha_d) \in P \cap \mathbb{Z}^d$, where $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$. Barvinok's goal was representing that formal sum of monomials in the multivariate polynomial ring $\mathbb{R}[z_1, \dots, z_d]$, as a "short" sum of rational functions in the same variables. Actually, Barvinok presented a polynomial-time algorithm when the dimension, n, is fixed, to compute those functions. A clear example to illustrate that approach is the polytope $P = [0, N] \subset \mathbb{R}$: the long expression of the generating function is $f(P,z) = \sum_{i=0}^{N} z^i$, and it is easy to see that its representation as sum of rational functions is the well known formula $\frac{1-z^{N+1}}{1-z}$. For further details the interested reader is referred to [18,27].

We recall the main results on short rational generating functions of rational polytopes that will be useful through this paper.

Let $P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ be a rational polytope in $\mathbb{R}^n_+$. The integer points inside P can be encoded in the following "long" sum of monomials:

$$f(P;z) = \sum_{\alpha \in P \cap \mathbb{Z}^n} z^\alpha$$

where $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. However, this long sum can be re-encoded in polynomial-time for fixed dimension in a "short" sum of rational functions in the form

$$f(P;z) = \sum_{i \in I} \varepsilon_i \frac{z^{u_i}}{\prod_{j=1}^{n}(1 - z^{v_{ij}})}$$

where I is a polynomial-size indexing set, and where $\varepsilon_i \in \{1, -1\}$ and $u_i, v_{ij} \in \mathbb{Z}^n$ for all i and j (Theorem 5.4 in [18]). This short expression is usually called the *short rational generating function* of the polytope P.

2.2. Multiobjective integer linear programming

The results on the complexity to compute short rational functions of rational polyhedra and those to perform basic operations with them have allowed to develop polynomial-time algorithms both for counting integer points inside polytopes (see [18]) and for solving single (see [28]) or multiobjective linear integer programming problems (see [19,29]). In this paper we use some of these polynomial-time procedures to solve some important families of fuzzy integer programming problems. The complexity results presented in this paper are based on performing convenient transformations of the original fuzzy problems to multiobjective ones.

A multiobjective integer linear programming problem can be formulated as:

$$\begin{aligned} &\max\ (c_1 x, \ldots, c_k x) =: Cx \\ &\text{s.t.}\quad Ax \leq b \\ &\qquad\ \ x \in \mathbb{Z}^n_+ \end{aligned} \tag{MOILP$_{A,C}(b)$}$$

where $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$ and $C = (c_1, \ldots, c_k) \in \mathbb{Z}^{k\times n}$, and such that $P = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$ is a convex polytope in $\mathbb{R}^n$. The solutions of (MOILP$_{A,C}(b)$) are called Pareto-optimal solutions. A vector $x^\star \in \mathbb{R}^n_+$ with $Ax^\star \leq b$ is a Pareto-optimal solution if there is no other feasible solution $y \in \mathbb{Z}^n$ such that $c_i x^\star \leq c_i y$ for all $i = 1, \ldots, k$ with at least one strict inequality.

The following result states that encoding the set of Pareto-optimal solutions of a multiobjective integer linear problem can be done in polynomial-time when the dimension is fixed. (It will be useful for our development.)

Theorem 2.1. *(See [19].) Let $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, $C = (c_1, \ldots, c_k) \in \mathbb{Z}^{k\times n}$, and assume that the number of variables n is fixed. Then, we can encode, in polynomial-time, the entire set of Pareto-optimal solutions for (MOILP$_{A,C}(b)$) in a short sum of rational functions.*

The above theorem does not assure to list all the Pareto-optimal solutions of (MOILP$_{A,C}(b)$) in polynomial-time, since it only provides a compact representation, as a short rational generating function, of the set of Pareto-optimal solutions. Addressing the enumeration, one has to consider a different notion of complexity that has been already used in the literature for slightly different problems. Computing maximal independent sets on graphs is known to be #P-hard [30,31], nevertheless there exist algorithms for obtaining these sets which ensure that the number of operations necessary to obtain two consecutive solutions of the problem is bounded by a polynomial in the problem input size. These procedures are called polynomial-delay algorithms. Formally, an algorithm is said *polynomial-delay* if the delay, which is the maximum computation time between two consecutive outputs, is bounded by a polynomial in the input size [32]. In our case, a polynomial-delay algorithm, in fixed dimension, for solving (MOILP$_{A,C}(b)$) means that once a Pareto-optimal solution is computed, either in polynomial-time, another Pareto-optimal solution is found or the termination of the algorithm is given as an output.

The following result ensures that there is a polynomial-delay algorithm for solving (MOILP$_{A,C}(b)$).

Theorem 2.2. *(See [19].) Assume n is a constant. There exists a polynomial-delay procedure to enumerate the entire set of Pareto-optimal solutions of* (MOILP$_{A,C}(b)$).

3. Integer programs with soft constraints

In this section we study integer programming problems where some imprecision is considered in the formulation of the constraints. This type of problems has been widely studied in the literature (see for example, [11,16,22,33–40]) and many different methods have been proposed for solving them. We provide here a new polynomial-time approach for solving this type of problems, by transforming first the single-objective fuzzy problem to a multiobjective crisp one; and then applying short rational generating functions theory to the transformed problems. The interaction between multiobjective and fuzzy optimization is not new, and it is usual to apply multiobjective optimization tools to solve fuzzy mathematical programs (see [6,41–44] and the references therein). However, these transformations by themselves do not ensure polynomiality in the solution approach. Our contribution is to have found the link between this theory and the new tools available to handle short rational generating functions. These tools allow to design a polynomial-time algorithm for the considered fuzzy integer programming problems.

Let $P = \{x \in \mathbb{R}^n : \sum_{i=1}^n d_{ij} x_i \leq e_j,\ \ j = 1, \ldots, \ell\}$ be a rational polytope with $d_{ij}, e_j \in \mathbb{Z}$, $i = 1, \ldots, n$, $j = 1, \ldots, \ell$, $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$. We also assume, w.l.o.g., that the number of fuzzy constraints, m, is upper bounded by a polynomial in n, $p(n)$. Consider the following integer program with soft constraints:

$$
\begin{aligned}
&\max\, cx \\
&\text{s.t.}\quad Ax \lesssim b \\
&\qquad\ \ x \in P \cap \mathbb{Z}^n_+
\end{aligned}
\qquad\qquad (\mathrm{FIP}^{\lesssim}_{A,c,P}(b))
$$

where $\lesssim$ means that the decision-maker is willing to permit some violations in the accomplishment of the constraints, that is, s/he assumes fuzzy constraints characterized by membership functions $\mu_i : \mathbb{R}^n \to [0, 1]$ for $i = 1, \ldots, m$, for each of the constraints in $Ax \leq b$. For each i, μ_i is the degree to which each $x \in \mathbb{R}^n$ accomplishes the i-th constraint (see [45] for further details about fuzzy relations). According to a general classification of fuzzy mathematical programming this problem would be into the family of flexible programming problems (see [20,21]).

Next, we consider an overall (aggregated) membership function for the system of linear constraints $Ax \leq b$. Although, our approach is independent of that choice, we have chosen to continue our analysis, one of the most standard membership functions already introduced by Bellman and Zadeh in the seventies [45], namely $\mu(x) = \min_i \mu_i(x)$.

Then, Problem (FIP$^{\lesssim}_{A,c,P}(b)$) consists in finding the pairs $(x, \mu(x))$ with $x \in P \cap \mathbb{Z}^+_n$ such that it does not exist $y \in P \cap \mathbb{Z}^+_n$ satisfying $(cx, \mu(x)) \leq (cy, \mu(y))$ and $(cx, \mu(x)) \neq (cy, \mu(y))$.

For the ease of readability and in order to simplifying the presentation we shall consider that for each inequality its membership function μ_i, $i = 1, \ldots, m$ is, as in [12], of the form:

$$
\mu_i(x) = \begin{cases} 0 & \text{if } a_i x > b_i + \delta_i, \\ \frac{(b_i+\delta_i)-a_i x}{\delta_i} & \text{if } b_i < a_i x \leq b_i + \delta_i, \\ 1 & \text{if } a_i x \leq b_i, \end{cases} \tag{1}
$$

where δ_i is a nonnegative real number that fixes the threshold to consider that a solution violates the i-th inequality.

Therefore, feasible solutions of (FIP$^{\lesssim}_{A,c,P}(b)$) are pairs $(x, \mu(x))$ where $x \in P \cap \mathbb{Z}^n_+$ and $\mu(x)$ is the overall degree of satisfaction of x to the system of inequalities $Ax \leq b$. In this context (see e.g. [22]), the greater the objective function the better but at the same time, the higher the membership function also the better. The reader may observe that any other choice for the aggregated membership function would not change our analysis. (See e.g. [47] for other admissible possibilities.)

Note that the feasible region of (FIP$^{\lesssim}_{A,c,P}(b)$) is given in a rather general form since it considers a set of fuzzy constraints ($Ax \lesssim b$), and also a set of crisp constraints ($x \in P \cap \mathbb{Z}^n_+$). Note that the crisp constraints may also be considered as fuzzy constraints with 0–1 membership functions. However, independently of the membership functions associated to the fuzzy constraints, assuming that the values are integer points inside a rational polytope assures that the number of feasible solution is finite, which will be needed for the development through the paper.

Fuzziness of constraints (also called *flexible constraints*) as in the model above, have been already considered in [10], and since then, they have been widely used in fuzzy mathematical programming (see [12,17,22,34,48–50]) and also applied for practical problems (see [51]).

According to the definition of feasible and optimal solutions of $(\mathrm{FIP}^{\lesssim}_{A,c,P}(b))$, it is clear that if $(x^*, \mu(x^*))$ is an optimal solution of $(\mathrm{FIP}^{\lesssim}_{A,c,P}(b))$, then it is a solution of the following biobjective mixed integer linear programming problem in $\mathbb{R}^{n+2m+1}$:

$$
\begin{array}{llll}
\max(cx, Z) & & & (\mathrm{BIP}^0_{P,\mu}) \\
\text{s.t.} & a_i x - b_i \le \mathrm{UB}_i(1-w_i), & i=1,\dots,m & (2) \\
& a_i x - b_i \ge \mathrm{LB}_i w_i, & i=1,\dots,m & (3) \\
& a_i x - b_i - \delta_i \le (\mathrm{UB}_i - \delta_i)(1-t_i), & i=1,\dots,m & (4) \\
& a_i x - b_i - \delta_i \ge (\mathrm{LB}_i - \delta_i)t_i, & i=1,\dots,m & (5) \\
& z_i \le t_i, & i=1,\dots,m & (6) \\
& z_i \ge w_i, & i=1,\dots,m & (7) \\
& z_i \le \dfrac{(b_i+\delta_i) - a_i x}{\delta_i} + (1 - t_i + w_i), & i=1,\dots,m & (8) \\
& z_i \le 1 & i=1,\dots,m & (9) \\
& Z \le z_i, & i=1,\dots,m & (10) \\
& Z, z_i \ge 0 & i=1,\dots,m & \\
& w_i, t_i \in \{0,1\} & i=1,\dots,m & \\
& x \in P \cap \mathbb{Z}^n_+ & &
\end{array}
$$

where $\mathrm{UB}_i > \max\{a_i x - b_i : x \in P \cap \mathbb{Z}^n_+\}$ and $\mathrm{LB}_i < \min\{a_i x - b_i : x \in P \cap \mathbb{Z}^n_+\}$.

The auxiliary binary variables t_i and w_i, are defined such that

$$
w_i = \begin{cases} 1 & \text{if } a_i x \le b_i, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad t_i = \begin{cases} 1 & \text{if } a_i x \le b_i + \delta_i, \\ 0 & \text{otherwise.} \end{cases}
$$

The correct definition of these variables is assured by constraints (2)–(5). Hence, the membership function for the ith constraint, μ_i, which is encoded in the above formulation with the variable $z_i \in [0,1]$, is defined based on the different combinations of the variables w_i and t_i. If $t_i = 0$ (being $ax_i \ge b_i + \delta_i$), by constraint (6), z_i takes value 0. In case $w_i = 1$ (being $a_i x \le b_i$), z_i takes value 1 (constraint (7)); and finally (note that no more cases are allowed), if $t_i = 1$ and $w_i = 0$, by constraint (8) and the maximization criterion, z_i takes value $\frac{(b_i+\delta_i)-a_i x}{\delta_i}$. The variable Z representing the overall membership function μ, is defined by constraint (10) as the minimum of all the z_i values.

We remark that this formulation allows us to represent the membership function as the minimum value of the μ_i functions, provided that they are smaller than the threshold value 1 assigned to the crisp polytope.

Furthermore, if $(\hat{x}, \hat{Z}, \hat{z}, \hat{w}, \hat{t}) \in \mathbb{R}^{n+3m+1}$ is a Pareto-optimal solution for $(\mathrm{BIP}^0_{P,\mu})$, then $(\hat{x}, \hat{Z} = \mu(\hat{x}))$ is an optimal solution for $(\mathrm{FIP}^{\lesssim}_{A,c,P}(b))$.

Indeed, since x^* is an optimal solution for $(\mathrm{FIP}^{\lesssim}_{A,c,P}(b))$, then, for any $y \in P \cap \mathbb{Z}^n_+$, either $cx^* \ge cy$ or $\mu(x^*) \ge \mu(y)$. This is equivalent to say that $(x^*, \mu(x^*))$ is a Pareto-optimal solution for $(\mathrm{BIP}^0_{P,\mu})$. On the other hand, if $(\hat{x}, \hat{Z}, \hat{\mu}, \hat{w}, \hat{t})$ is a Pareto-optimal solution for $(\mathrm{BIP}^0_{P,\mu})$, then, one has that $\hat{Z} = \min_i z_i = \mu(\hat{x})$; because on the contrary $\hat{Z} \le \mu_i(\hat{x})$ would imply $\hat{Z} < \mu(\hat{x})$, so $(\hat{x}, \mu(\hat{x}), \hat{z}, \hat{w}, \hat{t})$ would dominate $(\hat{x}, \hat{Z}, \hat{z}, \hat{w}, \hat{t})$. This fact contradicts the Pareto-optimality of $(\hat{x}, \hat{Z}, \hat{z}, \hat{w}, \hat{t})$. Thus, by definition of Pareto-optimality, there does not exist a feasible solution $(\tilde{x}, \tilde{Z}, \tilde{z}, \tilde{w}, \tilde{t})$ for $(\mathrm{BIP}^0_{P,\mu})$ such that $c\hat{x} \le c\hat{\tilde{x}}$ and $\hat{Z} \le \tilde{Z}$, and at least one strict inequality.

Hence, solving $(\mathrm{FIP}^{\lesssim}_{A,c,P}(b))$ is equivalent to solve the biobjective integer programming problem $(\mathrm{BIP}^0_{P,\mu})$.

It is not difficult to see that the number of Pareto-optimal solutions of $(\mathrm{BIP}^0_{P,\mu})$ is finite since $P \cap \mathbb{Z}^n$ is the set of integer points inside the polytope P. Moreover, if $\hat{x} \in P \cap \mathbb{Z}^n$ is the x-component of such a solution of $(\mathrm{BIP}^0_{P,\mu})$, then the z_i-components, $\hat{z}_i$, for $i = 1, \dots, m$, and the Z-component, $\hat{Z}$, of that solution give, respectively, the value $\mu_i(\hat{x})$, for $i = 1, \dots, m$, of the membership function of constraint i, and $\mu(\hat{x})$, namely the overall membership function of the system $Ax \lesssim b$. Therefore the number of solutions $(\hat{x}, \mu(\hat{x}))$ is finite. Furthermore, $\hat{z}_i$, for $i = 1, \dots, m$, and $\hat{Z}$ are rational because $x \in \mathbb{Z}^n$ and $\mu_i(x)$ are linear functions with rational coefficients. Thus, if we consider $y_i = M\hat{z}_i$, for $i = 1, \dots, m$, and $\bar{y} = M\hat{Z}$, with M the least common multiple of all the determinants of full rank submatrices of the system of inequalities defining the feasible region of Problem $(\mathrm{BIP}^0_{P,\mu})$, we have the equivalent biobjective integer programming problem:

$$\max(cx, \bar{y}) \qquad (\mathrm{BIP}_{P,\mu})$$

$$\begin{aligned}
\text{s.t.}\quad & a_i x - b_i \le \mathrm{UB}_i(1 - w_i), && i = 1, \dots, m,\\
& a_i x - b_i \ge -\mathrm{LB}_i w_i, && i = 1, \dots, m,\\
& a_i x - b_i - \delta_i \le (\mathrm{UB}_i - \delta_i)(1 - t_i), && i = 1, \dots, m,\\
& a_i x - b_i - \delta_i \ge (\mathrm{LB}_i - \delta_i)t_i, && i = 1, \dots, m,\\
& y_i \le M t_i, && i = 1, \dots, m,\\
& y_i \ge M w_i, && i = 1, \dots, m,\\
& y_i \le M\left(\frac{(b_i + \delta_i) - a_i x}{\delta_i} + 1 - t_i + w_i\right), && i = 1, \dots, m,\\
& \bar{y} \le y_i, && i = 1, \dots, m,\\
& \bar{y}, y_i \in [0, M] \cap \mathbb{Z} && i = 1, \dots, m,\\
& w_i, t_i \in \{0, 1\} && i = 1, \dots, m\\
& x \in P \cap \mathbb{Z}^n_+
\end{aligned}$$

Let $\overline{P}$ denote the feasible set of Problem $(\mathrm{BIP}_{P,\mu})$.

In this case, if $(\hat{x}, \hat{\bar{y}}, \hat{y}, \hat{w}, \hat{t})$ is a Pareto-optimal solution for $(\mathrm{BIP}_{P,\mu})$, then $\frac{\hat{y}_i}{M} = \mu_i(\hat{x})$ and $\hat{\bar{y}} = \min_i \hat{y}_i$. Moreover, if c is generic for the problem $\max\{cx : x \in P \cap \mathbb{Z}^n\}$, that is, if the crisp problem has a unique optimal solution, $(\mathrm{BIP}_{P,\mu})$ has at most $M + 1$ Pareto-optimal solutions since for each value of $\bar{y}$ there is exactly one solution in x. Then, this problem may be solved solving $M + 1$ single objective integer problems, one for each of the possible values of y.

The following example from [12] illustrates the proposed transformation between $(\mathrm{FIP}^{\lesssim}_{A,c,P}(b))$, $(\mathrm{BIP}^0_{P,\mu})$ and $(\mathrm{BIP}_{P,\mu})$.

Example 3.1. Consider the following problem:

$$\begin{aligned}
& \max\ 2x_1 + 5x_2\\
& \text{s.t.}\quad 2x_1 - x_2 \lesssim 9\\
& \qquad 2x_1 + 8x_2 \lesssim 31\\
& \qquad (x_1, x_2) \in [0, 8] \times [0, 5] \cap \mathbb{Z}^2_+ \qquad (11)
\end{aligned}$$

We use the membership functions proposed in (1) (already used in [12]) with $p_1 = p_2 = 1$ and $q_1 = 3$, $q_2 = 4$ being then $\delta_1 = 3$ and $\delta_2 = 4$ and we have taken $\mathrm{UB}_i = -\mathrm{LB}_i = 35$. Fig. 1 shows the crisp polytope induced by the fuzzy constraints (Q) and the area where the membership function is not zero ($\widetilde{Q} - Q$).

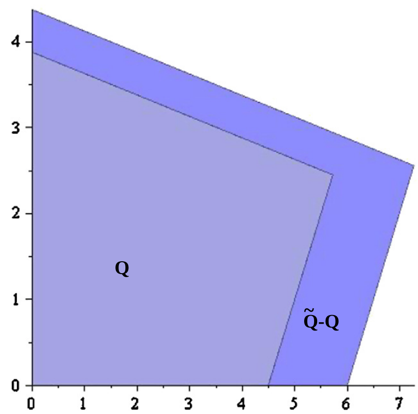

Fig. 1. The crisp polytope (Q) and its maximum deformation by the membership function ($\widetilde{Q}$) of Example 3.1.

Transforming the problem as in (BIP$^0_{P,\mu}$) we obtain the following biobjective mixed-integer problem:

$$
\begin{aligned}
\max\ & (2x_1 + 5x_2, Z)\\
\text{s.t.}\ \ & 2x_1 - x_2 - 9 \le 35(1 - w_1)\\
& 2x_1 - x_2 - 9 \ge -35w_1\\
& 2x_1 - x_2 - 12 \le 32(1 - t_1)\\
& 2x_1 - x_2 - 12 \ge -38t_1\\
& 2x_1 + 8x_2 - 31 \le 35(1 - w_2)\\
& 2x_1 + 8x_2 - 31 \ge -35Mw_2\\
& 2x_1 + 8x_2 - 35 \le 31M(1 - t_2)\\
& 2x_1 + 8x_2 - 35 \ge -39Mt_2\\
& z_1 \le t_1,\\
& z_2 \le t_2,\\
& z_1 \ge w_1,\\
& z_2 \ge w_2,\\
& z_1 \le \frac{12 - 2x_1 + x_2}{3} + (1 - t_1 + w_1)\\
& z_1 \le \frac{35 - 2x_1 - 8x_2}{4} + (1 - t_2 + w_2)\\
& Z \le z_1 \le 1\\
& Z \le z_2 \le 1\\
& (x_1, x_2) \in [0, 8] \times [0, 5] \cap \mathbb{Z}^2\\
& w_1, w_1, t_1, t_2 \in \{0, 1\}, \quad Z, z_1, z_2 \ge 0.
\end{aligned}
\tag{12}
$$

Fig. 2 shows the feasible region of the above biobjective problem. The bottom of that polytope ($z = 0$) coincides with the embedding of the crisp original polytope and on the top ($z = 1$) it appears the maximum deformation of the crisp polytope by the membership functions.

Now, the least common multiple of all the determinants of the full rank submatrices defined by the membership functions is $M = 12$, then, the problem above is equivalent to the biobjective integer problem:

$$
\begin{aligned}
\max\ & (2x_1 + 5x_2, \bar{y})\\
\text{s.t.}\ \ & 2x_1 - x_2 - 9 \le 35(1 - w_1),
\end{aligned}
$$

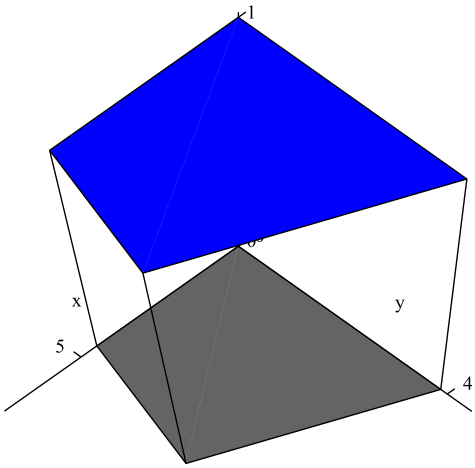

Fig. 2. Feasible polytope of (12).

$$
\begin{aligned}
&2x_1 - x_2 - 9 \geq -35w_1,\\
&2x_1 - x_2 - 12 \leq 32(1 - t_1),\\
&2x_1 - x_2 - 12 \geq -38t_1,\\
&2x_1 + 8x_2 - 31 \leq 35(1 - w_2),\\
&2x_1 + 8x_2 - 31 \geq -35Mw_2,\\
&2x_1 + 8x_2 - 35 \leq 31M(1 - t_2),\\
&2x_1 + 8x_2 - 35 \geq -39Mt_2,\\
&y_1 \leq 12t_1,\\
&y_2 \leq 12t_2,\\
&y_1 \geq 12w_1,\\
&y_2 \geq 12w_2,\\
&y_1 \leq 12\left(\frac{12 - 2x_1 + x_2}{3} + 1 - t_1 + w_1\right),\\
&y_1 \leq 12\left(\frac{35 - 2x_1 - 8x_2}{4} + 1 - t_2 + w_2\right),\\
&\bar{y} \leq y_1\\
&\bar{y} \leq y_2\\
&(x_1, x_2) \in [0, 8] \times [0, 5] \cap \mathbb{Z}^2\\
&w_1, w_1, t_1, t_2 \in \{0, 1\},\\
&\bar{y}, y_1, y_2 \in [0, M] \cap \mathbb{Z}.
\end{aligned} \tag{13}
$$

The Pareto-optimal solutions of Problem (13) are $x_1^* = 5$, $x_2^* = 3$, $y_1^* = 12$, $y_2^* = 3$, $\bar{y}^* = 3$, $w_1^* = 1$, $w_2^* = 0$, $t_1^* = 1$, $t_2^* = 1$ ($f^* = (25, 3)$), $x_1^* = 4$, $x_2^* = 3$, $y_1^* = 12$, $y_2^* = 9$, $\bar{y}^* = 9$, $w_1^* = 1$, $w_2^* = 0$, $t_1^* = 1$, $t_2^* = 1$ ($f^* = (23, 9)$), $x_1^* = 3$, $x_2^* = 3$, $y_1^* = 12$, $y_2^* = 12$, $\bar{y}^* = 12$, $w_1^* = 1$, $w_2^* = 1$, $t_1^* = 1$, $t_2^* = 1$ ($f^* = (21, 12)$) and $x_1^* = 8$, $x_2^* = 5$, $y_1^* = 0$, $y_2^* = 0$, $\bar{y}^* = 0$, $w_1^* = 0$, $w_2^* = 0$, $t_1^* = 1$, $t_2^* = 0$ ($f^* = (41, 0)$). Then, the set of solutions of the fuzzy problem, with their respective membership values (obtained dividing $\bar{y}$ by M) are:

$$\bigl(x^*, \mu\bigl(x^*\bigr)\bigr) \in \bigl\{\bigl((8, 5), 0\bigr), \bigl((5, 3), 0.25\bigr), \bigl((4, 3), 0.75\bigr), \bigl((3, 3), 1\bigr)\bigr\}.$$

Note that we get the additional solution (8, 5) that is not reported in [12] since the authors did not consider as feasible solutions those with zero membership function.

It is worth noting that, in general, fuzzy integer programming is NP-hard. Indeed, reduction comes from crisp integer programming. It is well-known that finding an optimal solution of a general integer program, when the dimension is part of the input, is NP-hard (see [1]). Thus, since we can state that $\hat{x}$ is an optimal solution to (BIP$_{P,\mu}$) if and only if there exists $\hat{y} \in [0, M] \cap \mathbb{Z}$ such that $(\hat{x}, \frac{\hat{y}}{M})$ is an optimal solution of (FIP$^{\lesssim}_{A,c,P}(b)$), the conclusion follows.

This shows that fuzzy integer programming is as hard as crisp integer programming. Nevertheless, the situation is even harder because crisp integer programming in fixed dimension is polynomial (see [25]) but fuzzy integer programming is equivalent to bicriteria integer linear programming (see [29]) which is also NP-hard.

In spite of that, there is a natural (not easy) way to find all the solutions to (FIP$^{\lesssim}_{A,c,P}(b)$) which is based on solving $M + 1$ crisp integer problems of the form (BIP$_{P,\mu}$), fixing $y = 0, 1, \dots, M$. However this approach does not ensure polynomiality even in fixed dimension. Here the problem comes from $M + 1$, the number of problems to be solved. This figure might be exponential in the input size and therefore even solving each subproblem in polynomial-time, the overall complexity will be only pseudopolynomial.

The best complexity result that we can state is given by the next theorem.

Theorem 3.1. *If the dimension n is fixed, the entire set of solutions for (FIP$^{\lesssim}_{A,c,P}(b)$) can be encoded in a short sum of rational functions in polynomial-time.*

Proof. Using Barvinok's algorithm (Theorems 1.7 and 5.4 in [18]), compute the following generating function in $2n + 2$ variables:

$$f(x_1, x_2) := \sum_{((u,y_u),(v,y_v)) \in Q \cap ((\mathbb{Z}^n_+ \times \mathbb{Z}_+) \times (\mathbb{Z}^n_+ \times \mathbb{Z}_+))} x_1^{(u,y_u)} x_2^{(v,y_v)} \tag{14}$$

where $Q = \{((u, y_u), (v, y_v)) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : (u, \bar{y}_u, y_u, w_u), (v, \bar{y}_v, y_v, w_v) \in \overline{P}, cu - cv \geq 0, y_u \leq y_v \text{ and } cu + y_u - cv - y_v \geq 1\}$, being $\overline{P}$ the feasible region of (BIP$_{P,\mu}$). Q is clearly a rational polytope (is a projection of the rational polyhedron $\overline{P}$). For fixed $u \in \mathbb{Z}^n$, the second components, (v, y_v), in the monomial $x_1^{(u,y_u)} x_2^{(v,y_v)}$ of $f(x_1, x_2)$ represent the solutions dominated by (u, y_u). f can be computed in polynomial-time by Theorems 1.7 and 5.4 in [18].

Now, for any function φ, let $\pi_{1,\varphi}, \pi_{2,\varphi}$ be the projections of $\varphi(x_1, x_2)$ onto the x_1- and x_2-variables, respectively. Thus $\pi_{2,f}(x_2)$ encodes all dominated feasible integral vectors (because the degree vectors of the x_1-variables dominate them, by construction), and it can be computed from $f(x_1, x_2)$ in polynomial-time by Theorem 1.7 in [18].

Furthermore, let $\mathrm{F}(x)$ be the short form of the generating function encoding the integer points in Q. Both, $\pi_{2,f}(x)$ and $\mathrm{F}(x)$ are computed in polynomial-time by Theorem 1.7 and Theorem 5.4 in [27] respectively. Compute the following difference:

$$\mathrm{h}(x) := \mathrm{F}(x) - \pi_{2,f}(x).$$

This is the sum over all monomials $x^{(u,y_u)}$ where $(u, y_u) \in \mathbb{Z}^n_+ \times \mathbb{Z}_+$ is a Pareto-optimal solution of (BIP$_{P,\mu}$), since we are deleting, from the total sum of feasible solutions, the set of dominated ones.

This construction gives us a short sum of rational functions associated with the sum over all monomials with degrees being the Pareto-optimal solutions of (BIP$_{P,\mu}$). The complexity of the entire construction being polynomial since we only use polynomial-time operations among two generating functions of lattice points insides rational polytopes (these operations are the computation of the short rational functions $\mathrm{F}(x)$ and $\pi_{2,f}(x)$). □

The above result states that the solutions of the fuzzy problem can be encoded in a short rational generating function in polynomial-time for fixed dimension. However, to obtain the explicit list of solutions we should expand, as a Laurent series, the rational functions that appear in that expression.

The combination of Theorem 3.1 and Theorem 2 in [19] results in the following consequence.

Theorem 3.2. *If the dimension, n, is fixed, there is a polynomial-delay method to enumerate all the solutions of (FIP$^{\lesssim}_{A,c,P}(b)$) by using short rational generating functions.*

Remark 3.1. One may think of using a different approach to enumerate the entire set of solutions of $(\mathrm{FIP}^{\lesssim}_{A,c,P}(b))$ that may lead to a simpler polynomial delay algorithm. The idea would be to use the equivalence between $(\mathrm{FIP}^{\lesssim}_{A,c,P}(b))$ and solving a series of $M+1$ crisp IP (see the equivalence above). In fixed dimension, it is known that solving each of these problems is polynomially doable [25]. Then, from one solution to the next one, generated in this way, the method would need a polynomial number of operations and therefore there would be, at most, a polynomial delay between two consecutive solutions found. However, this simple method does not guarantee the complete enumeration of the set of solution of $(\mathrm{FIP}^{\lesssim}_{A,c,P}(b))$ since each of these IP may have multiple optima. The reader may note that to ensure the entire enumeration of the set of optimal solutions the method would need to find *all* the alternative optima of each integer problem and this process is equivalent to enumeration of integer points in polyhedra which would lead us again to the starting point.

4. Fuzzy integer programs with fuzzy objective function coefficients

In this section we deal with integer linear problems where the coefficients of the linear objective functions are fuzzy numbers. In real-world problems it is usual to have only estimates of the true values of the objective function coefficients rather than the exact values of them. Moreover, those estimates can be given with some vagueness. Fuzzy Integer Programming with fuzzy objective functions deals with this lack of certainty when formulating an integer programming problem. This problem has been previously studied in [17,23,52], among others.

A general formulation for an integer programming problem where the cost vector is imprecise may be the following:

$$\begin{aligned} &\max \tilde{c}x \\ &\text{s.t.} \quad x \in P \cap \mathbb{Z}^n_+ \end{aligned} \tag{$\mathrm{FIP}_{P,\tilde{c}}$}$$

being P a rational polytope and $\tilde{c} = (\tilde{c}_1, \dots, \tilde{c}_n)$ a vector of fuzzy numbers with membership function $\mu_{\tilde{c}}$.

First of all, we describe an equivalent way to handle fuzzy numbers that will be useful to solve $(\mathrm{FIP}_{P,\tilde{c}})$. Let $\tilde{c}$ be a real fuzzy number, for each $\alpha \in [0,1]$ the α-cut of $\tilde{c}$ is the set

$$\tilde{c}_\alpha = \left\{c \in \mathbb{R} : \mu_{\tilde{c}}(c) \geq \alpha\right\}.$$

It is clear that $\tilde{c}$ is totally determined by its set of α-cuts for $\alpha \in [0,1]$ since the membership function that determines $\tilde{c}$ can be identified with this family of sets. Actually, the expression for the membership function, given the set of α-cuts is:

$$\mu_{\tilde{c}}(x) = \sup_{\alpha \in (0,1]} \min\left\{\alpha, \chi_{\tilde{c}_\alpha}(x)\right\}$$

where $\chi_{\tilde{c}_\alpha}$ is the characteristic function of the α-cut of $\tilde{c}$.

Furthermore, for each $\alpha \in (0,1]$, $\tilde{c}_\alpha$ is a closed interval in $\mathbb{R}$ (see for instance [22]):

$$\tilde{c}_\alpha = \left[c_1^\alpha, c_2^\alpha\right] \quad \alpha \in (0,1].$$

To have the exact representation of the fuzzy number, it is necessary the entire set of intervals that describes the α-cuts. However, in many well-known families of fuzzy numbers, we can describe a fuzzy number just giving a finite subset of α-cuts. For instance, interval fuzzy numbers are totally described using just one α-cut and triangular or trapezoidal ones using exactly two α-cuts.

The objective function of our problem, $(\mathrm{FIP}_{P,\tilde{c}})$, is $\tilde{c}x = \sum_{i=1}^n \tilde{c}_i x_i$ that is a fuzzy number. Let $[c_{i1}^\alpha, c_{i2}^\alpha]$ be the α-cut for $\tilde{c}_i$, $\alpha \in (0,1]$, $i = 1, \dots, n$. Then, using the addition and multiplication of a fuzzy number by an ordinary number (see [53]), the α-cut for $\tilde{c}x$ is given by:

$$(\tilde{c}x)_\alpha = \left[\sum_{i=1}^n c_{i1}^\alpha x_i, \sum_{i=1}^n c_{i2}^\alpha x_i\right].$$

Now, evaluating a feasible solution x in the objective function means to compute a fuzzy number or equivalently, its α-cuts. To compare two feasible solutions x and y, we have to compare fuzzy numbers (that are actually functions), but the orderings defined over functions are not total, but partial (there may exist functions that are not comparable). However, by means of α cuts, we can compare both fuzzy numbers (the evaluation by the objective function in x and y) using the equivalent way to treat them.

Let $\tilde{c}$ and $\tilde{d}$ be two real fuzzy numbers, we say that $\tilde{c} \leq \tilde{d}$ if $c_1^\alpha \leq d_1^\alpha$ and $c_2^\alpha \leq d_2^\alpha$ for all $\alpha \in (0, 1]$ (see [54]). Then, to compare two fuzzy numbers by this partial ordering we just need the extreme points of each of its α-cuts intervals, i.e., to compare $\tilde{c}$ and $\tilde{d}$ we only need to compare by the componentwise order in $\mathbb{R}^2$ the set of vectors $\{(c_1^\alpha, c_2^\alpha) : \alpha \in (0, 1]\}$ and $\{(d_1^\alpha, d_2^\alpha) : \alpha \in (0, 1]\}$ for each α-cut.

For our particular case, for two feasible solutions, x and y, we need to compare $\sum_{i=1}^n c_{i1}^\alpha x_i$ with $\sum_{i=1}^n c_{i1}^\alpha y_i$, and $\sum_{i=1}^n c_{i2}^\alpha x_i$ with $\sum_{i=1}^n c_{i2}^\alpha y_i$, for $\alpha \in (0, 1]$. Then, we can transform ($\mathrm{FIP}_{P,\tilde{c}}$) to a continuum family of biobjective problems:

$$\begin{aligned} &\max\ \left(c_1^\alpha x, c_2^\alpha x\right) \\ &\text{s.t.}\quad x \in P \cap \mathbb{Z}_+^n \end{aligned} \tag{$\mathrm{FIP}_{P,\tilde{c}}^\alpha$}$$

for each $\alpha \in (0, 1]$ and where $c_1^\alpha = (c_{11}^\alpha, \dots, c_{1n}^\alpha)$ and $c_2^\alpha = (c_{21}^\alpha, \dots, c_{2n}^\alpha)$ are the lower and upper end-points of the α-cuts of $\tilde{c}$, respectively.

The set of solutions of these problems are the points $x \in P \cap \mathbb{Z}_+^n$ such that there is no $y \in P \cap \mathbb{Z}_+^n$ with $c_1^\alpha x \leq c_1^\alpha y$ and $c_2^\alpha x \leq c_2^\alpha y$ for all $\alpha \in (0, 1]$. In [54,55] it is proposed a way to reduce this continuum (in α) family of problems to a discrete one, even at the price of loosing some precision. Let us consider $\alpha_1 < \cdots < \alpha_k \in (0, 1]$, that it is usually called *ranking system*. Then, one can assume that instead of considering the entire range of α-values in the interval $(0, 1]$, it is sufficient to consider a representative set of elements of this interval. In practice, many problems are completely determined by a finite subset of α-cuts, so that we are not loosing information assuming this “discretization”. In those cases, where all the α-cuts are needed, we can consider approximations to the corresponding fuzzy numbers with as many elements (finite) in the ranking system as we want.

The approximation by considering a finite number of $\alpha_1 < \cdots < \alpha_k$ in $(0, 1]$ instead of the whole interval $(0, 1]$ induces some lack of accuracy due to the approximation by linear interpolation of the membership function in the intervals $(\alpha_i, \alpha_i + 1)$, for $i = 1, \dots, k-1$. Further, this deviation can be controlled by introducing additional nodes into the representation or by using a sufficiently high number of nodes with $\max_i\{\alpha_{i+1} - \alpha_i\}$ sufficiently small. (Note that we can proceed by increasing the number $k+1$ of elements in the ranking system, decreasing in that way the approximation error. A possible strategy is to double the number of points by using $k = 2^s$ and by moving automatically to $k = 2^{s+1}$ if a better precision is necessary.)

However, in the most standard models (see [54–56]) a finite and pre-specified ranking system describes exactly the fuzzy numbers. For instance, trapezoidal fuzzy numbers (including triangular fuzzy numbers as a special case) are totally characterized by two elements in the ranking system $\{\alpha_0, 1\}$. In general, it is also true for piecewise linear fuzzy numbers where the ranking system is $\{\alpha_1 < \cdots < \alpha_k = 1\}$, being α_i each one of the vertices of the polygonal that gives the membership function. This ranking system describes completely the fuzzy number. The following example illustrates this idea.

Example 4.1. Let $\tilde{x}$ be a trapezoidal fuzzy number with membership function given by:

$$\mu_{\tilde{x}}(z) = \begin{cases} 0 & \text{if } z < a_1 \\ \frac{z-a_1}{a_2-a_1} & \text{if } a_1 \leq z \leq a_2 \\ 1 & \text{if } a_2 \leq z \leq a_3 \\ \frac{a_4-z}{a_4-a_3} & \text{if } a_3 \leq z \leq a_4 \\ 0 & \text{if } x > a_4 \end{cases}$$

Fig. 3 shows the membership function of a trapezoidal fuzzy number. Fig. 4 shows the α_0-cut and the 1-cut for some $\alpha_0 \in (0, 1)$. From the 1-cut, the elements a_2 and a_3 of the fuzzy number are determined. From the α_0-cut, $[l_1, l_2]$, the equation of the line that goes through the points (l_1, α_0) and $(a_2, 1)$ intersects with the x-axis in $(a_1, 0)$ and the line that goes through (l_2, α_0) and $(a_3, 1)$ intersects with the x-axis in $(a_4, 0)$. Then, we have completely determined the fuzzy number.

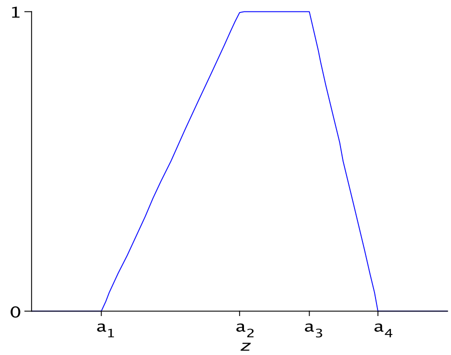

Fig. 3. Trapezoidal membership function.

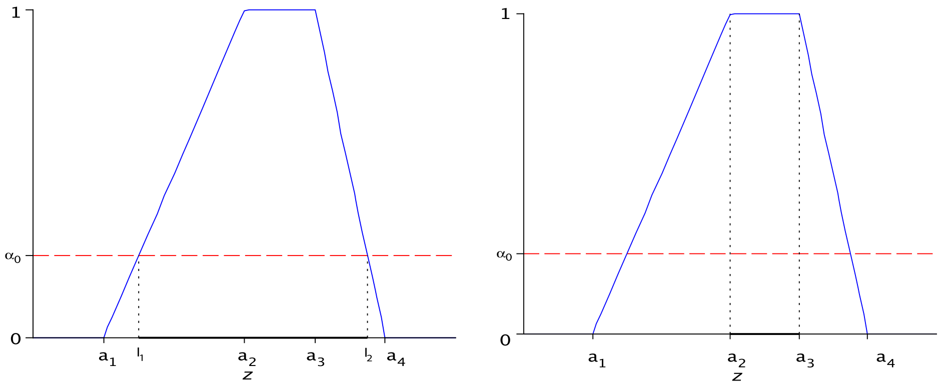

Fig. 4. α_0-cut and 1-cut for a trapezoidal fuzzy number.

The following result states the complexity of encoding the solutions of ($\mathrm{FIP}_{P,\tilde{c}}$) in a short sum of rational functions when the membership functions involved in the problem are piecewise linear (or equivalently, the problem is described by a finite ranking system).

Theorem 4.1. *Assume that the ordering between fuzzy numbers is induced by a finite ranking system and that the dimension of the decision space, n, is fixed. Then, the entire set of solutions for* ($\mathrm{FIP}_{P,\tilde{c}}$) *can be encoded in a short sum of rational functions in polynomial-time. Moreover, there exists a polynomial-delay algorithm for solving* ($\mathrm{FIP}_{P,\tilde{c}}$).

Proof. Since the partial order between fuzzy numbers is induced by a finite ranking system, $\{\alpha_1, \ldots \alpha_k\}$, ($\mathrm{FIP}_{P,\tilde{c}}$) is equivalent to the following problem:

$$\begin{aligned} &\max\left(c_1^{\alpha_1}x, c_2^{\alpha_1}x, \ldots, c_1^{\alpha_k}x, c_2^{\alpha_k}x\right) \\ &\text{s.t.} \quad x \in P \cap \mathbb{Z}_+^n \end{aligned} \tag{$\mathrm{FIP}_{P,\tilde{c}}^{\alpha_1,\ldots,\alpha_k}$}$$

Then, the result follows from Theorem 2.1.

The existence of a polynomial-delay algorithm for ($\mathrm{FIP}_{P,\tilde{c}}$) follows from the application of Theorem 2.2 to the above transformation. □

In general, fuzzy numbers are not totally described by finite ranking systems (this is the case of general LR fuzzy numbers). Nevertheless, we can approximate the solution of this problem up to any degree of accuracy. Let us consider LR-fuzzy numbers (see [47]), i.e., fuzzy numbers with membership function of the form:

$$\mu_{\tilde{x}}(z) = \begin{cases} L\left(\frac{a_1 - z}{a_1 - a_0}\right) & \text{if } z < a_1 \\ R\left(\frac{z - a_1}{a_2 - a_1}\right) & \text{if } z \geq a_1 \end{cases}$$

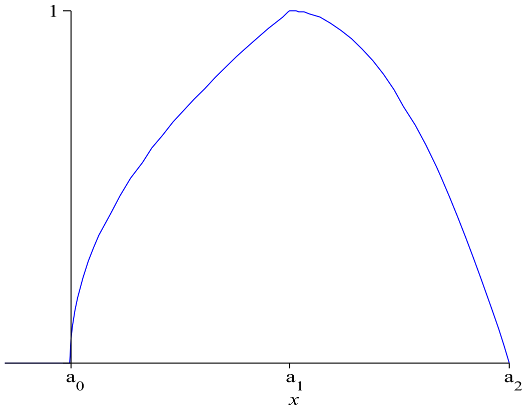

Fig. 5. LR membership function.

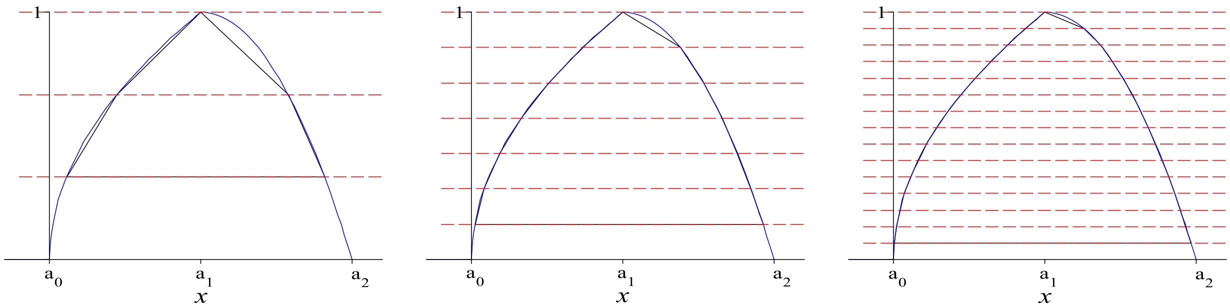

Fig. 6. Different choices for approximating a LR fuzzy number by a polygonal ($k = 3, 7, 15$).

where $a_0 \leq a_1 \leq a_2$ and $L, R : [0, 1] \to [0, 1]$ are non-increasing continuous mappings such that $L(0) = R(0) = 1$. An example of this type of fuzzy numbers is shown in Fig. 5.

Introducing as many elements as necessary in the ranking system, we can approximate any LR fuzzy number by a continuous piecewise linear fuzzy number. Let $\alpha_i = \frac{i}{k}$ for some $k \in \mathbb{N} \setminus \{0\}$. Fig. 6 shows different choices for the number of elements in the system of generators (in the form $\alpha_i = \frac{i}{k}$, with k the number of elements).

Theorem 4.2. *Assume that the dimension of the decision space, n, is fixed. Then, the entire set of solutions for the approximated problem* ($\mathrm{FIP}^{\alpha_1,\dots,\alpha_k}_{P,\tilde{c}}$) *can be encoded in a short sum of rational functions in polynomial-time.*

Moreover, there exists a polynomial-delay algorithm for solving the approximated problem ($\mathrm{FIP}^{\alpha_1,\dots,\alpha_k}_{P,\tilde{c}}$).

Note that Theorem 2.1 states that the complexity of multiobjective problems does not depends of the number of objective functions (provided finiteness of the ranking system). On the other hand, assuming finiteness of the ranking system does not suppose an actual loss of generality since we can increase the number of elements in the ranking system to obtain better approximations without increasing the theoretical complexity of the problem, i.e., maintaining the polynomiality of the algorithm.

The following example illustrates the approach described above.

Example 4.2. (See [12].) Consider the following problem:

$$
\begin{aligned}
\max\ & \widetilde{c}_1 x_1 + 5x_2 \\
\text{s.t.}\ \ & 2x_1 - x_2 \leq 12 \\
& 2x_1 + 8x_2 \leq 35 \\
& x_1, x_2 \in \mathbb{Z}_+
\end{aligned}
\tag{15}
$$

where $\widetilde{c}_1$ is the triangular fuzzy number given by the following membership function:

$$\mu_{\widetilde{c}_1}(z) = \begin{cases} \frac{z-1}{2} & \text{if } 1 \le z \le 3 \\ \frac{5-z}{2} & \text{if } 3 \le z \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Then, the α-cuts for the fuzzy number $\widetilde{c}x = \widetilde{c}_1 x_1 + 5x_2$ are:

$$(\widetilde{c}x)^\alpha = \big[(2\alpha+1)x_1, (5-2\alpha)x_1 + 5x_2\big].$$

These α-cuts define the triangular fuzzy number given by the following membership function:

$$\mu_{\widetilde{c}x}(z) = \begin{cases} \frac{z-x_1-5x_2}{2x_1} & \text{if } x_1 + 5x_2 \le z \le 3x_1 + 5x_2 \\ \frac{5x_1+5x_2-z}{2x_1} & \text{if } 3x_1 + 5x_2 \le z \le 5x_1 + 5x_2 \\ 0 & \text{otherwise} \end{cases}$$

The ranking system given by $\{\frac{1}{2}, 1\}$ is enough to solve the problem. By using the transformation ($\mathrm{FIP}^{\alpha_1,\dots,\alpha_k}_{P,\widetilde{c}}$) for a ranking system given by two α-cuts, we obtain the following 4-objective optimization problem which is equivalent to Problem (15):

$$\begin{aligned} \max\ & (3x_1 + 5x_2, 2x_1 + 5x_2, 3x_1 + 5x_2, 4x_1 + 5x_2) \\ \text{s.t.}\ \ & 2x_1 - x_2 \le 12 \\ & 2x_1 + 8x_2 \le 35 \\ & x_1, x_2 \in \mathbb{Z}_+ \end{aligned} \tag{16}$$

The entire set of Pareto-optimal solutions is $\{(4,3), (5,3), (7,2)\}$, and thus, these are also the solutions to the Fuzzy Problem (15).

In the following we present extensions of the above problems where similar complexity results can be stated.

Corollary 4.1. *Let (17) be an integer programming problem where the constraints are soft and the coefficients of the objective function are fuzzy numbers:*

$$\begin{aligned} \max\ & \widetilde{c}x \\ \textit{s.t.}\ \ & Ax \lesssim b \\ & x \in P \cap \mathbb{Z}^n_+ \end{aligned} \tag{17}$$

where the number of fuzzy constraints, m, is upper bounded by a polynomial in n, $p(n)$. Then,

1. *If the fuzzy numbers involved in (17) are totally described by a finite ranking system, then, the solutions of (17) can be encoded in a short sum of rational functions in polynomial-time for fixed dimension. Furthermore, those solutions can be enumerated using a polynomial delay algorithm.*
2. *If the fuzzy numbers involved in (17) are not totally described by a finite ranking system, then, the solutions of an approximated modification of (17) (with any fixed accuracy level) can be encoded in a short sum of rational functions in polynomial-time for fixed dimension. Those solutions can be enumerated using a polynomial delay algorithm.*

Proof. The result follows from the following equivalent reformulation of Problem (17):

$$\begin{aligned} \max\ & \big(c_1^\alpha x, c_2^\alpha x, \bar{y}\big) \\ \text{s.t.}\ \ & a_i x - b_i \le \delta_i w_i & i = 1,\dots,m \\ & y_i \le M\big(f_i(x) + (1 - w_i)\big) & i = 1,\dots,m \end{aligned}$$

$$\begin{aligned}
&\bar{y} \le y_i, && i = 1, \dots, m \\
&x \in P \cap \mathbb{Z}_+^n && \\
&y_i \in [0, M] \cap \mathbb{Z} && i = 1, \dots, m \\
&\bar{y} \in [0, M] \cap \mathbb{Z} && \\
&w_i \in \{0, 1\} && i = 1, \dots, m
\end{aligned}$$

with M as described for $(\mathrm{FIP}_{A,c,P}^{\lesssim}(b))$ in the equivalent formulation in $(\mathrm{BIP}_{P,\mu})$. □

Corollary 4.2 *(Multiobjective fuzzy integer programming). Let (18) be the following multiobjective integer programming problem with soft constraints and fuzzy coefficients in the objective functions:*

$$\begin{aligned}
&\max \widetilde{C}x \\
&\text{s.t.} \quad Ax \lesssim b \\
&\qquad x \in P \cap \mathbb{Z}_+^n
\end{aligned} \tag{18}$$

where $\widetilde{C}$ is a $k \times m$ matrix of rational fuzzy numbers and the number of fuzzy constraints, m, is upper bounded by a polynomial in n, $p(n)$.

Then,

1. *If the fuzzy numbers involved in (18) are totally described by a finite ranking system, then, the solutions of (18) can be encoded in a short sum of rational functions in polynomial-time for fixed dimension. Furthermore, those solutions can be enumerated using a polynomial delay algorithm.*
2. *If the fuzzy numbers involved in (18) are not totally described by a finite ranking system, then, the solutions of an approximated modification of (18) (with any fixed accuracy level) can be encoded in a short sum of rational functions in polynomial-time for fixed dimension. Those solutions can be enumerated using a polynomial delay algorithm.*

Proof. The result follows from the following equivalent reformulation of Problem (18):

$$\begin{aligned}
&\max \left(c_{11}^{\alpha}x, c_{12}^{\alpha}x, \dots, c_{k1}^{\alpha}x, c_{k2}^{\alpha}x, \bar{y}\right) && \\
&\text{s.t.} \quad a_i x - b_i \le \delta_i w_i && i = 1, \dots, m \\
&\qquad y_i \le M\big(f_i(x) + (1 - w_i)\big) && i = 1, \dots, m \\
&\qquad \bar{y} \le y_i, && i = 1, \dots, m \\
&\qquad x \in P \cap \mathbb{Z}_+^n && \\
&\qquad y_i \in [0, M] \cap \mathbb{Z} && i = 1, \dots, m \\
&\qquad \bar{y} \in [0, M] \cap \mathbb{Z} && \\
&\qquad w_i \in \{0, 1\} &&
\end{aligned}$$

where, c_{j1}^{α} and c_{j2}^{α} are the lower and upper extremes of the α-cut of the j-th row of $\widetilde{C}$ and M takes the value already described in Problem $(\mathrm{BIP}_{P,\mu})$. □

5. Conclusions

In this paper we present a new approach for solving different models of fuzzy integer programs analyzing their theoretical complexity. We deal with fuzzy integer programs with soft constraints and imprecise costs. The proofs of the results presented in this paper are based on the transformations of the fuzzy problems to (crisp) multiobjective integer programs and the use of generating functions of rational polytopes. We prove new complexity results about fuzzy integer programming, concluding that: (1) Encoding the entire set of optimal solutions of a broad class of fuzzy integer programs in a short sum of rational functions is doable in polynomial-time for fixed dimension; and (2) Enumerating these solutions can be done using a polynomial-delay algorithm. For problems with imprecise cost where the fuzzy numbers involved in the problem are not totally described by a finite ranking system, we present similar results for approximated problems. The advantage of the results valid for the approximated problems is that their theoretical complexity does not depend of the number of α-cuts in the approximation (provided finite and constant number), and then we can choose any accuracy level maintaining the polynomiality result. Finally, we give similar results also for fuzzy multiobjective integer programs where both the constraints and the coefficients of the objective functions are fuzzified.

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