Notas de Aeroelasticidad

tomadas del artículo y del Material Suplementario asociado a
"A note on the thrust of airfoils", Journal of Fluid Mechanics, Volume 1012 , 10 June 2025 , A6
DOI: https://doi.org/10.1017/jfm.2025.10177

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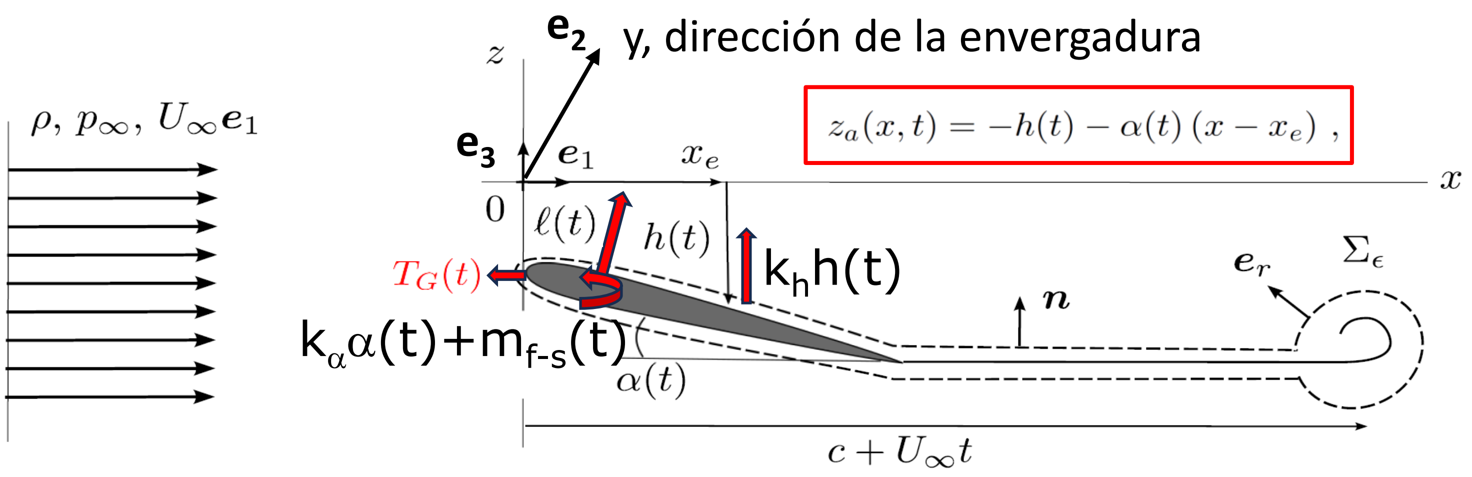
Para citar el contenido de estas notas, hágase referencia a
"A note on the thrust of airfoils", Journal of Fluid Mechanics, Volume 1012 , 10 June 2025 , A6
DOI: https://doi.org/10.1017/jfm.2025.10177

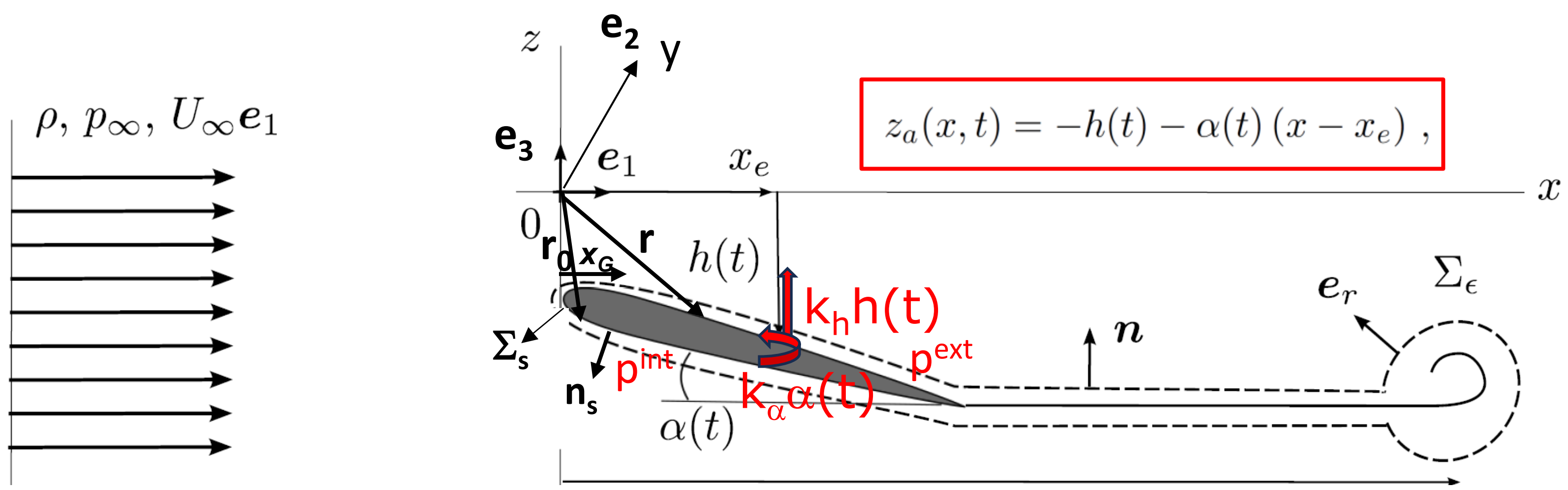
El objetivo del curso es cuantificar la dinámica acoplada entre cargas y momentos aerodinámicos y una estructura elástica cuando ésta es una superficie sustentadora. Dedicaremos una buena parte de la asignatura a cuantificar el fenómeno denominado flameo, 'flutter' en inglés.

https://youtu.be/qpJBvQXQC2M https://youtu.be/I6ZAgjm2Wuo

Cuantificaremos el fenómeno para el caso simplificado en el que existen únicamente dos grados de libertad y el ala tiene una envergadura b=2L tal que 2L/c>>1 así que el problema, en primera aproximación, es 2D:



Las constantes elásticas del muelle de torsión, k_α, y del muelle lineal, k_h, son conocidas de la parte primera de la asignatura. Aquí nos ocuparemos de calcular las fuerzas y momentos aerodinámicos, l(t) y $m_{f\text{-}s}$(t) en función de h(t) y α(t), así como de resolver el sistema de ecuaciones diferenciales ordinarias para los grados de libertad: en esta parte de la asignatura podremos cuantificar si se produce el flameo o no cuando b/c>>1 en el límite de flujo incompresible.



Flujo a altos números de Reynolds

$$Re = \frac{\rho U_\infty c}{\mu} \gg 1 \tag{1}$$

con la capa límite adherida; es decir, con gradientes de presión desfavorables pequeños, para lo cual es necesario que

$$\alpha(t) \ll 1; z_p(x) = z_c(x) \pm z_e(x); \quad z_p(x)/c \ll 1; \quad h(t)/c \ll 1; \tag{2}$$

$$\mathbf{f_{f-s}} = \int_{\Sigma_s} (p - p_\infty)\,(-\mathbf{n_s}) d\sigma; \quad \mathbf{e_3} \cdot \mathbf{f_{f-s}} = \ell(t) \simeq \int_0^c \left(p^{int} - p^{ext}\right) dx \qquad \mathbf{m_{f-s}}(\mathbf{r_0}) = \int_{\Sigma_s} (\mathbf{r} - \mathbf{r_0}) \times (-\mathbf{n_s})\,(p - p_\infty)\, d\sigma$$

$$M = \int_{-L}^{L} m\, dy = 2Lm; \quad \text{con} \quad L/c \gg 1; \quad (-\mathbf{e_3}) \cdot \left(m \frac{d\mathbf{v_g}}{dt} = k_h h(t) \mathbf{e_3} + \mathbf{f_{f-s}} \right)$$

$$\mathbf{v_G} = (-\mathbf{e_3}) \left(\dot{h}(t) + \dot{\alpha}(t)(x_G - x_e) \right)$$

$$m\left(\ddot{h}+\ddot{\alpha}\left(x_G-x_e\right)\right)+k_h h(t)=-\ell(t) \tag{7}$$

$$\omega(t)=\dot{\alpha}(t)\mathbf{e_2};\quad \mathbf{I_0}=\int_{\Omega_s}\rho_s\begin{pmatrix} y^2+z^2 & -xy & -xz\\ -xy & x^2+z^2 & -yz\\ -xz & -yz & y^2+z^2\end{pmatrix}dV\Rightarrow \mathbf{I_0}\cdot\omega(t)=2L\left(I_G+m\left(x_0-x_g\right)^2\right)\dot{\alpha}\,\mathbf{e_2} \tag{8}$$

$$I=I_G+m(x_0-x_G)^2; \tag{9}$$

Por ejemplo, si $dm/dx=2\rho_s(x)z_e(x)\neq f(x)\Rightarrow$

$$I_G=\int_{-c/2}^{c/2}x^2\,dm=c\frac{dm}{dx}\frac{c^2}{12}=m\frac{c^2}{12} \tag{10}$$

$$\mathbf{e_2}\cdot\left(\frac{d}{dt}\left[m\left(\mathbf{r_G}-\mathbf{r_0}\right)\times\mathbf{v_0}+I\dot{\alpha}\mathbf{e_2}\right]=-k_\alpha\alpha(t)\mathbf{e_2}+(x_e-x_0)\mathbf{e_1}\times\mathbf{e_3}k_h h(t)+\mathbf{m_{f-s}}\right) \tag{11}$$

Caso 1, $\mathbf{r_0}=\mathbf{r_G}$:

$$I_G\ddot{\alpha}+k_\alpha\,\alpha(t)+\left(x_e-x_G\right)k_h h(t)=\int_0^c\left(x_G-x\right)\left(p^{int}-p^{ext}\right)dx=x_G\ell(t)-m_{fs}(x=0) \tag{12}$$

siendo

$$m_{fs}(x=0)=\int_0^c x\left(p^{int}-p^{ext}\right)dx \tag{13}$$

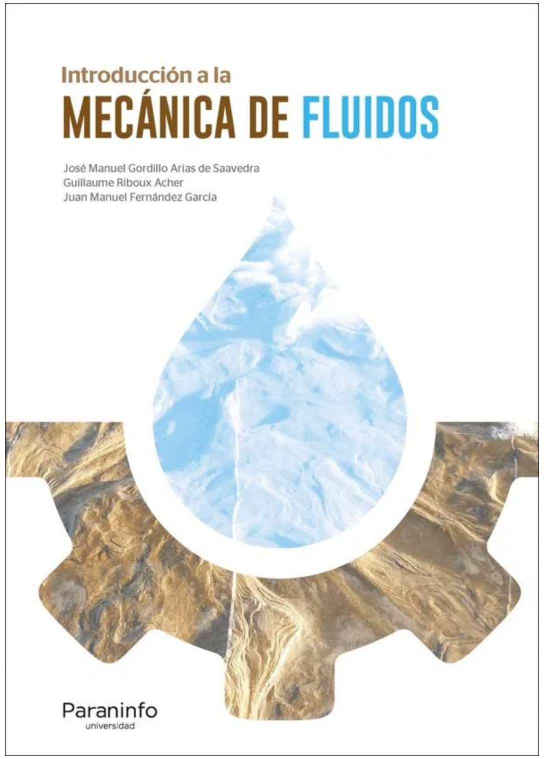
Caso 2, $\mathbf{r_0}=\mathbf{r_e}$:

$$\left(I_G+m\left(x_G-x_e\right)^2\right)\ddot{\alpha}+m(x_G-x_e)\ddot{h}(t)+k_\alpha\,\alpha(t)=\int_0^c\left(x_e-x\right)\left(p^{int}-p^{ext}\right)dx=x_e\ell(t)-m_{fs}(x=0) \tag{14}$$

Nótese que la ecuación (14) es una combinación lineal de las ecuaciones (7) y (12): en efecto, (14) resulta de sumar a (12) la ecuación (7) multiplicada por (x_G-x_e).

Los objetivos de esta parte del curso son los siguientes: calcular las fuerzas y los momentos aerodinámicos en función del tiempo y de los N grados de libertad, así como la resolución de las N EDOs para los N grados de libertad. El problema a resolver es análogo al visto en la asignatura de vibraciones, con la novedad de que en este caso las fuerzas y los momentos aerodinámicos son ejercidos por flujos a altos números de Reynolds no estacionarios y no desprendidos. Ya habéis resuelto un problema conceptualmente análogo que acopla fluido y estructura: en efecto, el amortiguamiento en la EDO de un grado de libertad que describe el movimiento de una masa acoplada a un muelle, proviene de la caída de presión asociada al flujo relativo al sólido cuando éste está dominado por fuerzas viscosas. En este curso deduciremos ecuaciones diferenciales ordinarias para los grados de libertad en las que los términos de fuerza y momento ejercido por el fluido sobre el perfil aerodinámico resultan de integrar la distribución de presiones calculadas sobre la superficie de un sólido fuselado para el caso de flujos irrotacionales a altos números de Reynolds no estacionarios y no desprendidos.

PROBLEMAS ANÁLOGOS YA VISTOS EN LAS ASIGNATURAS DE MECÁNICA DE FLUIDOS Y VIBRACIONES



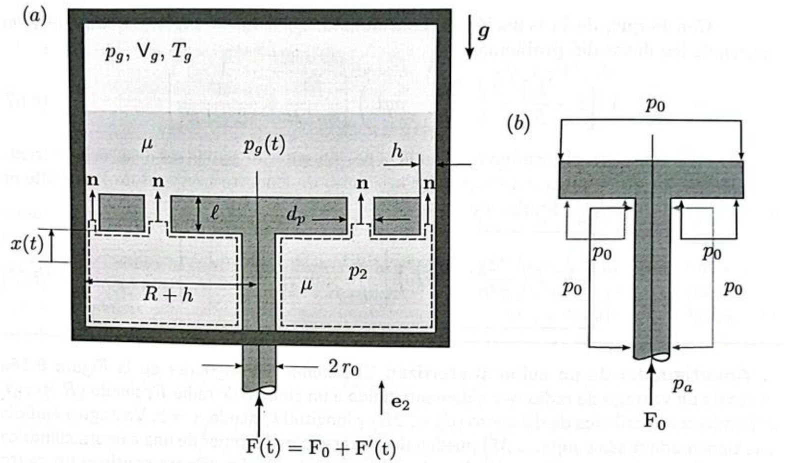


Figura 8.16. (*a*) Esquema de un amortiguador de un avión. (*b*) Diagrama de fuerzas que actúan sobre el émbolo y el vástago del amortiguador, cuando el émbolo está en la posición de equilibrio, $x = 0$.

$$M\,\ddot{x}(t) + \left[\pi(R^2 - r_0^2)\right]\left[n\left(\frac{\pi d_p^4}{128\mu\ell}\right) + 2\pi R\left(\frac{h^3}{12\mu\ell}\right)\right]^{-1}\dot{x}(t) + p_0\left(\frac{(\pi r_0^2)^2}{V_{g0}}\right)x(t) = F'(t)\,. \tag{8.105}$$

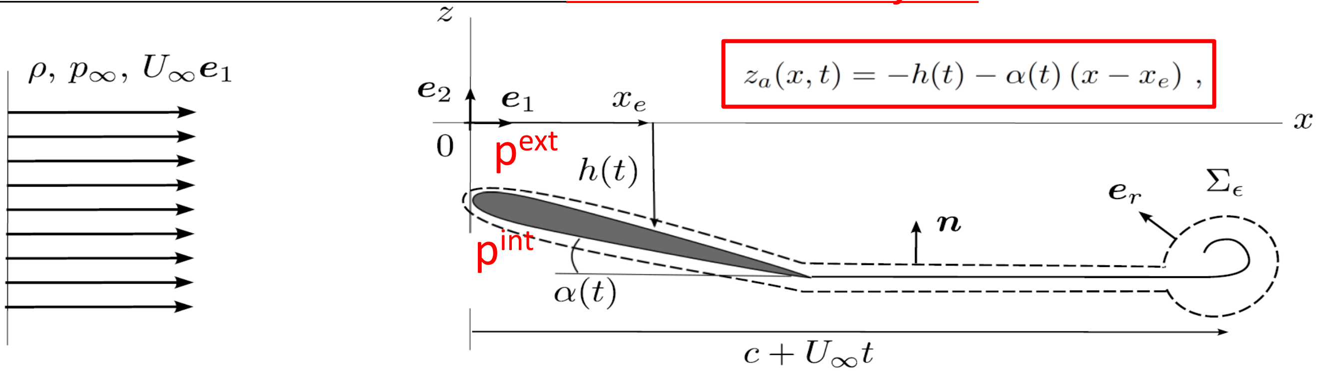
En este problema se muestra que el origen del término de amortiguación que aparece en la clásica ecuación diferencial utilizada en el análisis de las oscilaciones amortiguadas de los sistemas mecánicos,

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = F(t)\,, \tag{8.106}$$

tiene su origen en el flujo dominado por la viscosidad a través de los orificios o ranuras que posee un sólido en movimiento relativo respecto a un líquido.

ESTRUCTURA DE ESTAS 10 PRIMERAS CLASES: 1.- RESOLVER LAS ECUACIONES DE CONTINUIDAD Y DE CANTIDAD DE MOVIMIENTO EN EL LÍMITE DE ALTOS NÚMEROS DE REYNOLDS PARA FLUJOS NO DESPRENDIDOS: CONCEPTOS DE AEI y AEII

$$\alpha(t) \ll 1; \quad h(t)/c \ll 1; \quad Re = \frac{\rho U_\infty c}{\mu} \gg 1$$



CON EL PROPÓSITO DE 2.- CALCULAR LA DISTRIBUCIÓN DE PRESIONES SOBRE EL PERFIL ASÍ COMO 3.- LAS INTEGRALES

$$\mathbf{f_{f-s}} = \int_{\Sigma_s} (p - p_\infty)\,(-\mathbf{n_s})d\sigma; \quad \mathbf{e_3}\cdot\mathbf{f_{f-s}} = \ell(t) \simeq \int_0^c \left(p^{int} - p^{ext}\right) dx$$

$$\mathbf{m_{f-s}}(\mathbf{r_0}) = \int_{\Sigma_s} (\mathbf{r} - \mathbf{r_0}) \times (-\mathbf{n_s})\,(p - p_\infty)\ d\sigma$$

FINALMENTE, CON ESTA INFORMACIÓN, 4.- RESOLVEREMOS EL SISTEMA DE EDOS (1) Y (2) O (1) Y (2') TANTO NUMÉRICA COMO ANALÍTICAMENTE

$$m\left(\ddot{h} + \ddot{\alpha}\,(x_G - x_e)\right) + k_h h(t) = -\ell(t) \qquad \textbf{(1)}$$

$$I_G\ddot{\alpha} + k_\alpha\,\alpha(t) + (x_e - x_G)\,k_h h(t) = \int_0^c (x_G - x)\left(p^{int} - p^{ext}\right) dx = x_G\ell(t) - m_{fs}(x=0) \qquad \textbf{(2)}$$

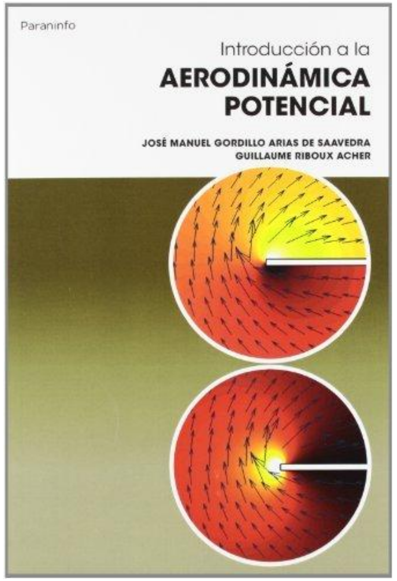
$$m_{fs}(x=0) = \int_0^c x\left(p^{int} - p^{ext}\right) dx$$

EN EFECTO, LA SEGUNDA EDO PUEDE SER SUSTITUIDA POR LA COMBINACIÓN LINEAL DE AMBAS,

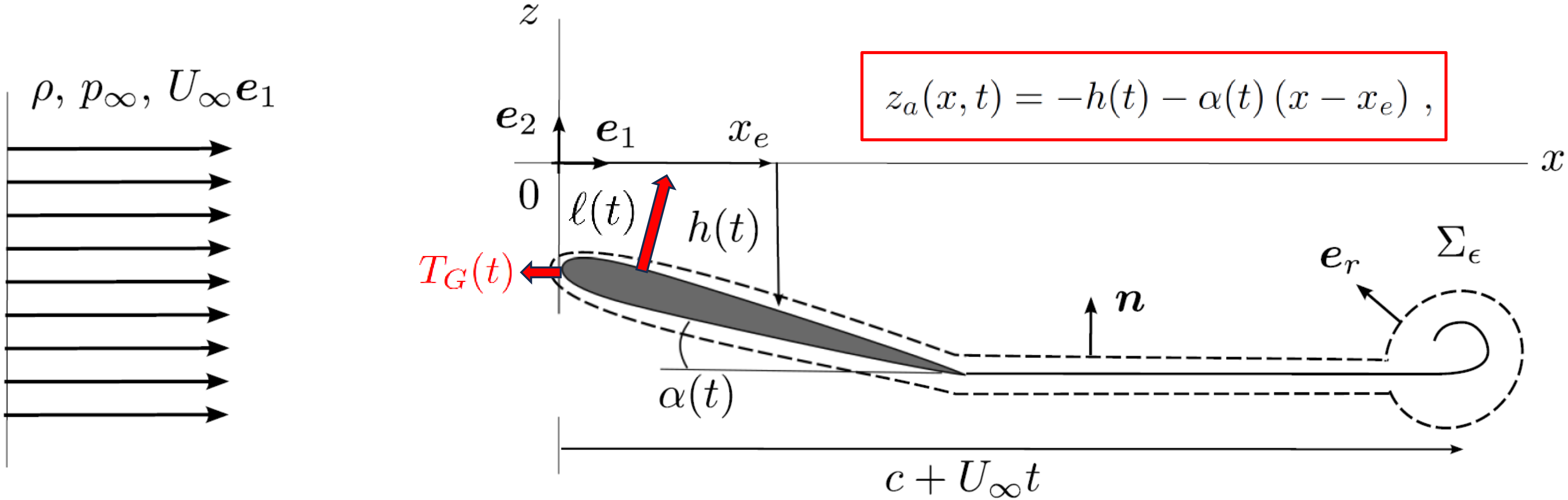
(2')

$$\left(I_G + m\,(x_G - x_e)^2\right)\ddot{\alpha} + m(x_G - x_e)\ddot{h}(t) + k_\alpha\,\alpha(t) = \int_0^c (x_e - x)\left(p^{int} - p^{ext}\right) dx = x_e\ell(t) - m_{fs}(x=0) \qquad (12)$$

<u>Para seguir los desarrollos analíticos de estas notas, así como las clases, es necesario recordar los conceptos de AEI y también es conveniente recordar algunos de los resultados de AEII, con lo que es recomendable hacer uso, bien de los apuntes de las clases de AEI y AEII o bien del manual (hay muchas copias disponibles en la biblioteca de la ETSI)</u>



QUESTION: **What's the value of the force and torque over an airfoil of chord c performing arbitrary heaving and pitching motions in an uniform incompressible stream calculated using the linearized potential flow theory?**



The linearized potential flow theory describes the irrotational flow outside the attached BLs, for which it is neccesary that

$$\alpha(t) \ll 1,\ z_{a,w}/c \ll 1,\ h/c \ll 1$$

Under these starting hipotheses, the linearized impenetrability condition is imposed at z=0, the wake is also located at z=0 and the BL thickness is $\delta/c \propto Re^{-1/2} \ll 1$

Equations and boundary conditions for the irrotational flow outside the BLs

$\mathbf{v} = \nabla\phi$ with $\phi = U_\infty x + \phi'$ and ϕ' indicating the perturbed velocity potential

Which satisfies the Laplace equation $\boxed{\nabla^2\phi' = 0\,,}$

which must be solved subjected to the boundary condition at infinity $\phi' \to 0$ and to the linearized impenetrability condition, which can be expressed as

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + (U_\infty \mathbf{e_1} + \nabla\phi')\cdot\nabla F = 0\,, \quad \text{with} \quad F = z - z_{a,w}(x,t) \tag{2.3}$$

Hence $$w'_{a,w}(x, z = 0^\pm, t) = \frac{\partial z_{a,w}}{\partial t} + U_\infty \frac{\partial z_{a,w}}{\partial x}$$ with $z_a(x,t) = -h(t) - \alpha(t)\,(x - x_e)$,

with $z = 0^\pm$ indicating the upper and lower sides of the airfoil and the wake

and $\mathbf{v}' = \nabla\phi' = u'\mathbf{e_1} + w'\mathbf{e_2}$

We seek for antisymmetric solutions of the Laplace equation in the form of a vortex sheet extending along the airfoil and the wake. We can express the pressure difference across the airfoil in terms of the circulation density

making use of the notation $\phi'^{\pm} = \phi'(x, z = 0^{\pm}, t)$, $\Gamma(x,t) = \oint_C (U_\infty \mathbf{e_1} + \nabla\phi') \cdot d\ell =$ $(\phi^+ - \phi^-) = 2\phi^+(x,t)$ and $\gamma(x,t) = u'^+ - u'^- = \partial\Gamma/\partial x$, in this contribution

$$\Gamma(x,t) = 2\phi^+(x,t) = \int_0^x \gamma(x_0,t) dx_0$$

refers to the clockwise circulation along any closed loop encircling the leading edge of the airfoil and connecting the points $(x, z = 0^-)$ and $(x, z = 0^+)$, whereas $\gamma(x,t)$ indicates the circulation density. In the following, $\Gamma_{a,w}(x,t)$ and $\gamma_{a,w}(x,t)$ will denote the values of the circulation and of the circulation density on the airfoil or at the wake.

The equation governing the pressure jump at $z = 0$ namely, $\Delta p(x, z = 0, t) = p'(x, z = 0^-, t) - p'(x, z = 0^+, t) = p'^-(x,t) - p'^+(x,t)$, with $p' = p - p_\infty$ indicating the perturbed pressure, can be deduced from the linearized Bernoulli equation particularized at $z = 0^\pm$,

$$z = 0^\pm: \quad \rho\frac{\partial\phi'^\pm}{\partial t} + \rho U_\infty \frac{\partial\phi'^\pm}{\partial x} + p'^\pm = 0. \tag{2.6}$$

Hence, the subtraction of the two equations in (2.6) yields

$$\rho\frac{\partial\Gamma}{\partial t} + \rho U_\infty\frac{\partial\Gamma}{\partial x} = \rho\frac{\partial}{\partial t}\left(\int_0^x \gamma\, dx_0\right) + \rho U_\infty \gamma = G(x,t) \quad \text{with} \quad G(0 \leqslant x \leqslant c, t) = \Delta p_a(x,t) \tag{2.7}$$

with Δp_a the pressure jump at the airfoil and, since $\Delta p = 0$ for $x < 0$ and $x > c$, we conclude that $G(x < 0, t) = G(x > c, t) = 0$ in equation (2.7), a fact implying that the material derivatives of both Γ and γ are zero at $z = 0$ for $x < 0$ and $x > c$ namely (Ashley & Landahl 1985),

$$\frac{D\Gamma}{Dt} = \frac{\partial}{\partial x}\left(\frac{D\Gamma}{Dt}\right) = 0 \Rightarrow \frac{\partial\Gamma}{\partial t} + U_\infty\gamma = 0, \quad \frac{\partial\gamma}{\partial t} + U_\infty\frac{\partial\gamma}{\partial x} = 0 \quad \text{for} \quad x < 0, \quad x > c. \tag{2.8}$$

$$\Delta p_a(x,t) = \rho\frac{\partial\Gamma_a}{\partial t} + \rho U_\infty\frac{\partial\Gamma_a}{\partial x} = \rho\frac{\partial}{\partial t}\left(\int_0^x \gamma_a(x_0,t)\, dx_0\right) + \rho U_\infty\gamma_a(x,t)$$

Taking into account that $\Gamma = 0$ for $x \to -\infty$ and also for instants $t < 0$ and that the circulation at the origin of the wake is prescribed by the circulation around the airfoil namely,

$$\Gamma_w(x = c, t) = \Gamma_a(x = c, t) = \Gamma_e(t) = \int_0^c \gamma_a(x,t)\, dx, \tag{2.9}$$

with $\Gamma_e(t)$ the circulation around the airfoil, we deduce from equations (2.8)-(2.9) that

$$\Gamma(x < 0, t) = 0, \quad \Gamma(x > c + U_\infty t, t) = 0, \quad \Gamma_w(x = c + U_\infty(t - t_0), t) = \int_0^c \gamma_a(x, t_0)\, dx,$$
$$\text{and} \quad \gamma_w(x = c + U_\infty(t - t_0), t) = \gamma_w(x = c, t_0), \tag{2.10}$$

with $\gamma_w(x = c, t_0)$ given by equations (2.8) and (2.10):

$$\frac{d}{dt}\left(\int_0^c \gamma_a(x,t)\, dx\right)(t_0) + U_\infty\gamma_w(c, t_0) = 0, \tag{2.11}$$

from which we conclude that:

$$\gamma_w(x_0 = c + U_\infty(t - t_0), t) = \gamma_w(x = c, t_0) = -\frac{1}{U_\infty}\frac{d}{dt}\left[\int_0^c \gamma_a(x,t)\, dx\right](t_0) = -\frac{1}{U_\infty}\frac{d\Gamma_e}{dt}(t_0), \tag{2.12}$$

with the circulation around the airfoil $\Gamma_e(t)$ defined in equation (2.9). Equations (2.7) and (2.12) indicate that the unsteady lift force and the torque,

$$\ell(t) = \int_0^c \Delta p_a(x,t)\, dx \quad \text{and} \quad m(t) = \int_0^c x\, \Delta p_a(x,t)\, dx, \tag{2.13}$$

as well as the density of circulation along the wake, $\gamma_w(x,t)$, can be expressed as a function of $\gamma_a(x,t)$.

Integral equation for the circulation density $\gamma_a(x,t)$

Finally, the density of circulation at the airfoil, $\gamma_a(x,t)$, is deduced imposing that the perturbed vertical velocity induced by the vortex sheet extending along $z=0$, $0 \leqslant x \leqslant c+U_\infty t$ satisfies the linearized impenetrability condition given by equations (2.1) and (2.4) namely (Ashley & Landahl 1985),

$$w'_a(x,z=0^\pm,t) = -\frac{dh}{dt} - U_\infty \alpha(t) - \frac{d\alpha}{dt}(x-x_e) = \\ = \frac{1}{2\pi}\int_0^c \frac{\gamma_a(x_0,t)}{x_0-x}\,dx_0 + \frac{1}{2\pi}\int_c^{c+U_\infty t}\frac{\gamma_w(x_0,t)}{x_0-x}\,dx_0\,. \tag{2.14}$$

Introducing the change of variables

$$x_0 = c + U_\infty(t-t_0) \Rightarrow dx_0 = -U_\infty dt_0 \tag{2.15}$$

and taking into account that the second integral at the right hand side of equation (2.14) can be expressed solely in terms of γ_a by means of equation (2.12), the equation for $\gamma_a(x,t)$ reads

$$w'_a(x,z=0^\pm,t) = \frac{1}{2\pi}\int_0^c \frac{\gamma_a(x_0,t)}{x_0-x}\,dx_0 - \frac{1}{2\pi}\int_0^t \frac{d\Gamma_e/dt_0}{c+U_\infty(t-t_0)-x}\,dt_0 \\ \text{with}\quad \Gamma_e(t) = \int_0^c \gamma_a(x,t)\,dx\,. \tag{2.16}$$

Esta es la ecuación clave de la aerodinámica no estacionaria, pues permite calcular $\gamma_a(x,t)$

Una vez conocida $\gamma_a(x,t)$, las fuerzas y momentos pueden ser calculados de la siguiente manera:

$$\Delta p_a(x,t) = \rho\frac{\partial \Gamma_a}{\partial t} + \rho U_\infty \frac{\partial \Gamma_a}{\partial x} = \rho\frac{\partial}{\partial t}\left(\int_0^x \gamma_a(x_0,t)\,dx_0\right) + \rho U_\infty \gamma_a(x,t)$$

$$\ell(t) = \int_0^c \Delta p_a(x,t)\,dx \quad \text{and} \quad m(t) = \int_0^c x\,\Delta p_a(x,t)\,dx$$

La ecuación integral para $\gamma_a(x,t)$ así como las fuerzas y momentos aerodinámicos pueden ser calculados numéricamente haciendo uso del código numérico descrito en las próximas transparencias, donde explica la implementación numérica del código que resuelve la ecuación integral para $\gamma_a(x,t)$ particularizada para el caso de la respuesta al escalón: función de Wagner

En efecto, como se verá más adelante, la expresión general de las fuerzas y momentos aerodinámicos puede ser obtenida de manera analítica en función de la respuesta a un cambio súbito en el ángulo de ataque del perfil. La función adimensional que describe la evolución temporal de la fuerza de sustentación como respuesta a un cambio súbito del ángulo de ataque recibe el nombre de función de Wagner y es, en realidad, la única función de la aerodinámica no estacionaria que tenemos que hallar de forma numérica. Una vez hallada esta función, tanto la fuerza de sustentación como el momento aerodinámico pueden ser expresados de manera analítica en función de los grados de libertad y de la función de Wagner.

Para resolver la ecuación integral para $\gamma_a(x,t)$ podéis usar el código numérico basado en el método del vortex-lattice 2D que vimos en AEII, cuya extensión al caso de flujo no estacionario es

IV. MATLAB CODES FOR: WAGNER PROBLEM AND FOR THE CASE OF AN OSCILLATING PLUNGING PLATE, THE MEAN THRUST OF OSCILLATING AIRFOILS CALCULATED USING THE VORTEX LATTICE METHOD AND THE MEAN THRUST CALCULATED USING EQUATIONS (4.20)-(4.22) IN THE MAIN TEXT.

A. Vortex Lattice method for unsteady flows: numerical values of the Wagner function and of the suction force at the leading edge

Equation (3) can be written as

$$w(x,t) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{c+U_\infty t}\frac{\gamma(x_0,t)}{x_0-x+\epsilon}\,dx_0 = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{c+U_\infty t}\frac{\partial}{\partial x_0}\left(\frac{\Gamma(x_0,t)}{x_0-x+\epsilon}\right)+\frac{\Gamma(x_0,t)}{(x_0-x+\epsilon)^2}\,dx_0 = \tag{69}$$
$$= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{c+U_\infty t}\frac{\Gamma(x_0,t)}{(x_0-x+\epsilon)^2}\,dx_0\,,$$

where we have made use of the fact that $\Gamma(x=0)=\Gamma(x=c+U_\infty t)=0$. Next, we divide both the airfoil and the wake in N panels of identical width h, bounded by x_{0i} and $x_{0i}+h$ where Γ is constant and equal to the value at the midpoint of the panel, Γ_i. Hence, the integral in (69) can be approximated as

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{c+U_\infty t}\frac{\Gamma(x_0,t)}{(x_0-x+\epsilon)^2}\,dx_0 = \frac{1}{2\pi}\sum_{i=1}^{N}\Gamma_i(t)\left(\frac{1}{x_{0i}-x}-\frac{1}{x_{0i}+h-x}\right)\,. \tag{70}$$

The number of unknowns is the number of panels at the airfoil, N_p. In order to calculate the N_p values of Γ at the airfoil at the instant τ, the values of x in equation (70) are particularized at the midpoints of the N_p panels namely, $x=x_j$, with j varying from 1 to N_p, where the perturbed vertical velocity is known, $w'(x_j,t)=w'_a(x_j,t)$. The values of Γ at the wake panels can be expressed as a function of $\Gamma(x=c,t)=\Gamma_{i=Np}(t)$ because, by virtue of the Euler Bernouilli equation,

$$\frac{\partial\Gamma}{\partial t}+U_\infty\frac{\partial\Gamma}{\partial x}=0\Rightarrow\Gamma(x=c+U_\infty(t-t_0),t)=\Gamma(x=c,t=t_0)=\Gamma_{i=Np}(t_0)\,. \tag{71}$$

```
% JM GORDILLO, UNSTEADY VORTEX-LATTICE: WAGNER PROBLEM
%
clear all; close all; clc;
%
N=200;
h=1/N;
x(1:N)=h/4+((1:N)-1)*h;

x0(1:N)=3*h/4+((1:N)-1)*h;
alpha=5*pi/180;
for j=1:N
for i=1:N
R(j,i)=-1/(x0(j)-x(i))+1/(x0(j)-(x(i)+h));
end
end

R=R/(2*pi);
Rinv=inv(R);
dtau=2*h;
Nsteps=20/dtau;

Clunsteadym1=0;
Cmunsteadym1=0;

for I=0:Nsteps
for j=1:N
if I >0
tau=I*dtau;
tv(I)=tau;
b(j)=-alpha;

for i=1:I
b(j)=b(j)+(1/(2*pi))*Gammaestela(i)*(1/(1+0.25*h+0.5*dtau*(I-i+1)-x0(j))-1/(1+0.25*h+0.5*dtau*(I-i)-
x0(j)));
end

else
b(j)=-alpha;
end
end
gammai=Rinv*b';
Gammaestela(I+1)=gammai(N);
Clunsteady=0;
Cmunsteady=0;
for j=1:N-1
Clunsteady=Clunsteady+h*gammai(j);
end


for j=2:N-1
Cmunsteady=Cmunsteady+h*gammai(j)*x0(j);
end


Cl =2*( Clunsteady - Clunsteadym1 )/ dtau + gammai (N);
Cm =2*( Cmunsteady - Cmunsteadym1 )/ dtau + gammai (N)-Clunsteady ;


Clunsteadym1 = Clunsteady ;
Cmunsteadym1 = Cmunsteady ;


if I>0
Clv(I)=Cl/(pi*alpha);
Clteor1(I)=1-0.165*exp(-0.0455*tau)-0.335*exp(-0.3*tau);
end


end


figure
plot (tv,Clv,'-','linewidth',2,'Color','b');
hold on;
plot (tv,Clteor1,'-','linewidth',2,'Color','r');
```

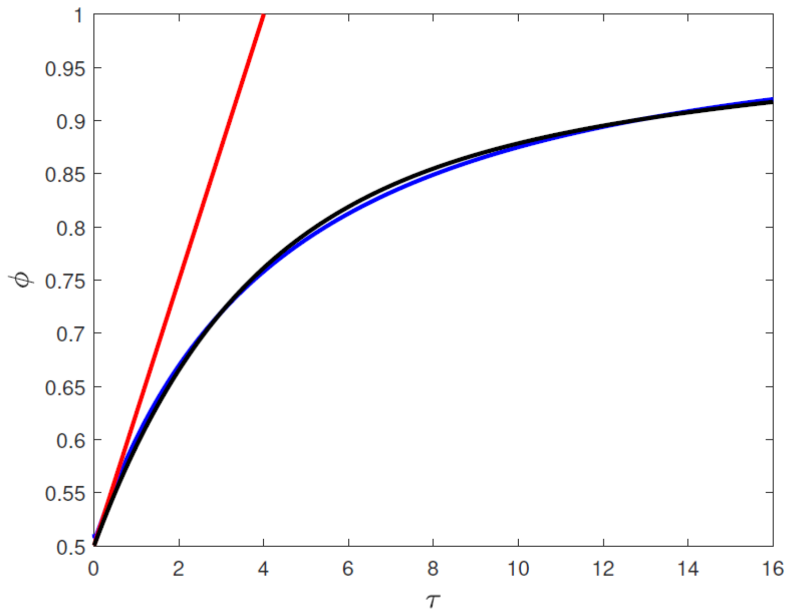
La función de Wagner representa la fuerza adimensional de sustentación que resulta de la respuesta a un escalón (función de Heaviside).

$$\tau = t\frac{2U_\infty}{c} \quad \text{and} \quad \tau_0 = t_0\frac{2U_\infty}{c}$$

$\phi(\tau)$ is the so-called Wagner function, which is the dimensionless lift force in response to the linearized impenetrability condition

$$-w'_{3/4}/U_\infty = \alpha\, H(\tau)$$

Namely $$\ell_{cW}(\tau) = \rho U_\infty^2 c\pi\alpha\phi(\tau) \Rightarrow \phi(\tau) = \frac{\ell_{cW}(\tau)}{1/2\rho U_\infty^2 c 2\pi\alpha} = \frac{\ell_{cW}(\tau)}{\ell_{cW}(\tau \to \infty)}$$



The figure shows the numerical solution of the Wagner function obtained using the numerical method explained previously. The numerical solution can be very well approximated using the expression due to Jones:

$$\phi(\tau) = 1 - 0.165\, e^{-0.0455\,\tau} - 0.335\, e^{-0.3\tau}$$

$$\tau = t\frac{2U_\infty}{c} \quad \text{and} \quad \tau_0 = t_0\frac{2U_\infty}{c}$$

The figure also shows a comparison with the two-term analytical solution, valid for $\tau \ll 1$, described in the Supplementary Material

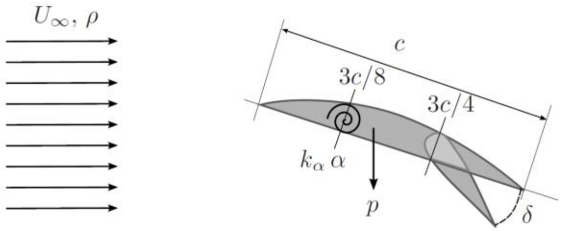
Usando el método del VL para la variable Γ en vez de para γ no es necesario usar la ecuación adicional dada por el teorema de Bjerness-Kelvin

Las fuerzas y momentos aerodinámicos correspondientes al flujo irrotacional, linealizado e incompresible pueden ser también calculados de manera analítica siguiendo los desarrollos de estas notas. Como paso previo es conveniente repasar los conceptos de AEI y, más concretamente, el método de Glauert, que permite resolver de manera analítica la ecuación integral para $\gamma_a(x,t)$

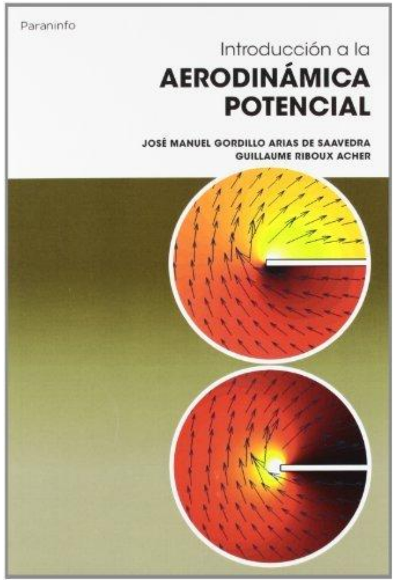
Antes de tratar el caso de flujo no estacionario, recordemos cómo resolver primero el siguiente problema:

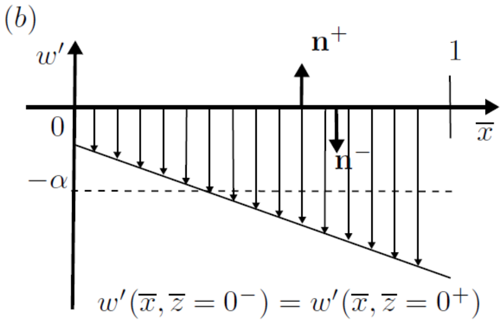
DPTO. INGENIERÍA AEROESPACIAL Y MECÁNICA DE FLUIDOS	*AERODINÁMICA I* *18/06/25 - Segunda convocatoria*	***Problema*** **2 h 15 min**

1. Un flujo uniforme de densidad conocida ρ y velocidad horizontal U_∞ incide sobre un perfil aerodinámico de cuerda y peso conocidos c y p respectivamente, estando el centro de masa situado a una distancia $c/2$ del borde de ataque del perfil, cuyos intradós y extradós se encuentran situados, respectivamente, en $\bar{z}_{int} = 0$, $\bar{z}_{ext} = 2\epsilon\left(\bar{x} - \bar{x}^2\right)$ siendo $\bar{x} = x/c$ y ϵ conocido. Como muestra la figura adjuntada, el perfil se encuentra articulado a un muelle de torsión situado a una distancia del borde de ataque de $3c/8$ que ejerce un momento $k_\alpha\,\alpha$ en sentido antihorario, siendo α el ángulo de ataque y k_α la constante elástica, que tiene un valor conocido. Asimismo, el perfil incorpora un flap articulado a una distancia del borde de ataque de $3c/4$ que puede deflectarse en sentido horario un ángulo $\delta \ll 1$. Se pide: (*i*) Determinar el valor de la denominada velocidad de divergencia, U_∞^*, a partir de la cual se rompe el muelle de torsión (*2 puntos*), (*ii*) para $U_\infty < U_\infty^*$ y $\delta = 0$, proporcionen la expresión del coeficiente de presiones tanto en el extradós (*1.5 puntos*) como en el intradós del perfil (*1.5 puntos*), (*iii*) para $U_\infty < U_\infty^*$ y $\delta \neq 0$ conocido, den la expresión de la fuerza de sustentación en función de los datos del problema y, asimismo, determine la denominada velocidad de inversión de mando, por encima de la cual, la contribución a la fuerza de sustentación asociada a la defexión del flap disminuye al aumentar U_∞ (*5 puntos*).



Problema sustentador, antisimétrico, campo de velocidades generado por una superposición de torbellinos





$$w'(\overline{x},\overline{z}=0) = U_\infty\left(-\alpha+\frac{\mathrm{d}\,\overline{z}_c}{\mathrm{d}\overline{x}}\right) = $$

$$= -\frac{1}{2\pi}\lim_{z\to 0}\int_0^1 \frac{2\,u'(\overline{x}_0,\overline{z}=0^+)(\overline{x}-\overline{x}_0)}{(\overline{x}-\overline{x}_0)^2+\overline{z}^2}\,\mathrm{d}\overline{x}_0 = \qquad (5.24)$$

$$= -\frac{1}{\pi}\mathrm{VP}\int_0^1 \frac{u'(\overline{x}_0,\overline{z}=0^+)}{\overline{x}-\overline{x}_0}\,\mathrm{d}\overline{x}_0\,,$$

γ(x)

$\overline{x} = 1/2\,(1+\cos\theta)$ $\qquad \overline{x}_0 = 1/2\,(1+\cos\theta_0)$. **Borde de salida, θ=0 y borde de ataque θ=π**

Rebordeo de borde de ataque $A_0\tan(\theta/2) = A_0/\sqrt{\overline{x}}$

Método de Glauert: solución de la ecuación integral expresando γ(x) como una serie infinita en la que el flujo rebordea el borde de ataque y no rebordea el borde de salida: u'(x=c,z=0⁺)=0

$$u'(\theta) = A_0\tan(\theta/2) + \sum_{n=1}^{\infty} A_n\sin(n\theta)\,.$$

$$U_\infty\left(-\alpha+\frac{\mathrm{d}\,\overline{z}_c}{\mathrm{d}\overline{x}}\right) = \frac{A_0}{\pi}\int_0^\pi \frac{1-\cos\theta_0}{\cos\theta_0-\cos\theta}\,\mathrm{d}\theta_0 + \sum_{n=1}^{\infty}\frac{A_n}{\pi}\int_0^\pi \frac{\sin(n\theta_0)\sin\theta_0}{\cos\theta_0-\cos\theta}\,\mathrm{d}\theta_0 \qquad (5.35)$$

El término de la derecha de 5.35 se calcula fácilmente usando el valor de la integral de Glauert

$$\mathrm{VP}\int_0^\pi \frac{\cos(n\theta_0)}{\cos\theta_0-\cos\theta}\,\mathrm{d}\theta_0 = \pi\frac{\sin n\theta}{\sin\theta}\,,$$

… obteniéndose la siguiente ecuación para los coeficientes A_n, cuyos valores se hallan proyectando la ecuación 5.37 en el conjunto ortogonal de autofunciones cos(nθ)

$$U_\infty\left(-\alpha+\frac{\mathrm{d}\,\overline{z}_c}{\mathrm{d}\overline{x}}\right)=-A_0+\sum_{n=1}^{\infty}\frac{A_n}{2}\frac{\sin(n-1)\theta-\sin(n+1)\theta}{\sin\theta}= \\ =-A_0+\sum_{n=1}^{\infty}\frac{A_n}{2}\frac{-2\cos(n\theta)\sin\theta}{\sin\theta}=-A_0-\sum_{n=1}^{\infty}A_n\cos(n\theta)\ . \tag{5.37}$$

Puesto que el conjunto de autofunciones $\cos n\theta$ es ortogonal; es decir, ya que

$$\int_0^\pi\cos(m\theta)\cos(n\theta)\,\mathrm{d}\theta=\int_0^\pi\frac{\cos[(m-n)\theta]+\cos[(m+n)\theta]}{2}\,\mathrm{d}\theta=\frac{\pi}{2}\delta_{m,n}\,, \tag{5.38}$$

donde $\delta_{m,n}=1$ si $m=n\neq 0$, $\delta_{0,0}=2$ y $\delta_{m,n}=0$ si $m\neq n$, el valor de los coeficientes A_n puede hallarse sin más que multiplicar ambos lados de la ecuación (5.37) por $\cos(m\theta)$ e integrar la ecuación resultante entre 0 y π. En efecto, el resultado para $m=0$ proporciona la siguiente ecuación para el coeficiente A_0

Llegándose a que los coeficientes A_n pueden ser calculados de la siguiente manera como función de la condición de contorno de impenetrabilidad linealizada; es decir, como función del valor de w'(x,z=0)

$$\frac{A_0}{U_\infty} = \alpha - \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}\,\overline{z}_c}{\mathrm{d}\overline{x}}\,\mathrm{d}\theta \qquad \frac{A_m}{U_\infty} = -\frac{2}{\pi}\int_0^\pi \frac{\mathrm{d}\,\overline{z}_c}{\mathrm{d}\overline{x}}\cos(m\theta)\,\mathrm{d}\theta$$

u'(x,z=0^+) ya es conocido: $u'(\theta) = A_0\tan(\theta/2) + \sum_{n=1}^{\infty} A_n \sin(n\theta)\,.$ **Y puesto que** $C_p(\overline{x}) = \frac{p - p_\infty}{1/2\,\rho U_\infty^2} = 1 - \left(1 + \frac{u'}{U_\infty}\right)^2 \simeq -2\frac{u'}{U_\infty}$

$$C_p = C_p^{curv} + C_p^{esp}\,.$$

$$C_{p,curv}^{\mathrm{ext}} = -2\left(\frac{A_0}{U_\infty}\tan(\theta/2) + \sum_{n=1}^{\infty}\frac{A_n}{U_\infty}\sin(n\theta)\right) \quad \text{y}$$

$$C_{p,curv}^{\mathrm{int}} = -C_{p,curv}^{\mathrm{ext}} = 2\left(\frac{A_0}{U_\infty}\tan(\theta/2) + \sum_{n=1}^{\infty}\frac{A_n}{U_\infty}\sin(n\theta)\right)$$

Ya podemos calcular el coeficiente de sustentación y de momentos en función del valor de los coeficientes A_n

$$C_l = \int_0^1 \left(C_p^{\mathrm{int}} - C_p^{\mathrm{ext}}\right)\mathrm{d}\overline{x} = 2\pi\left(\frac{A_0}{U_\infty} + \frac{A_1}{2\,U_\infty}\right) = 2\pi\left[\alpha - \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}\,\overline{z}_c}{\mathrm{d}\overline{x}}(1+\cos\theta)\,\mathrm{d}\theta\right]$$

$$C_{m,\overline{x}=0} = \int_0^1 \left(C_p^{\mathrm{int}} - C_p^{\mathrm{ext}}\right)\overline{x}\,\mathrm{d}\overline{x} = \frac{\pi}{2}\left(\frac{A_0}{U_\infty} + \frac{A_1}{U_\infty} + \frac{A_2}{2\,U_\infty}\right)$$

Cálculo analítico del valor inicial de la función de Wagner: las velocidades inducidas por el torbellino de arranque disminuyen al ángulo de ataque efectivo. Además, inicialmente, la solución es simétrica respecto a x=c/2, con rebordeo de borde de ataque, no rebordeo de borde de salida como indica la condición de Kutta pero 'rebordeo' en el torbellino de arranque, que está en x= c+U$_{oo}$dt

It is interesting to note that the expression of the circulatory lift force experienced by an airfoil whose angle of attack varies suddenly at $t = 0$ i.e., $\alpha(\tau) = \alpha H(\tau)$ can be easily calculated at $\tau \to 0^+$ using the alternative procedure described next, which makes use of the result in Appendix A of the main text, reproduced here for clarity purposes. Indeed, the fact that the value of the circulation at $\tau = 0^+$ namely, right after the angle of attack has changed to a value $\alpha \neq 0$, is $\Gamma_e(\tau = 0^+) = 0$, means that the flow around the airfoil is symmetric with respect to $x = c/2$. The potential flow which satisfies this requirement corresponds to the one generated by a distribution of vortices with a circulation per unit length given by

$$\gamma(x) = U_\infty A_0' \left(\sqrt{\frac{1-x/c}{x/c}} - \sqrt{\frac{x/c}{1-x/c}}\right) = U_\infty A_0' \frac{1-2x/c}{\sqrt{x/c\,(1-x/c)}}, . \tag{54}$$

Indeed, notice that Eq. (54) expresses a symmetric distribution of γ around $x = c/2$ and also that the flow turns around both the leading and trailing edges of the airfoil.

The value of A_0' in Eq. (54) is determined by imposing the linearized impenetrability condition namely,

$$w' = -U_\infty\alpha = \frac{U_\infty A_0'}{2\pi}\int_0^c \frac{1-2x_0/c}{\sqrt{x/c\,(1-x/c)}}\frac{dx_0}{x_0 - x} = \frac{U_\infty A_0'}{2\pi}\int_0^\pi \frac{\sin\theta_0}{2}\frac{2\cos\theta_0}{\sin\theta_0}\frac{d\theta_0}{1/2\,(\cos\theta - \cos\theta_0)} = \frac{-2\pi U_\infty A_0'}{2\pi}, \tag{55}$$

where we have made use of the value of the Glauert integral (20) and, consequently, the perturbed potential satisfying the condition that the circulation around the airfoil is zero is given by, see Eq. (54),

$$\gamma(x) = U_\infty\alpha\frac{1-2x/c}{\sqrt{x/c\,(1-x/c)}}. \tag{56}$$

However, the solution expressed by Eq. (56) does not satisfy the Kutta condition because $\gamma(x = c) \to \infty$ as a consequence of the fact that, as it was explained above, the solution in Eq. (56) the flow is symmetric around $x = c/2$ and, hence, the flow turns around both the leading and trailing edges, a fact implying that the Kutta condition is not fulfilled by the potential flow generated by the distribution of vortices with $\gamma(x)$ given by Eq. (56). Then, how is it possible to fulfill at the same time the following two conditions namely, a zero initial circulation, which implies a symmetric flow around the airfoil and also the Kutta condition? The solution to this apparent paradox is the following: it is possible to comply with both conditions when we seek for a symmetric potential flow around an airfoil with a length increasing in time as $dc/dt = U_\infty$: in this case, the flow turns around the leading edge of the airfoil *but not the trailing edge*. Indeed, whereas the trailing edge is located at any instant of time at $x = c$, the potential flow generated by the distribution of $\gamma(x,t)$ given by

$$\gamma_a(x,t) = U_\infty\alpha\frac{1-2x/c(t)}{\sqrt{x/c(t)\,(1-x/c(t))}} \quad \text{with} \quad \frac{dc}{dt} = U_\infty. \tag{57}$$

corresponds to a potential flow which turns around $x = c + U_\infty\,dt$ namely, a potential flow which turns around the *stating vortex*, which is located downstream the leading edge and it is transported with a velocity U_∞.

Hence,

$$\ell_c(t = 0^+) = \int_0^c \Delta p_a(x,t)\,dx \quad \text{with} \quad \Delta p_a(x,t) = \rho\frac{\partial\Gamma_a}{\partial t} + \rho U_\infty\frac{\partial\Gamma_a}{\partial x} \Rightarrow \ell(t = 0^+) = \rho\frac{d}{dt}\int_0^c \Gamma_a\,dx \tag{58}$$

because

$$\int_0^c \frac{\partial \Gamma_a}{\partial x} dx = \Gamma_a(x=c) - \Gamma_a(x=0) = 0 \tag{59}$$

for the potential flow generated by the symmetric distribution given in Eq. (57). Now, notice that

$$\Gamma_a(x,t) = \int_0^x \gamma_a dx = c(t) \int_0^{x/c(t)} U_\infty \alpha \, \frac{1-2x/c(t)}{\sqrt{x/c(t)\,(1-x/c(t))}} d\left(x/c(t)\right) = 2\, U_\infty \alpha\, c(t) \sqrt{x/c(t) - \left(x/c(t)\right)^2} \tag{60}$$

with $dc(t)/dt = U_\infty$ and, therefore, the substitution of the result in Eq. (60) in Eq. (58) yields,

$$\begin{aligned} \ell_c(t=0^+) &= 2\rho\, U_\infty\, \alpha \frac{d}{dt}\left(c^2(t) \int_0^{x/c(t)} \sqrt{x/c(t)\,(1-x/c(t))}\; d\left(x/c(t)\right)\right) = 2\rho\, U_\infty\, \alpha \frac{d}{dt}\left(c^2(t) \int_0^{\pi} \sqrt{\frac{1-\cos^2\theta}{4}}\; d\theta\right) = \\ &= 2\rho\, U_\infty\, \alpha \frac{d}{dt}\left(c^2(t) \int_0^{\pi} \frac{\sin^2\theta}{4} d\theta\right) = 2\rho\, U_\infty\, \alpha \frac{d}{dt}\left(c^2(t) \int_0^{\pi} \frac{1-\cos(2\theta)}{8} d\theta\right) = 2\rho\, U_\infty\, \alpha \frac{d}{dt}\left(\frac{\pi\, c^2(t)}{8}\right) = \\ &= \frac{\pi}{2}\rho U_\infty \frac{dc}{dt} c(t=0+)\alpha = \frac{\pi}{2}\alpha\rho U_\infty^2 c\,, \end{aligned} \tag{61}$$

where we have made use of the change of variables $x/c(t) = (1-\cos\theta)/2 \rightarrow d(x/c(t)) = 1/2 \sin\theta \, d\theta$. The result in Eq. (61) shows that the initial lift around an airfoil which experiences a sudden change of the angle of attack is, indeed, one-half the lift force corresponding to steady flow.

One of the advantages of using this alternative way of finding the value of $\ell(t=0^+)$, is that it reveals the idea that, in order to satisfy Kutta's condition, the potential flow needs to turn around, not at the trailing edge but at the *starting vortex*, which is convected downstream at a velocity U_∞. Notice also that the local distribution of $\gamma(x,t)$ around the starting vortex is identical to that at the leading edge of the airfoil.

Solution of the integral equation for $\gamma_a(x,t)$ using Glauert's method (II)

Finally, the density of circulation at the airfoil, $\gamma_a(x,t)$, is deduced imposing that the perturbed vertical velocity induced by the vortex sheet extending along $z=0$, $0 \leqslant x \leqslant c+U_\infty t$ satisfies the linearized impenetrability condition given by equations (2.1) and (2.4) namely (Ashley & Landahl 1985),

$$w'_a(x,z=0^\pm,t) = -\frac{dh}{dt} - U_\infty \alpha(t) - \frac{d\alpha}{dt}(x-x_e) = \\ = \frac{1}{2\pi}\int_0^c \frac{\gamma_a(x_0,t)}{x_0-x}\,dx_0 + \frac{1}{2\pi}\int_c^{c+U_\infty t}\frac{\gamma_w(x_0,t)}{x_0-x}\,dx_0 \,. \tag{2.14}$$

Introducing the change of variables

$$x_0 = c + U_\infty (t-t_0) \Rightarrow dx_0 = -U_\infty dt_0 \tag{2.15}$$

and taking into account that the second integral at the right hand side of equation (2.14) can be expressed solely in terms of γ_a by means of equation (2.12), the equation for $\gamma_a(x,t)$ reads

$$w'_a(x,z=0^\pm,t) = \frac{1}{2\pi}\int_0^c \frac{\gamma_a(x_0,t)}{x_0-x}\,dx_0 - \frac{1}{2\pi}\int_0^t \frac{d\Gamma_e/dt_0}{c+U_\infty(t-t_0)-x}\,dt_0 \\ \text{with} \quad \Gamma_e(t) = \int_0^c \gamma_a(x,t)\,dx \,. \tag{2.16}$$

In order to solve the integral equation (2.16) notice first that, since $\Gamma(x\to-\infty,t)=0$ then, by virtue of equation (2.8), $\Gamma(x<0,t)=0$ and hence, $\phi'(z=0^\pm,x<0)=0$. Consequently, the local solution of the Laplace equation (2.2) at the leading edge of the airfoil is the one corresponding to the flow around a wedge of angle 2π namely,

$$\phi' = U_\infty c A_0(t)\,(r/c)^{1/2}\cos(\beta/2)\,, \tag{2.17}$$

with $A_0(t)$ a dimensionless time-dependent constant, $r/c \ll 1$ the radial distance to the leading edge -which is located at $z=0$, $x=0$ in the linearized theory- and $0 \leqslant \beta \leqslant 2\pi$ indicating the polar angle measured in counterclockwise manner from the horizontal axis.

Taking into account: i) that the Kutta condition ensures that $\gamma_a(x=c,t)$ is finite in order to avoid that the flow turns around the trailing edge of the airfoil and ii) that $\gamma_a(x/c \ll 1,t)$ is given by

$$\frac{\partial\phi'}{\partial r}(\beta=0,r/c\ll 1,t) = u'(z=0,x/c\ll 1,t) = \frac{U_\infty}{2}A_0(t)\,(x/c)^{-1/2} \Rightarrow \\ \Rightarrow \gamma_a(x/c\ll 1,t) = U_\infty A_0(t)\,(x/c)^{-1/2}\,, \tag{2.18}$$

where use of equation (2.17) has been made, it can be concluded that the integral equation (2.16) can be solved using Glauert's method, which relies on expressing the unknown function $\gamma_a(x,t)$ as the infinite series (Glauert 1983):

$$\frac{\gamma_a(x,t)}{U_\infty} = A_0(t)\sqrt{\frac{1-x/c}{x/c}} + \sum_{n=1}^{\infty} A_n(t)\sin(n\theta) = A_0(t)\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n(t)\sin(n\theta)\,, \tag{2.19}$$

where we have introduced the change of variables

$$\frac{x}{c} = \frac{1-\cos\theta}{2} \tag{2.20}$$

$$\Gamma_e(t) = \Gamma_a(x=c,t) = \frac{\pi U_\infty c}{2}\left(A_0 + \frac{A_1}{2}\right)$$

$$\Delta p_a(x,t) = \rho\frac{\partial \Gamma_a}{\partial t} + \rho U_\infty \frac{\partial \Gamma_a}{\partial x} = \rho\frac{\partial}{\partial t}\left(\int_0^x \gamma_a(x_0,t)\,dx_0\right) + \rho U_\infty \gamma_a(x,t)$$

$$\ell(t) = \int_0^c \Delta p_a(x,t)\,dx \quad \text{and} \quad m(t) = \int_0^c x\,\Delta p_a(x,t)\,dx$$

Solution of the integral equation for $\gamma_a(x,t)$ using Glauert's method (II)

$$\ell(t) = \int_0^c \Delta p_a(x,t)\, dx \quad \text{and} \quad m(t) = \int_0^c x\, \Delta p_a(x,t)\, dx$$

$$\Delta p_a(x,t) = \rho \frac{\partial \Gamma_a}{\partial t} + \rho\, U_\infty \frac{\partial \Gamma_a}{\partial x} = \rho \frac{\partial}{\partial t}\left(\int_0^x \gamma_a(x_0,t)\, dx_0\right) + \rho\, U_\infty \gamma_a(x,t)$$

$$\tau = t\frac{2\,U_\infty}{c} \quad \text{and} \quad \tau_0 = t_0 \frac{2\,U_\infty}{c}$$

$$\Gamma_a(x,t) = \int_0^x \gamma_a(x_0,t)\, dx_0 = \frac{U_\infty\, c}{2}\int_0^\theta \left(A_0 \frac{1+\cos\theta_0}{\sin\theta_0} + \sum_{n=1}^{\infty} A_n \sin(n\theta_0)\right)\sin\theta_0\, d\theta_0 =$$

$$= \frac{U_\infty\, c}{2}\left[A_0\,(\theta + \sin\theta) + \sum_{n=1}^{\infty}\frac{A_n}{2}\int_0^\theta \left(\cos[(n-1)\theta_0] - \cos[(n+1)\theta_0]\right)\, d\theta_0\right] =$$

$$= \frac{U_\infty\, c}{2}\left[A_0\,(\theta+\sin\theta) + \frac{A_1}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right) + \sum_{n=2}^{\infty}\frac{A_n}{2}\left(\frac{\sin((n-1)\theta)}{n-1} - \frac{\sin((n+1)\theta)}{n+1}\right)\right],$$

$$\Gamma_e(t) = \Gamma_a(x=c,t) = \frac{\pi\, U_\infty c}{2}\left(A_0 + \frac{A_1}{2}\right)$$

Circulation around th airfoil

$$\ell(t) = \frac{\rho U_\infty c^2}{4}\frac{d}{dt}\int_0^\pi \sin\theta\left(A_0\,(\theta+\sin\theta) + \frac{A_1}{2}\theta + \frac{A_2}{2}\sin\theta\right)d\theta + \rho U_\infty \Gamma_e(t) =$$

$$= \frac{\rho U_\infty^2 c\,\pi}{2}\left(\frac{3\,\dot{A}_0}{2} + \frac{\dot{A}_1}{2} + \frac{\dot{A}_2}{4} + \left(A_0 + \frac{A_1}{2}\right)\right) \quad \text{with} \quad \Gamma_e(t) = \Gamma_a(\theta=\pi,t)\,,$$

$$m(t) = \rho\frac{d}{dt}\int_0^c x\,\Gamma_a\, dx + \rho\, U_\infty \int_0^c x \frac{\partial \Gamma_a}{\partial x}\, dx = \rho \frac{d}{dt}\int_0^c x\,\Gamma_a\, dx + \rho\, U_\infty \Gamma_e\, c - \rho\, U_\infty \int_0^c \Gamma_a\, dx$$

$$= \frac{\rho U_\infty^2 c^2 \pi}{4}\left(\frac{3\,\dot{A}_0}{2} + \frac{\dot{A}_1}{2} + \frac{\dot{A}_2}{4}\right) + \frac{\rho U_\infty^2 c^2 \pi}{4}\left(\frac{\dot{A}_0}{4} + \frac{3\dot{A}_1}{16} - \frac{\dot{A}_3}{16}\right) - \frac{\rho U_\infty^2 c^2 \pi}{4}\left(\frac{3A_0}{2} + \frac{A_1}{2} + \frac{A_2}{4}\right) + \rho\, U_\infty \Gamma_e(t)\, c,$$

Solution of the integral equation for $\gamma_a(x,t)$ using Glauert's method (III)

$$w'_a(x, z=0^{\pm}, t) = \frac{1}{2\pi}\int_0^c \frac{\gamma_a(x_0,t)}{x_0 - x}\,dx_0 - \frac{1}{2\pi}\int_0^t \frac{d\Gamma_e/dt_0}{c + U_\infty(t-t_0) - x}\,dt_0$$

$$\text{with} \quad \Gamma_e(t) = \int_0^c \gamma_a(x,t)\,dx\,.$$

$$\Gamma_e(t) = \Gamma_a(x=c,t) = \frac{\pi U_\infty c}{2}\left(A_0 + \frac{A_1}{2}\right)$$

$$w'(x, z=0^{\pm}, t) = \frac{1}{2\pi}\int_0^c \frac{\gamma_a(x_0,t)}{x_0 - x}\,dx_0 - \frac{\pi U_\infty c}{4\pi}\int_0^t \frac{d\left(A_0 + A_1/2\right)/dt_0}{c + U_\infty(t-t_0) - x}\,dt_0 =$$

$$\frac{U_\infty}{2\pi}\int_0^\pi \frac{\sin\theta_0}{\cos\theta - \cos\theta_0}\left(A_0\frac{1+\cos\theta_0}{\sin\theta_0} + \sum_{n=1}^\infty A_n \sin(n\theta_0)\right)d\theta_0 - \frac{U_\infty}{2}\int_0^\tau \frac{\dot{A}_0 + \dot{A}_1/2}{1 + (\tau - \tau_0) + \cos\theta}\,d\tau_0\,,$$

$$I_G(n) = \int_0^\pi \frac{\cos(n\theta_0)\,d\theta_0}{\cos\theta_0 - \cos\theta} = \pi\frac{\sin(n\theta)}{\sin\theta}$$

$$\frac{U_\infty}{2\pi}\int_0^\pi \frac{\sin\theta_0}{\cos\theta - \cos\theta_0}\left(A_0\frac{1+\cos\theta_0}{\sin\theta_0} + \sum_{n=1}^\infty A_n \sin(n\theta_0)\right)d\theta_0 = -\frac{U_\infty}{2}\left(A_0 - \sum_{n=1}^\infty A_n\cos(n\theta)\right)$$

$$\int_0^\pi w'(\theta, z=0^{\pm}, t)\cos(m\theta)\,d\theta = -\frac{U_\infty}{2}\left(A_0\frac{\pi}{2}F(0,m) - A_m\frac{\pi}{2}F(n,m)\right) - $$
$$- \frac{U_\infty}{2}\int_0^\tau \frac{d}{d\tau_0}\left(A_0 + \frac{A_1}{2}\right)\left(\int_0^\pi \frac{\cos(m\theta)\,d\theta}{1 + (\tau-\tau_0) + \cos\theta}\right)d\tau_0$$

Before solving equation (12) we first need to calculate the value of integrals of the type

$$\int_0^{\pi} \frac{\cos(n\theta)\, d\theta}{1+(\tau-\tau_0)+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{e^{i\,n\theta}}{1+(\tau-\tau_0)+\cos\theta}\,, \tag{13}$$

which can be easily evaluated using the calculus of residues by applying Cauchy's theorem once the integral in equation (13) is evaluated carrying out the line integral along the unit circle $z = e^{i\theta}$ in the complex plane. Indeed, introducing the change of variables

$$z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta \Rightarrow d\theta = \frac{-i\,dz}{z}\,, \quad e^{in\theta} = z^n\,, \quad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)\,, \tag{14}$$

equation (13) can be written as

$$\frac{1}{2}\int_0^{2\pi} \frac{e^{i\,n\theta}\, d\theta}{1+(\tau-\tau_0)+\cos\theta} = -i\int_0^{2\pi} \frac{z^n\, dz}{z^2+2\,z\,(1+(\tau-\tau_0))+1} = \int_0^{2\pi} \frac{-i\, z^n\, dz}{(z-z_1)\,(z-z_2)}\,, \tag{15}$$

with

$$z_1 = -\left(1+(\tau-\tau_0)\right) + \sqrt{(\tau-\tau_0)^2+2\,(\tau-\tau_0)} \quad \text{and} \quad z_2 = -\left(1+(\tau-\tau_0)\right) - \sqrt{(\tau-\tau_0)^2+2\,(\tau-\tau_0)}\,. \tag{16}$$

Since $\tau_0 < \tau$, $|z_1| < 1$, i.e., the pole z_1 is included within the unit circle and hence, the calculus of residues yields that

$$\int_0^{2\pi} \frac{-i\, z^n\, dz}{(z-z_1)\,(z-z_2)} = 2\pi\, i \frac{-i\, z_1^n}{z_1-z_2} = \pi \frac{\left(-B+\sqrt{B^2-1}\right)^n}{\sqrt{B^2-1}} \quad \text{with} \quad B = 1+(\tau-\tau_0) \tag{17}$$

and, therefore,

$$\begin{aligned}
n=0\,, &\quad \int_0^{\pi} \frac{d\theta}{1+(\tau-\tau_0)+\cos\theta} = \pi\frac{1}{\sqrt{B^2-1}}\,,\\
n=1\,, &\quad \int_0^{\pi} \frac{\cos(\theta)d\theta}{1+(\tau-\tau_0)+\cos\theta} = \pi\left(1-\frac{B}{\sqrt{B^2-1}}\right)\,,\\
n=2\,, &\quad \int_0^{\pi} \frac{\cos(2\theta)d\theta}{1+(\tau-\tau_0)+\cos\theta} = \pi\left(-2B+\frac{2B^2-1}{\sqrt{B^2-1}}\right)\,,\\
n=3\,, &\quad \int_0^{\pi} \frac{\cos(3\theta)d\theta}{1+(\tau-\tau_0)+\cos\theta} = \pi\left(4B^2-1-2B\sqrt{B^2-1}-B\frac{2B^2-1}{\sqrt{B^2-1}}\right)\,,
\end{aligned} \tag{18}$$

with $\quad B = 1+(\tau-\tau_0)\;.$

Solution of the integral equation for $\gamma_a(x,t)$ using Glauert's method (IV)

Equations (24)

$$m=0\,,\quad \frac{1}{U_\infty}\left(U_\infty \alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{c}{2}\right)\frac{d\alpha}{dt}\right)=\frac{A_0}{2}+\frac{I_1}{2}$$

$$m=1\,,\quad \frac{c}{4U_\infty}\frac{d\alpha}{dt}=-\frac{A_0}{2}+\frac{I_1}{2}+\frac{I_2}{2}$$

$$m=2\,,\quad 0=\frac{A_2}{4}+A_0+\frac{A_1}{2}-\frac{I_1}{2}+\int_0^\tau(\tau-\tau_0)\left(\dot A_0+\dot A_1/2\right)d\tau_0-\int_0^\tau\left(\dot A_0+\dot A_1/2\right)\sqrt{B^2-1}\,d\tau_0$$

$$m=3\,,\quad 0=\frac{A_3}{4}-\frac{3}{2}\left(A_0+\frac{A_1}{2}\right)+\frac{I_1+I_2}{2}+2\int_0^\tau\left(\dot A_0+\dot A_1/2\right)\sqrt{B^2-1}\,d\tau_0+$$

$$+2\int_0^\tau(\tau-\tau_0)\left(\dot A_0+\dot A_1/2\right)\sqrt{B^2-1}\,d\tau_0-2\int_0^\tau(\tau-\tau_0)^2\left(\dot A_0+\dot A_1/2\right)d\tau_0-4\int_0^\tau(\tau-\tau_0)\left(\dot A_0+\dot A_1/2\right)d\tau_0$$

$$\text{with}\quad I_1(\tau)=\int_0^\tau\left(\dot A_0(\tau_0)+\frac{\dot A_1(\tau_0)}{2}\right)\frac{d\tau_0}{\sqrt{B^2-1}}\,,\quad I_2(\tau)=\int_0^\tau\left(\dot A_0(\tau_0)+\frac{\dot A_1(\tau_0)}{2}\right)\frac{\tau-\tau_0}{\sqrt{B^2-1}}d\tau_0\,,\quad B=1+(\tau-\tau_0)$$

Notice also that the addition of the first and second of the equations in (24) yields that

$$\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{3c}{4}\right)\frac{d\alpha}{dt}\right)=I_1(\tau)+\frac{I_2(\tau)}{2}\Rightarrow I_1(\tau)+\frac{I_2(\tau)}{2}=\frac{-w'(x=3c/4,z=0^\pm,t)}{U_\infty}$$

The solution of this integral equation provides the value of

$$\Gamma_e(t)=\Gamma_a(x=c,t)=\frac{\pi U_\infty c}{2}\left(A_0+\frac{A_1}{2}\right)$$

With the purpose of finding the expressions for $\ell(t)$ and $m(t)$ notice first that Leibniz's rule for the derivative of time-dependent integrals yields:

$$
\begin{aligned}
&\frac{d}{d\tau}\left(\int_0^{\tau}(\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0\right)=\int_0^{\tau}\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0=A_0+\frac{A_1}{2}\,,\\
&\frac{d}{d\tau}\left(\int_0^{\tau}(\tau-\tau_0)^2\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0\right)=2\int_0^{\tau}(\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0\,,\\
&\frac{d}{d\tau}\left(\int_0^{\tau}\left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0\right)=I_1(\tau)+I_2(\tau)\,,\\
&\frac{d}{d\tau}\left(\int_0^{\tau}(\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0\right)=2\int_0^{\tau}\left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0-I_2(\tau).
\end{aligned}
\tag{27}
$$

Notice also that the addition of the first and second of the equations in (24) yields that

$$
\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{3c}{4}\right)\frac{d\alpha}{dt}\right)=I_1(\tau)+\frac{I_2(\tau)}{2}\Rightarrow I_1(\tau)+\frac{I_2(\tau)}{2}=\frac{-w'(x=3c/4,z=0^{\pm},t)}{U_\infty}\,,
\tag{28}
$$

where we have made use of the expression for w' given in equation (3). It will be discussed below that the solution of the integral equation (28) provides with the circulation around the airfoil $\Gamma_e(t)$. Moreover, the addition of the first and third equations in equation (24) yields,

$$
\begin{aligned}
&\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{c}{2}\right)\frac{d\alpha}{dt}\right)=\frac{3A_0}{2}+\frac{A_1}{2}+\frac{A_2}{4}+\int_0^{\tau}(\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)d\tau_0-\\
&-\int_0^{\tau}\left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0\Rightarrow\\
&\frac{c}{2U_\infty}\frac{d}{dt}\left(\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{c}{2}\right)\frac{d\alpha}{dt}\right)\right)=\frac{3\dot{A}_0}{2}+\frac{\dot{A}_1}{2}+\frac{\dot{A}_2}{4}+A_0+\frac{A_1}{2}-(I_1+I_2)
\end{aligned}
\tag{29}
$$

where we have made use of the results in equation (27).

The substitution of the results in equations (28)-(29) into the expression for $\ell(t)$ given in equation (9) provides with the following equation for the unsteady lift:

$$
\ell(t)=\frac{\rho c^2\pi}{4}\frac{d}{dt}\left(U_\infty\alpha(t)+\frac{dh}{dt}+\left(\frac{c}{2}-x_e\right)\frac{d\alpha}{dt}\right)+\frac{\rho U_\infty^2 c\,\pi}{2}\left(I_1(\tau)+I_2(\tau)\right)=\ell_a(t)+\ell_c(t)\,.
\tag{30}
$$

The result for the unsteady lift in equation (30) reveals that $\ell(t)$ results from the addition of two different terms: the first term at the right hand side of equation (30), $\ell_a(t)$, represents the contribution to the lift associated with the acceleration of the airfoil in the vertical direction; it will be termed, in what follows, as added mass term. The second term at the right hand side of equation (30), ℓ_c, is the so-called circulatory lift and represents the contribution of the wake to the lift force. For those cases in which the airfoil does not accelerate, the only lift force experienced by the airfoil is associated with the effect of the wake vortices, quantified through the integrals $I_1(\tau)$ and $I_2(\tau)$.

With the purpose of finding the expression for the time-dependent torque $m(t)$, notice first that the subtraction of the equations corresponding to $m = 1$ and $m = 3$ in (24) yields,

$$A_0(\tau) + \frac{3A_1(\tau)}{4} - \frac{A_3(\tau)}{4} = \frac{c}{4\,U_\infty}\frac{d\alpha}{dt} + 2\int_0^\tau \left(\dot{A}_0 + \dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0 + 2\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0 + \dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0 -$$
$$-2\int_0^\tau (\tau-\tau_0)^2\left(\dot{A}_0 + \dot{A}_1/2\right)d\tau_0 - 4\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0 + \dot{A}_1/2\right)d\tau_0 \Rightarrow \dot{A}_0(\tau) + \frac{3\dot{A}_1(\tau)}{4} - \frac{\dot{A}_3(\tau)}{4} = \frac{c^2}{8\,U_\infty^2}\frac{d^2\alpha}{d\,t^2} +$$
$$+ 2\,I_1 + 4\int_0^\tau \left(\dot{A}_0 + \dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0 - 4\int_0^\tau (\tau-\tau_0)\left(\dot{A}_0 + \dot{A}_1/2\right)d\tau_0 - 4\left(A_0 + \frac{A_1}{2}\right), \tag{31}$$

where we have made use of the results in equation (27). In addition, equation (29) expresses that:

$$\frac{3A_0}{2} + \frac{A_1}{2} + \frac{A_2}{4} = \frac{1}{U_\infty}\left(U_\infty\alpha(t) + \frac{dh}{dt} - \left(x_e - \frac{c}{2}\right)\frac{d\alpha}{dt}\right) - \int_0^\tau (\tau-\tau_0)\left(\dot{A}_0 + \dot{A}_1/2\right)d\tau_0 + \int_0^\tau \left(\dot{A}_0 + \dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0\,. \tag{32}$$

The substitution of equations (31)-(32) into the equation for the torque calculated at $x=0$, $m(t)$, given in equation (9), and using the expression for $\ell(t)$ also given in equation (9) yields:

$$
\begin{aligned}
m(t) &= \frac{\rho U_\infty^2 c^2 \pi}{4}\left(\frac{3\,\dot{A}_0}{2}+\frac{\dot{A}_1}{2}+\frac{\dot{A}_2}{4}\right)+ \\
&+\frac{\rho U_\infty^2 c^2\pi}{16}\left(\frac{c^2}{8\,U_\infty^2}\frac{d^2\alpha}{d\,t^2}+2\,I_1+4\int_0^\tau\left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0-4\int_0^\tau(\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)\,d\tau_0-4\left(A_0+\frac{A_1}{2}\right)\right)- \\
&-\frac{\rho U_\infty^2 c^2\pi}{4}\left(\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{c}{2}\right)\frac{d\alpha}{dt}\right)-\int_0^\tau(\tau-\tau_0)\left(\dot{A}_0+\dot{A}_1/2\right)\,d\tau_0+\int_0^\tau\left(\dot{A}_0+\dot{A}_1/2\right)\sqrt{B^2-1}\,d\tau_0\right)+ \\
&+\frac{\rho\,U_\infty^2\,c^2}{2}\left(A_0+\frac{A_1}{2}\right)=\frac{\rho U_\infty^2 c^2\pi}{4}\left(\frac{3\,\dot{A}_0}{2}+\frac{\dot{A}_1}{2}+\frac{\dot{A}_2}{4}+A_0+\frac{A_1}{2}\right)+\frac{\rho c^4\pi}{128}\frac{d^2\alpha}{d\,t^2}+ \\
&+\frac{\rho U_\infty^2 c^2\pi}{4}\left(\frac{I_1}{2}-\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{c}{2}\right)\frac{d\alpha}{dt}\right)\right)=\frac{c\,\ell(t)}{2}+\frac{\rho c^4\pi}{128}\frac{d^2\alpha}{d\,t^2}+\frac{\rho U_\infty c^3\pi}{16}\frac{d\alpha}{d\,t}-\frac{\rho U_\infty^2 c^2\pi}{8}\left(I_1+I_2\right),
\end{aligned} \tag{33}
$$

where the last term in equation (33) has been deduced adding the equations corresponding to $m=0$ and $m=1$ in equation (24). Hence, making use of the result in equation (30) for $\ell(t)$, the torque $m(t)$ can be calculated as,

$$
\begin{aligned}
m(t) &= \frac{c}{2}\times\frac{\rho c^2\,\pi}{4}\frac{d}{dt}\left(U_\infty\alpha(t)+\frac{dh}{dt}+\left(\frac{c}{2}-x_e\right)\frac{d\alpha}{dt}\right)+\frac{\rho c^4\pi}{128}\frac{d^2\alpha}{d\,t^2}+\frac{\rho U_\infty c^3\pi}{16}\frac{d\alpha}{d\,t}+\frac{\rho U_\infty^2 c^2\pi}{8}\left(I_1(\tau)+I_2(\tau)\right) \\
&= \frac{c}{2}\ell_a(t)+\frac{c}{4}\ell_c(t)+\frac{\rho c^4\pi}{128}\frac{d^2\alpha}{d\,t^2}+\frac{\rho U_\infty c^3\pi}{16}\frac{d\alpha}{d\,t}\,.
\end{aligned} \tag{34}
$$

II. THE CIRCULATORY LIFT $\ell_c(t)$

The values of $\ell(t)$ and $m(t)$ in equations (30) and (34) depend on the value of the circulation around the airfoil $\Gamma_e(\tau)$, given by -see equations (8) and (11):

$$
\Gamma_e(\tau)=\frac{\pi\,U_\infty c}{2}\left(A_0(\tau)+\frac{A_1}{2}(\tau)\right)\,. \tag{35}
$$

The circulation around the airfoil, $\Gamma_e(\tau)$, is calculated solving the integral equation (28), reproduced here for clarity purposes:

$$
\begin{aligned}
&\frac{1}{U_\infty}\left(U_\infty\alpha(t)+\frac{dh}{dt}-\left(x_e-\frac{3c}{4}\right)\frac{d\alpha}{dt}\right)=I_1(\tau)+\frac{I_2(\tau)}{2}\Rightarrow \\
&I_1(\tau)+\frac{I_2(\tau)}{2}=\frac{-w'(x=3c/4,z=0^\pm,t)}{U_\infty}=\frac{-w'_{3/4}}{U_\infty}\quad\text{with}\quad w'(x=3c/4,z=0^\pm,t)=w'_{3/4}\,.
\end{aligned} \tag{36}
$$

The integrals $I_1(\tau)$ and $I_2(\tau)$ in equation (36), which depend on $\dot{\Gamma}_e(\tau)$, are defined in equation (24). We will firstly solve equation (36) for the case of a general time-dependent function $-w'_{3/4}$ making use of the fact that *any* function $F(t)$ can be expressed as:

$$F(t) = F(0)H(t) + \int_0^t \frac{dF}{dt'} H(t-t')\,dt' \,, \tag{37}$$

with the Heaviside function $H(\tau)$ defined as

$$H(\tau) = 1 \quad \text{if} \quad \tau \geq 0 \quad \text{and} \quad H(\tau) = 0 \quad \text{if} \quad \tau < 0 \,, \tag{38}$$

and hence, the right hand side of equation (36) can be expressed as:

$$\frac{-w'_{3/4}}{U_\infty}(\tau) = \frac{-w'_{3/4}}{U_\infty}(0)H(\tau) - \int_0^\tau \frac{d\,w'_{3/4}/U_\infty}{d\tau_0} H(\tau-\tau_0)\,d\tau_0 \,. \tag{39}$$

$$\frac{1}{U_\infty}\left(U_\infty \alpha(t) + \frac{dh}{dt} - \left(x_e - \frac{3c}{4}\right)\frac{d\alpha}{dt}\right) = I_1(\tau) + \frac{I_2(\tau)}{2} \Rightarrow$$
$$I_1(\tau) + \frac{I_2(\tau)}{2} = \frac{-w'(x=3c/4, z=0^{\pm}, t)}{U_\infty} = \frac{-w'_{3/4}}{U_\infty} \quad \text{with} \quad w'(x=3c/4, z=0^{\pm}, t) = w'_{3/4} \,. \tag{36}$$

$$\text{with} \quad I_1(\tau) = \int_0^\tau \left(\dot{A}_0(\tau_0) + \frac{\dot{A}_1(\tau_0)}{2}\right)\frac{d\tau_0}{\sqrt{B^2-1}} \,, \quad I_2(\tau) = \int_0^\tau \left(\dot{A}_0(\tau_0) + \frac{\dot{A}_1(\tau_0)}{2}\right)\frac{\tau-\tau_0}{\sqrt{B^2-1}}d\tau_0 \,, \quad B = 1 + (\tau-\tau_0) \tag{24}$$

Therefore, since the integral equation (36) is linear in the unknown $\dot{A}_0(\tau) + \dot{A}_1(\tau)/2$, the addition of solutions is also a solution of equation (36). Consequently, due to the fact that equation (39) expresses that *any* function $-w'_{3/4}(\tau)$ can be expressed as a linear combination of Heaviside functions $H(\tau)$, the general solution of equation (36) can be expressed as the linear combination of the function $g_e(\tau)$ which results from the solution of equation (36) particularized for the case in which the forcing term is a Heaviside function:

$$I_{1W} + \frac{I_{2W}}{2} = \int_0^\tau \dot{g}_e(\tau_0) \frac{1 + (\tau - \tau_0)/2}{\sqrt{(\tau - \tau_0)^2 + 2(\tau - \tau_0)}} \, d\tau_0 = H(\tau) = 1\,, \tag{40}$$

where we have made use of the definition of the integrals I_1 and I_2 in equation (24).

Once $g_e(\tau)$ with $g_e(\tau < 0) = 0$ is known from the solution of equation (40), we first notice that the solution of equation (40) when the right hand side is $H(\tau - \tau_0)$ is nothing but $g_e(\tau - \tau_0)$. This said, we can straightforwardly calculate the general expression of the circulatory lift. Indeed, let us first define the so-called Wagner function as:

$$\phi(\tau) = \frac{I_{1W}(\tau) + I_{2W}(\tau)}{2} = \frac{1}{2} \int_0^\tau \dot{g}_e(\tau_0) \frac{1 + (\tau - \tau_0)}{\sqrt{(\tau - \tau_0)^2 + 2(\tau - \tau_0)}} \, d\tau_0\,, \tag{41}$$

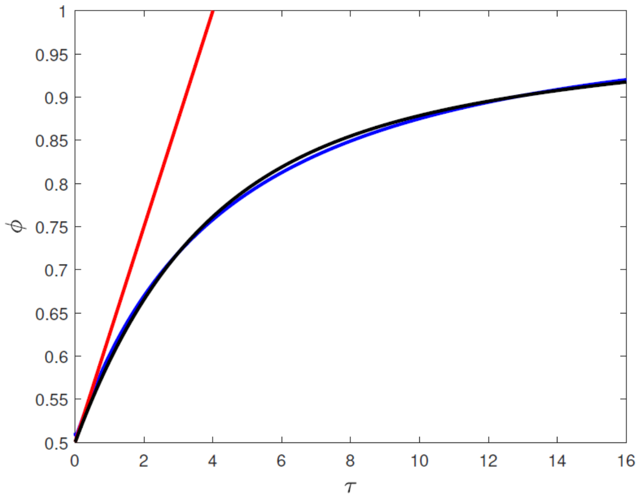
which is now a known function because $g_e(\tau)$ is determined solving the integral equation (40). Next, notice that, in view of equation (39), the general solution of equation (36) can be expressed in terms of the following linear combination of the known function $g_e(\tau)$ as:

$$\begin{aligned}\dot{A}_0(\tau_1) + \frac{\dot{A}_1(\tau_1)}{2} &= \frac{-w'_{3/4}}{U_\infty}(0)\, \dot{g}_e(\tau_1) - \int_0^{\tau_1} \frac{d\, w'_{3/4}/U_\infty}{d\tau_0} \, \dot{g}_e(\tau_1 - \tau_0)\, d\tau_0 = \\ &= \frac{-w'_{3/4}}{U_\infty}(0)\, \dot{g}_e(\tau_1) - \int_0^{\tau} \frac{d\, w'_{3/4}/U_\infty}{d\tau_0} \, \dot{g}_e(\tau_1 - \tau_0)\, d\tau_0\,,\end{aligned} \tag{42}$$

where we have taken into account that, for $\tau > \tau_1$, $\dot{g}_e(\tau_1 - \tau_0) = 0$ when $\tau_0 > \tau_1$. Therefore, since the circulatory lift in equation (30) is given by

$$\ell_c(\tau) = \frac{\rho U_\infty^2 c\pi}{2} \int_0^\tau \left(\dot{A}_0(\tau_1) + \frac{\dot{A}_1(\tau_1)}{2} \right) \frac{1 + (\tau - \tau_1)}{\sqrt{(\tau - \tau_1)^2 + 2(\tau - \tau_1)}} \, d\tau_1\,, \tag{43}$$

CUANDO LA VELOCIDAD VERTICAL EN 3/4C ES UNA FUNCIÓN DE HEAVISIDE LA FUERZA DE SUSTENTACIÓN ADIMENSIONAL ES LA FUNCIÓN DE WAGNER:



$$-w'_{3/4}/U_\infty = \alpha \bar{H}(\tau) \Rightarrow \ell_{cW}(\tau) = \rho U_\infty^2 c\pi\alpha\phi(\tau) \Rightarrow \phi(\tau) = \frac{\ell_{cW}(\tau)}{1/2\rho U_\infty^2 c 2\pi\alpha} = \frac{\ell_{cW}(\tau)}{\ell_{cW}(\tau \to \infty)}.$$

The figure shows the numerical solution of the Wagner function obtained using the numerical method explained previously. The numerical solution can be very well approximated using the expression due to Jones:

$$\phi(\tau) = 1 - 0.165\, e^{-0.0455\,\tau} - 0.335\, e^{-0.3\tau}$$

The figure also shows a comparison with the two-term analytical solution, valid for $\tau \ll 1$, described in the Supplementary Material

GRACIAS A LA LINEALIDAD DEL PROBLEMA, LA RESPUESTA DADA A LA CIRCULACIÓN QUE RESULTA DE UNA SUMA DE FUNCIONES DE HEAVISIDE (VÉASE LA ECUACIÓN 42), ES LA SUMA DE LAS RESPUESTAS A CADA UNA DE LAS FUNCIONES DE HEAVISIDE Y, POR TANTO, COMO LA RESPUESTA DE LA SUSTENTACIÓN A LA FUNCIÓN DE HEAVISIDE ES LA FUNCIÓN DE WAGNER, SE TIENE QUE

$$\ell_c(\tau) = \rho U_\infty^2 c\pi \left(\frac{-w'_{3/4}(0)}{U_\infty} \phi(\tau) \right) - \rho U_\infty^2 c\pi \int_0^\tau \frac{d\, w'_{3/4}(\tau_0)/U_\infty}{d\tau_0} \phi(\tau - \tau_0)\, d\tau_0 \,. \qquad (45)$$

EXPRESIONES DE LA FUERZA DE SUSTENTACIÓN Y MOMENTO EN FUNCIÓN DE LOS GRADOS DE LIBERTAD

$$\frac{1}{U_\infty}\left(U_\infty \alpha(t) + \frac{dh}{dt} - \left(x_e - \frac{3c}{4}\right)\frac{d\alpha}{dt}\right) = \frac{-w'_{3/4}}{U_\infty}$$

$$\ell_c(\tau) = \rho U_\infty^2 c\pi \left(\frac{-w'_{3/4}(0)}{U_\infty}\phi(\tau)\right) - \rho U_\infty^2 c\pi \int_0^\tau \frac{d\, w'_{3/4}(\tau_0)/U_\infty}{d\tau_0}\phi(\tau - \tau_0)\, d\tau_0 \,. \qquad (45)$$

Función de Wagner

$$\phi(\tau) = 1 - 0.165\, e^{-0.0455\,\tau} - 0.335\, e^{-0.3\tau} \,,$$

$$\ell(t) = \frac{\rho c^2 \pi}{4}\frac{d}{dt}\left(U_\infty \alpha(t) + \frac{dh}{dt} + \left(\frac{c}{2} - x_e\right)\frac{d\alpha}{dt}\right) + \frac{\rho U_\infty^2 c\,\pi}{2}\left(I_1(\tau) + I_2(\tau)\right) = \ell_a(t) + \ell_c(t)$$

$$\begin{aligned} m(t) &= \frac{c}{2} \times \frac{\rho c^2 \pi}{4}\frac{d}{dt}\left(U_\infty \alpha(t) + \frac{dh}{dt} + \left(\frac{c}{2} - x_e\right)\frac{d\alpha}{dt}\right) + \frac{\rho c^4 \pi}{128}\frac{d^2\alpha}{d t^2} + \frac{\rho U_\infty c^3 \pi}{16}\frac{d\alpha}{d t} + \frac{\rho U_\infty^2 c^2 \pi}{8}\left(I_1(\tau) + I_2(\tau)\right) \\ &= \frac{c}{2}\ell_a(t) + \frac{c}{4}\ell_c(t) + \frac{\rho c^4 \pi}{128}\frac{d^2\alpha}{d t^2} + \frac{\rho U_\infty c^3 \pi}{16}\frac{d\alpha}{d t} \,. \end{aligned}$$

YA PODEMOS RESOLVER EL SISTEMA DE EDOs PARA h(t) y α(t): ECUACIONES 7 Y 12 O BIEN, ECUACIONES 7 Y 14

$$m\left(\ddot{h}+\ddot{\alpha}\left(x_G-x_e\right)\right)+k_h h(t)=-\ell(t) \tag{7}$$

$$\omega(t)=\dot{\alpha}(t)\mathbf{e_2};\quad \mathbf{I_0}=\int_{\Omega_s}\rho_s\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & y^2+z^2\end{pmatrix} dV \Rightarrow \mathbf{I_0}\cdot\omega(t)=2L\left(I_G+m\left(x_0-x_g\right)^2\right)\dot{\alpha}\,\mathbf{e_2} \tag{8}$$

$$I=I_G+m(x_0-x_G)^2; \tag{9}$$

Por ejemplo, si $dm/dx=2\rho_s(x)z_e(x)\neq f(x)\Rightarrow$

$$I_G=\int_{-c/2}^{c/2}x^2\,dm=c\frac{dm}{dx}\frac{c^2}{12}=m\frac{c^2}{12} \tag{10}$$

$$\mathbf{e_2}\cdot\left(\frac{d}{dt}\left[m\left(\mathbf{r_G}-\mathbf{r_0}\right)\times\mathbf{v_0}+I\dot{\alpha}\mathbf{e_2}\right]=-k_\alpha\alpha(t)\mathbf{e_2}+(x_e-x_0)\mathbf{e_1}\times\mathbf{e_3}k_h h(t)+\mathbf{m_{f-s}}\right) \tag{11}$$

Caso 1, $\mathbf{r_0}=\mathbf{r_G}$:

$$I_G\ddot{\alpha}+k_\alpha\,\alpha(t)+(x_e-x_G)\,k_h h(t)=\int_0^c(x_G-x)\left(p^{int}-p^{ext}\right)dx=x_G\ell(t)-m_{fs}(x=0) \tag{12}$$

siendo

$$m_{fs}(x=0)=\int_0^c x\left(p^{int}-p^{ext}\right)dx \tag{13}$$

Caso 2, $\mathbf{r_0}=\mathbf{r_e}$:

$$\left(I_G+m\left(x_G-x_e\right)^2\right)\ddot{\alpha}+m(x_G-x_e)\ddot{h}(t)+k_\alpha\,\alpha(t)=\int_0^c(x_e-x)\left(p^{int}-p^{ext}\right)dx=x_e\ell(t)-m_{fs}(x=0) \tag{14}$$

Nótese que la ecuación (14) es una combinación lineal de las ecuaciones (7) y (12): en efecto, (14) resulta de sumar a (12) la ecuación (7) multiplicada por (x_G-x_e).

ANTES DE RESOLVER EL SISTEMA DE EDOs, HALLAMOS LAS EXPRESIONES DE l(t) y m(t) CORRESPONDIENTES A UN MOVIMIENTO PERIÓDICO: FUNCIÓN DE THEODORSEN

V. THEODORSEN FUNCTION

In order to determine whether or not the oscillations experienced by an airfoil are damped by the outer stream or, on the contrary, grow in time, giving rise to the dangerous phenomenon known as flutter, we seek for the real part of the time-dependent, complex functions $h(\tau)$, $\alpha(\tau)$ and the time derivative of the circulation around the airfoil,

$$h(\tau) = H\, e^{i\omega\tau}\,, \quad \alpha(\tau) = \bar{\alpha}\, e^{i\omega\tau}\,, \quad \dot{A}_0 + \dot{A}_1/2 = G\, e^{i\omega\tau} \tag{76}$$

with ω the dimensionless frequency, which is related with the dimensional frequency ω^* through the equation

$$\omega\tau = \omega^* t \Rightarrow \omega\, t \frac{2\,U_\infty}{c} = \omega^*\, t \Rightarrow \omega = \frac{\omega^* c}{2\,U_\infty}\,. \tag{77}$$

In this case, the substitution of $\dot{\Gamma}_e$ in Eq. (76) into the integral equation (28) provides with the following equation for G:

$$\begin{aligned} I_1 + \frac{I_2}{2} &= G \int_0^\tau \frac{e^{i\omega\tau_0}\,(1 + 1/2\,(\tau-\tau_0))}{\sqrt{(\tau-\tau_0)^2 + 2\,(\tau-\tau_0)}}\, d\tau_0 = \frac{-\bar{w'}_{3/4}}{U_\infty}\, e^{i\omega\tau} \Rightarrow \\ G\, e^{i\omega\tau} \int_0^\tau & \frac{(1 + 1/2\,(\tau-\tau_0))}{\sqrt{(\tau-\tau_0)^2 + 2\,(\tau-\tau_0)}}\, e^{-i\omega(\tau-\tau_0)}\, d\tau_0 = \frac{-\bar{w'}_{3/4}}{U_\infty}\, e^{i\omega\tau} \Rightarrow \\ G &= \frac{-\bar{w'}_{3/4}}{U_\infty} \left(\int_0^\tau \frac{(1 + 1/2\,(\tau-\tau_0))}{\sqrt{(\tau-\tau_0)^2 + 2\,(\tau-\tau_0)}}\, e^{-i\omega(\tau-\tau_0)}\, d\tau_0 \right)^{-1}, \end{aligned} \tag{78}$$

where

$$\frac{1}{U_\infty}\left(U_\infty \alpha(t) + \frac{dh}{dt} - \left(x_e - \frac{3c}{4} \right) \frac{d\alpha}{dt} \right) = \frac{-w'_{3/4}}{U_\infty}(\tau) \quad \text{and} \quad \frac{-w'_{3/4}}{U_\infty}(\tau) = e^{i\omega\tau} \frac{-\bar{w'}_{3/4}}{U_\infty}\,. \tag{79}$$

We are looking for solutions corresponding to dimensionless values of τ large enough so that the solution is not affected by initial conditions and, hence, since τ is such that $1+\tau \to \infty$, we introduce the change of variables $\xi = \tau - \tau_0 + 1$ into Eq. (78) for G, which reads:

$$G = \frac{-\bar{w}'_{3/4}}{U_\infty} \left(\frac{1}{2} \int_1^\infty \frac{(1+\xi)}{\sqrt{\xi^2 - 1}} \, e^{-i\omega(\xi-1)} \, d\xi \right)^{-1} . \tag{80}$$

Now that G is known through Eq. (80), the circulatory component of the lift for $\tau + 1 \to \infty$ can be calculated in terms of $I_1 + I_2$ using Eq. (30):

$$\begin{aligned}
(I_1 + I_2) &= e^{i\omega\tau} \, G \int_0^\tau \frac{1+\tau-\tau_0}{\sqrt{(\tau-\tau_0)^2 + 2\,(\tau-\tau_0)}} \, e^{-i\omega(\tau-\tau_0)} \, d\tau_0 = e^{i\omega\tau} \, G \int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}} \, e^{-i\omega(\xi-1)} \, d\xi = \\
&= e^{i\omega\tau} \frac{-\bar{w}'_{3/4}}{U_\infty} \left(\int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}} \, e^{-i\omega\,\xi} \, d\xi \right) \left(\frac{1}{2} \int_1^\infty \frac{(1+\xi)}{\sqrt{\xi^2-1}} \, e^{-i\omega\xi} \, d\xi \right)^{-1} = \\
&= \frac{-w'_{3/4}}{U_\infty} \left(\int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}} \, e^{-i\omega\,\xi} \, d\xi \right) \left(\frac{1}{2} \int_1^\infty \frac{(1+\xi)}{\sqrt{\xi^2-1}} \, e^{-i\omega\xi} \, d\xi \right)^{-1} ,
\end{aligned} \tag{81}$$

where we have made use of the equation for G in Eq. (80) and of Eq. (79). Then, introducing Eq. (81) into Eq. (30), for the case of oscillatory motion, the lift force and the torque calculated at $x = 0$ are calculated as the real parts of -see Eqs. (30) and (34):

$$\begin{aligned}
\ell(t) &= \frac{\rho c^2 \, \pi}{4} \frac{d}{dt} \left(U_\infty \alpha(t) + \frac{dh}{dt} + \left(\frac{c}{2} - x_e \right) \frac{d\alpha}{dt} \right) + \rho U_\infty c \, \pi \left(-w'_{3/4} \right) C(\omega) = \ell_a(t) + \ell_c(t) \\
m(t) &= \frac{c}{2} \ell_a(t) + \frac{c}{4} \ell_c(t) + \frac{\rho c^4 \pi}{128} \frac{d^2\alpha}{dt^2} + \frac{\rho U_\infty c^3 \pi}{16} \frac{d\alpha}{dt},
\end{aligned} \tag{82}$$

with

$$\ell_c(t) = \rho U_\infty c \, \pi \left(-w'_{3/4} \right) C(\omega) \quad \text{and} \quad C(\omega) = \left(\int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}} \, e^{-i\omega\,\xi} \, d\xi \right) \left(\int_1^\infty \frac{(1+\xi)}{\sqrt{\xi^2-1}} \, e^{-i\omega\xi} \, d\xi \right)^{-1} , \tag{83}$$

the so-called Theodorsen function, which can be expressed in terms of modified Bessel functions of the second kind. Indeed, using the integral form of the K_n-Bessel function of order n, Theodorsen's function, defined in Eq. (83), can be expressed as:

$$C(\omega) = \frac{K_1(i\omega)}{K_0(i\omega) + K_1(i\omega)} . \tag{84}$$

Theodorsen's function, which arises as the contribution of the wake to the force and torque, can also be expressed in terms of Wagner's function $\phi(\tau)$ using the Duhamel's integral in Eq. (45). Indeed, the numerical solution of Wagner's problem, depicted in figure 1, reveals that $\phi(\tau)$ can be well approximated by:

$$\phi(\tau) = 1 - 0.165\, e^{-0.0455\,\tau} - 0.335\, e^{-0.3\tau}\,, \tag{85}$$

and, hence, by virtue of Eq. (45), the oscillatory part of $\ell_c(t)$, which can be calculated taking $-w'_{3/4}/U_\infty = Ce^{i\omega\tau}$ in the limit $\tau \to \infty$, is given by

$$\begin{aligned}
\ell_c(\tau) &= -\rho\, U_\infty c\pi\, w'_{3/4}(0)\phi(\tau) - \rho\, U_\infty^2 c\pi \int_0^\tau \frac{d\, w'_{3/4}(\tau_0)/U_\infty}{d\tau_0} \phi(\tau-\tau_0)\, d\tau_0 = \\
&= -\rho\, U_\infty c\pi\, w'_{3/4}(0)\phi(\tau) - \rho\, U_\infty c\pi\, w'_{3/4} \int_0^\tau i\omega\, e^{-i\omega(\tau-\tau_0)}\, \phi(\tau-\tau_0)\, d\tau_0 \simeq \\
&\simeq -\rho\, U_\infty c\pi\, w'_{3/4}(0)\phi(\tau) - \rho\, U_\infty c\pi\, w'_{3/4} \int_0^\tau i\omega\, e^{-i\omega(\tau-\tau_0)} \left(1 - 0.165\, e^{-0.0455\,(\tau-\tau_0)} - 0.335\, e^{-0.3(\tau-\tau_0)}\right) d\tau_0 \\
&= -\rho\, U_\infty c\pi\, w'_{3/4}(0)\phi(\tau) - \rho\, U_\infty c\pi\, w'_{3/4}(0) e^{i\omega\tau} \int_0^\tau i\omega\, e^{-i\omega\tau'} \left(1 - 0.165\, e^{-0.0455\tau'} - 0.335\, e^{-0.3\tau'}\right) d\tau'\,,
\end{aligned} \tag{86}$$

where we have made use of Eq. (85). Then, since $\phi(\tau \to \infty) \to 1$, see Eq. (85),

$$\begin{aligned} \ell_c(\tau \to \infty) &= -\rho\, U_\infty c\pi\, w'_{3/4}(0) - \rho\, U_\infty c\pi\, w'_{3/4}(0) \left(-1 + e^{i\omega\tau} - \frac{0.165\, i\omega\, e^{i\omega\tau}}{i\omega + 0.0455} - \frac{0.335\, i\omega\, e^{i\omega\tau}}{i\omega + 0.3} \right) = \\ &= -\rho\, U_\infty c\pi\, w'_{3/4} \left(1 - \frac{0.165\, i\omega}{i\omega + 0.0455} - \frac{0.3355\, i\omega}{i\omega + 0.3} \right) . \end{aligned} \tag{87}$$

Therefore, the comparison between Eq. (82) and Eq. (87) indicates that Theodorsen's function, given in Eq. (84) can be approximated by

$$C(\omega) = \frac{K_1(i\omega)}{K_0(i\omega) + K_1(i\omega)} = F(\omega) + iG(\omega) \simeq 1 - \frac{0.165\, i\omega}{i\omega + 0.0455} - \frac{0.335\, i\omega}{i\omega + 0.3} \,, \tag{88}$$

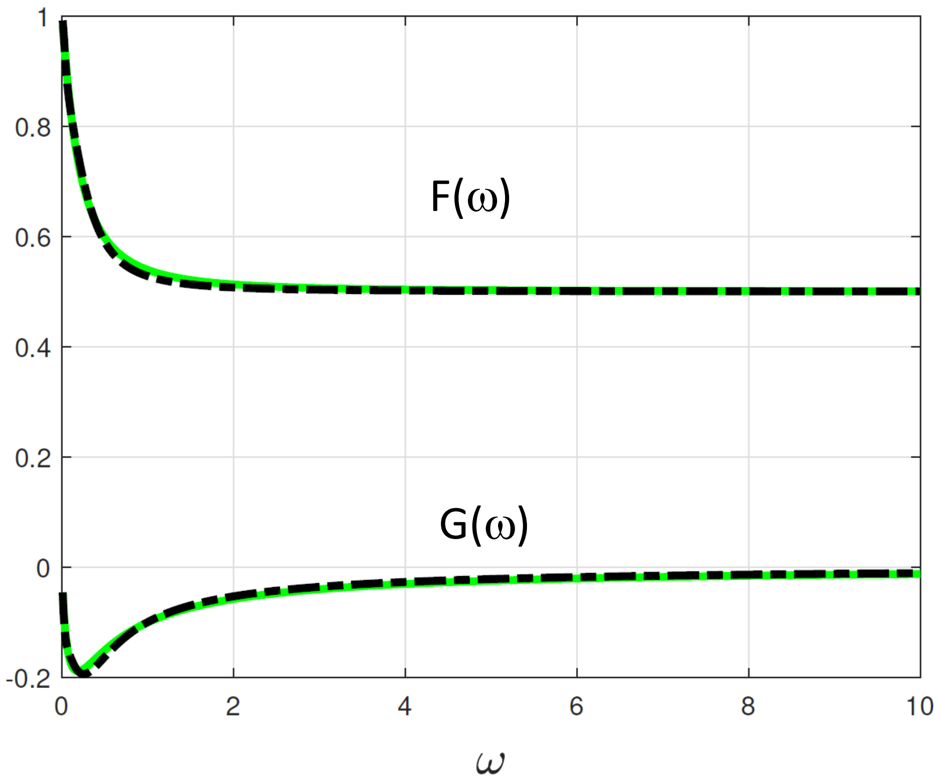
which is an excellent approximation to the exact result, as it is shown in figure 2, where the real and imaginary parts of Theodorsen's function, $C(\omega)$ in equation (88), are compared with the approximate expression, also given in this equation.

Summary of equations for the lift and torque for the case of purely periodic oscillations: THEODORSEN FUNCTION AND APPROXIMATION DEDUCED USING WAGNER'S FUNCTION

$$\begin{aligned} \ell(t) &= \frac{\rho c^2\,\pi}{4}\frac{d}{dt}\left(U_\infty \alpha(t) + \frac{dh}{dt} + \left(\frac{c}{2} - x_e\right)\frac{d\alpha}{dt}\right) + \rho U_\infty c\pi\left(-w'_{3/4}\right) C(\omega) = \ell_a(t) + \ell_c(t) \\ m(t) &= \frac{c}{2}\ell_a(t) + \frac{c}{4}\ell_c(t) + \frac{\rho c^4\pi}{128}\frac{d^2\alpha}{dt^2} + \frac{\rho U_\infty c^3\pi}{16}\frac{d\alpha}{dt}, \end{aligned} \quad (82)$$

$$C(\omega) = \frac{K_1(i\omega)}{K_0(i\omega) + K_1(i\omega)} = F(\omega) + iG(\omega) \simeq 1 - \frac{0.165\, i\omega}{i\omega + 0.0455} - \frac{0.335\, i\omega}{i\omega + 0.3}, \quad (88)$$

The approximate equation (88) for Theodorsen's function C(ω), deduced using Wagner's function ϕ(τ), will be used next in order to determine the flutter threshold speed instead of the exact Theodorsen function, and the reason is that the approximate expression is very similar to the exact value. Indeed:



```
% JM Gordillo: Theodorsen functions F and G, approximate Theodorsen
% functions Fw and Gw, new functions Fg and Gg, our thrust coefficient CT
% calculated by means of the vortex impulse theory using equation (4.20),
% FF's thrust coefficient CTFF calculated using equation (4.22) and
% thrust coefficient using Garrick's theory, CTG, using equation (4.21)
clear all;
close all;
clc;


i=sqrt(-1);
Nsteps=600;
Deltak=0.01;
a0=6*pi/180;
h0c=6*pi/180;
phi=pi/2;
a=-1/2;
%h0c=0;
%a0=0;


for s=1:Nsteps
    K(s)=s*Deltak;
    F(s)=real(besselk(1,i*K(s))/(besselk(0,i*K(s))+besselk(1,i*K(s))));
    G(s)=imag(besselk(1,i*K(s))/(besselk(0,i*K(s))+besselk(1,i*K(s))));
    Gg(s)=real((exp(-i*K(s))/(i*K(s)))/(besselk(0,i*K(s))+besselk(1,i*K(s))));
    Fg(s)=-imag((exp(-i*K(s))/(i*K(s)))/(besselk(0,i*K(s))+besselk(1,i*K(s))));
    Fw(s)=real(1-0.165*i*K(s)/(i*K(s)+0.0455)-0.335*i*K(s)/(0.3+i*K(s)));
    Gw(s)=imag(1-0.165*i*K(s)/(i*K(s)+0.0455)-0.335*i*K(s)/(0.3+i*K(s)));
end
```

AHORA SÍ PODEMOS RESOLVER EL SISTEMA DE EDOS LINEALES BUSCANDO SOLUCIONES EXPONENCIALES PARA EL PROBLEMA HOMOGÉNEO, QUE DA LUGAR A UN PROBLEMA DE AUTOVALORES. A CONTINUACIÓN, SE DEDUCE EL PROBLEMA DE AUTOVALORES A RESOLVER Y, LO QUE ES MÁS IMPORTANTE, SE PROPORCIONA LA HERRAMIENTA QUE RESUELVE EL SISTEMA, LO QUE PERMITE DETERMINAR SI LAS PERTURBACIONES SE AMPLIFICAN EN EL TIEMPO (FLUTTER) O NO. LA HERRAMIENTA QUE SE PROPORCIONA SÍMPLEMENTE CONSISTE EN HALLAR LAS RAÍCES DE UN POLINOMIO, Y ESTE CÁLCULO LO PUEDE REALIZAR MATLAB DE MANERA MUY SENCILLA.

LOS RESULTADOS QUE SIGUEN CORRESPONDEN A DOS GRADOS DE LIBERTAD, $h(t)$ y $\alpha(t)$, PERO ÉSTOS SON FÁCILMENTE GENERALIZABLE PARA UN CONJUNTO DE N PLACAS CON DOS GRADOS DE LIBERTAD CADA UNA ACOPLADAS ENTRE SÍ POR UN MUELLE DE TORSIÓN Y OTRO LINEAL, LO QUE DARÍA UN SISTEMA DE 2N GRADOS DE LIBERTAD. EL PROBLEMA DE AUTOVALORES CORRESPONDERÍA A HALLAR LAS RAÍCES DE UN POLINOMIO DE GRADO MAYOR, PERO MATLAB NO TENDRÍA NINGÚN PROBLEMA PARA RESOLVER EL PROBLEMA DE AUTOVALORES RESULTANTE

VI. FLUTTER: THE RESULTS PRESENTED IN THIS SECTION DO NOT AFFECT *AT ALL* TO ANY OF THE RESULTS IN THE MAIN TEXT OR IN THE REST OF THIS SUPPLEMENTARY MATERIAL. IT IS ADDED HERE ONLY FOR THE PURPOSE OF COMPLETENESS.

This section is devoted to determine the conditions under which small perturbations on the vertical distance $h(t)$ and on the angle of attack $\alpha(t)$ of the airfoil depicted in figure 1 of the main text, whose elastic axis and center of mass are respectively located at the distances x_e and x_g from the leading edge respectively, either grow or decay in time. For this purpose, we first write the two equations characterizing the time evolution of the two degrees of freedom, which result from projecting the force and momentum balances over the unit cartesian vectors, finding that:

$$\begin{aligned} & m\left(\ddot{h}+\ddot{\alpha}\left(x_g-x_e\right)\right)=-\ell(t)-k_h\,h(t) \Rightarrow m\left(\ddot{h}+\ddot{\alpha}\left(x_g-x_e\right)\right)+k_h\,h(t)+\ell(t)=0 \\ & I_e\ddot{\alpha}+m\ddot{h}\left(x_g-x_e\right)=-m_e(t)-k_\alpha\alpha(t) \Rightarrow I_e\ddot{\alpha}+m\ddot{h}\left(x_g-x_e\right)+k_\alpha\alpha(t)+m_e(t)=0\,, \end{aligned} \tag{89}$$

where m refers to the mass per unit length of the airfoil, $I_e = I_g + m(x_e - x_g)^2$ is the moment of inertia of the airfoil calculated at x_e, with I_g indicating the moment of inertia at the center of mass, k_h and k_α indicate the elastic constants corresponding to the vertical and angular deflections, $\ell(t)$ is the aerodynamic lift force and $m_e(t)$ refers to the aerodynamic torque calculated at x_e. Using the following definitions for the semi-chord, for the dimensionless distance a, and for the dimensionless distance x_α

$$b=\frac{c}{2}\,,\quad x_e=b\,(1+a)\,,\quad x_g-x_e=bx_\alpha \tag{90}$$

equations (30) and (34) read

$$m_e(t)=m(t)-x_e\ell(t)=b\ell_a(t)+\frac{b}{2}\ell_c(t)-b((1+a)\,(\ell_a(t)+\ell_c(t))+\frac{\rho b^4\pi}{8}\frac{d^2\alpha}{dt^2}+\frac{\rho U_\infty b^3\pi}{2}\frac{d\alpha}{dt}\,, \tag{91}$$

with

$$\ell(t)=\rho b^2\,\pi\frac{d}{dt}\left(U_\infty\alpha(t)+\frac{dh}{dt}+(b-b(1+a))\,\frac{d\alpha}{dt}\right)+\ell_c(t)=\rho b^2\,\pi\left(U_\infty\dot{\alpha}+\ddot{h}-ba\ddot{\alpha}\right)+\ell_c(t)=\ell_a(t)+\ell_c(t) \tag{92}$$

and, therefore, the substitution of Eq. (92) into Eq. (91) yields

$$\begin{aligned} m_e(t) &= \rho b^3 \pi a \left(-U_\infty \dot{\alpha} - \ddot{h} + b a \ddot{\alpha}\right) - b\left(\frac{1}{2} + a\right)\ell_c(t) + \frac{\rho b^4 \pi}{8}\frac{d^2\alpha}{dt^2} + \frac{\rho U_\infty b^3 \pi}{2}\frac{d\alpha}{dt} = \\ &= -\rho b^3 \pi a \ddot{h} + \rho b^4 \pi\left(\frac{1}{8} + a^2\right)\ddot{\alpha} - \rho b^3 \pi U_\infty \left(a - \frac{1}{2}\right)\dot{\alpha} - b\left(\frac{1}{2} + a\right)\ell_c(t) \end{aligned} \tag{93}$$

Equations (92)-(93) reveal that both the lift and the torque depend on the time-dependent variables $\alpha(t)$ and $h(t)$. Consequently, the substitution of Eqs. (92)-(93) into the two linear ordinary differential equations in Eq. (89) indicates that the resulting system for the two unknowns $h(t)$ and $\alpha(t)$, is homogeneous. Then, in order to determine whether perturbations grow in time or not, we just seek for the real parts of solutions of the type

$$\alpha(\tau) = \bar{\alpha} e^{i\omega\tau}, \; h(\tau) = b\bar{h} e^{i\omega\tau} \tag{94}$$

which implies

$$\left(\dot{h}, \dot{\alpha}\right) = \frac{U_\infty}{b}\left(\frac{dh}{d\tau}, \frac{d\alpha}{d\tau}\right) = i\omega\frac{U_\infty}{b}\left(b\bar{h}, \bar{\alpha}\right)e^{i\omega\tau} \quad \text{and} \quad \left(\ddot{h}, \ddot{\alpha}\right) = \left(\frac{U_\infty}{b}\right)^2\left(\frac{d^2h}{d\tau^2}, \frac{d^2\alpha}{d\tau^2}\right) = -\omega^2\left(\frac{U_\infty}{b}\right)^2\left(b\bar{h}, \bar{\alpha}\right)e^{i\omega\tau}. \tag{95}$$

Moreover, the type of solutions (94) imply that the circulatory lift is given by -see Eq. (83)

$$\begin{aligned} \ell_c(t) &= 2\pi\rho U_\infty b\left(-w'_{3/4}\right)C(\omega) = 2\pi\rho U_\infty b\left(\dot{h} + \alpha U_\infty + \dot{\alpha}\left(\frac{3}{2}b - b(1+a)\right)\right)C(\omega) = \\ &= 2\pi\rho U_\infty b\left(\dot{h} + \alpha U_\infty + \dot{\alpha} b\left(\frac{1}{2} - a\right)\right)C(\omega), \end{aligned} \tag{96}$$

The substitution of the type of harmonic solutions (94)-(95) into the linear system of ODEs (89) and into the equations describing both $m_e(t)$ and $\ell(t)$, given by Eqs. (92)-(93) and (96) yields:

$$\begin{aligned} & m\left(-\frac{U_\infty^2}{b}\omega^2\bar{h} - \frac{U_\infty^2}{b}x_\alpha\omega^2\bar{\alpha}\right) + k_h b\bar{h} + \rho b^2\pi\left(\frac{U_\infty^2}{b}i\omega\bar{\alpha} - \frac{U_\infty^2}{b}\omega^2\bar{h} + \frac{U_\infty^2}{b}a\,\omega^2\bar{\alpha}\right) + \\ & + 2\pi\rho U_\infty b\left(U_\infty i\omega\bar{h} + U_\infty\bar{\alpha} + U_\infty i\omega\left(\frac{1}{2} - a\right)\bar{\alpha}\right)C(\omega) = 0 \\ & - I_e\frac{U_\infty^2}{b^2}\omega^2\bar{\alpha} - mU_\infty^2 x_\alpha\omega^2\bar{h} + k_\alpha\bar{\alpha} + \rho b^2\pi a\,U_\infty^2\,\omega^2\bar{h} - \rho b^2\pi U_\infty^2\left(\frac{1}{8} + a^2\right)\omega^2\bar{\alpha} - \rho b^2\pi U_\infty^2 i\omega\left(a - \frac{1}{2}\right)\bar{\alpha} - \\ & - 2\pi\rho U_\infty b^2\left(\frac{1}{2} + a\right)\left(U_\infty i\omega\bar{h} + U_\infty\bar{\alpha} + U_\infty i\omega\left(\frac{1}{2} - a\right)\bar{\alpha}\right)C(\omega) = 0 \end{aligned} \tag{97}$$

Grouping terms, the system (97) for the two unknowns, $\bar{h}$ and $\bar{\alpha}$, depends on the following dimensionless parameters,

$$\bar{m} = \frac{m}{\rho\pi b^2}\,, \quad \bar{\Omega}^2 = \frac{k_h b^2}{m\, U_\infty^2}\,, \quad \bar{I} = \frac{I_e}{\rho\pi b^4}\,, \quad \bar{k} = \frac{k_\alpha}{b^2\, k_h} \tag{98}$$

and reads

$$\begin{aligned}
&\left((\bar{m}+1)\,\omega^2 - 2i\omega C(\omega) - \bar{m}\bar{\Omega}^2\right)\bar{h} + \left((\bar{m}\,x_\alpha - a)\,\omega^2 - i\omega - 2\left(1 + i\omega\left(\frac{1}{2} - a\right)\right)C(\omega)\right)\bar{\alpha} = 0\\
&\left((\bar{m}\,x_\alpha - a)\,\omega^2 + 2\left(\frac{1}{2} + a\right)i\omega\, C(\omega)\right)\bar{h} +\\
&+\left(\left(\bar{I} + \frac{1}{8} + a^2\right)\omega^2 + i\omega\left(a - \frac{1}{2}\right) - \bar{k}\bar{m}\bar{\Omega}^2 + 2\left(\frac{1}{2} + a\right)C(\omega) + 2i\omega\left(\frac{1}{4} - a^2\right)C(\omega)\right)\bar{\alpha} = 0\,.
\end{aligned} \tag{99}$$

Notice that the system (99) will possess a solution different from the trivial one, $\bar{h} = \bar{\alpha} = 0$, only for certain values of $\omega(a, x_\alpha, \bar{m}, \bar{I}, \bar{\Omega}^2, \bar{k})$, with these values constituting the eigenvalues of the system. In order to simplify as much as possible the determination of such eigenvalues, we will make use here of the approximate expression of the Theodorsen function given in Eq. (88). Then, if the two algebraic equations in (99) are multiplied by $(0.0455 + i\omega)\,(0.3 + i\omega)$ and we define the function

$$C'(\omega) = (0.0455 + i\omega)\,(0.3 + i\omega)\,C(\omega) = (0.0455 + i\omega)\,(0.3 + i\omega) - 0.165i\omega\,(0.3 + i\omega) - 0.335i\omega\,(0.0455 + i\omega)\ , \tag{100}$$

the solution of the system (99) will be different from the trivial one only for those values of ω satisfying the equation

$$\begin{vmatrix} A & C \\ D & B \end{vmatrix} = 0 \tag{101}$$

namely, for values of ω satisfying the equation

$$AB - CD = 0 \tag{102}$$

where

$$
\begin{aligned}
A &= (0.0455 + i\omega)\,(0.3 + i\omega)\,\left((\bar{m}+1)\,\omega^2 - \bar{m}\bar{\Omega}^2\right) - 2i\omega C'(\omega) \\
B &= (0.0455 + i\omega)\,(0.3 + i\omega)\left(\left(\bar{I} + \frac{1}{8} + a^2\right)\omega^2 + i\omega\left(a - \frac{1}{2}\right) - \bar{k}\bar{m}\bar{\Omega}^2\right) + 2\left(\frac{1}{2} + a\right)\left(1 + i\omega\left(\frac{1}{2} - a\right)\right) C'(\omega) \\
C &= (0.0455 + i\omega)\,(0.3 + i\omega)\,\left((\bar{m}\,x_\alpha - a)\,\omega^2 - i\omega\right) - 2\left(1 + i\omega\left(\frac{1}{2} - a\right)\right) C'(\omega) \\
D &= (0.0455 + i\omega)\,(0.3 + i\omega)\,\left((\bar{m}\,x_\alpha - a)\,\omega^2\right) + 2\left(\frac{1}{2} + a\right) i\omega\, C'(\omega)\,.
\end{aligned}
\tag{103}
$$

The solution of equation (102) will provide with eight different values of ω namely, with eight different eigenvalues, for a given set of values of the dimensionless parameters $(a, x_\alpha, \bar{m}, \bar{I}, \bar{\Omega}^2, \bar{k})$. The phenomenon known as flutter will take place when any of these eight eigenvalues possesses a negative imaginary part with a real part different from zero.

Notice that, if the imaginary part of an eigenvalue is negative, the small perturbations on h and α will grow exponentially in time due to the fact that the value of any of such eigenvalues can be written as

$$
\omega = \omega_r + i\omega_i \tag{104}
$$

and, by virtue of Eq. (94)

$$
\frac{h(\tau)}{b} = \Re\left(\bar{h}e^{i\omega_r\tau}\right)\, e^{-\omega_i\tau} \quad \text{and} \quad \alpha(\tau) = \Re\left(\bar{\alpha}e^{i\omega_r\tau}\right)\, e^{-\omega_i\tau}\,. \tag{105}
$$

In order to determine the minimum value of U_∞ for which an airfoil flutters for a given set of dimensionless parameters $(a, x_\alpha, \bar{m}, \bar{I}, \bar{k})$ we will proceed as follows: equation (102)-(103) is solved for a very large value of $\bar{\Omega}^2$, which can be viewed as the value of the dimensionless frequency corresponding to a very small value of U_∞. If all eight eigenvalues possess positive imaginary parts, then, Eq. (102)-(103) is solved for decreasing values of $\bar{\Omega}^2$ until at least one of the eigenvalues possesses an imaginary part equal to zero for a critical value of $\bar{\Omega}^2$, which we will denote here as $\bar{\Omega^*}^2 = \bar{\Omega^*}^2(a, x_\alpha, \bar{m}, \bar{I}, \bar{k})$. The critical flutter velocity is then determined as:

$$\bar{\Omega}^{*2} = \frac{k_h b^2}{m\, U_\infty^{*2}} \Rightarrow U_\infty^* = b\sqrt{\frac{k_h}{m\bar{\Omega}^{*2}}} \tag{106}$$

whereas the flutter frequency is calculated from

$$\omega^*_{flutter} = \omega^*_r \frac{U^*_\infty}{b}\,, \tag{107}$$

where ω_r^* indicates the real part of the first eigenvalue whose imaginary part is zero.

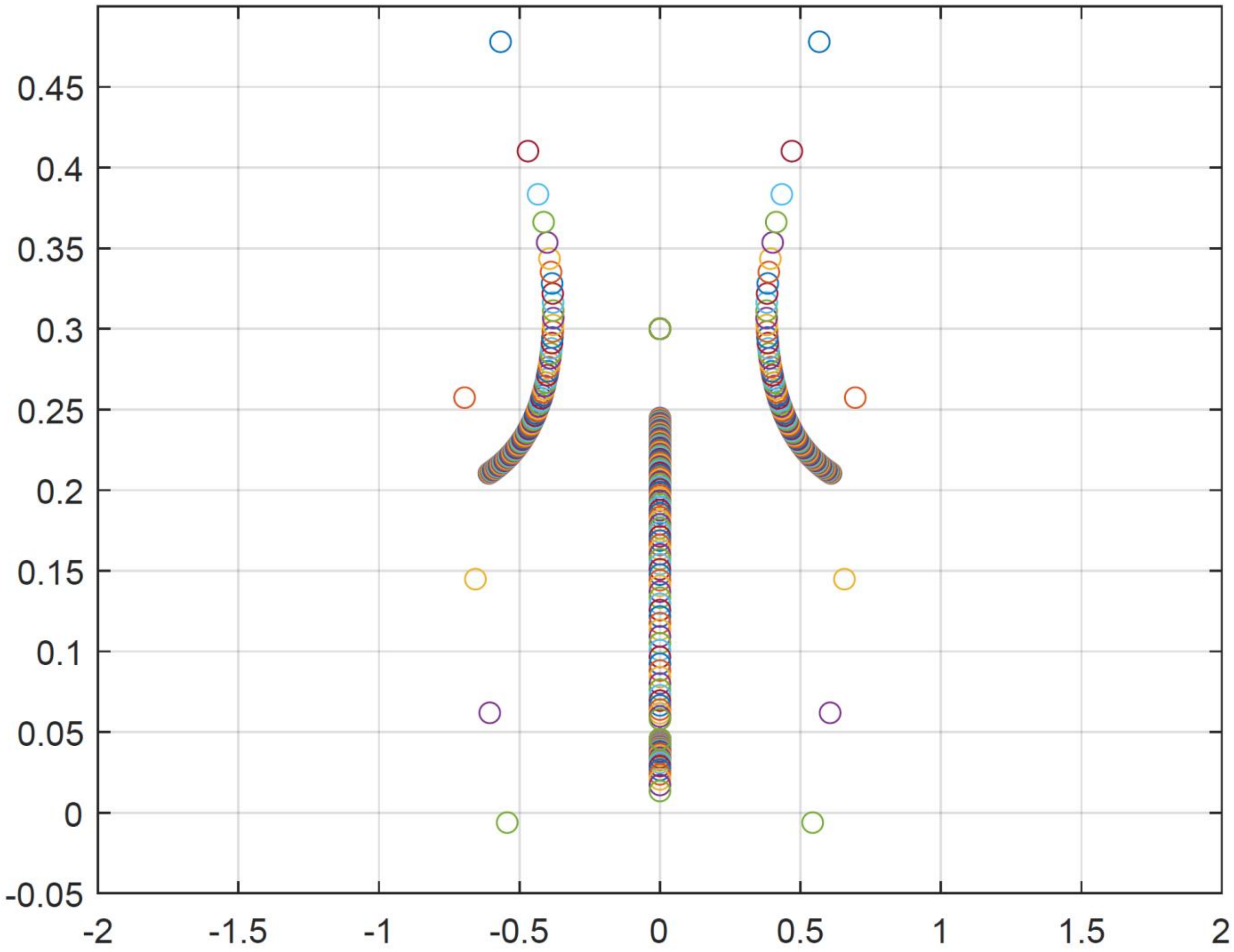
The numerical code provided below calculates the eight eigenvalues for a given set of dimensionless control parameters.

EL SIGUIENTE CÓDIGO PERMITE CALCULAR DE MANERA SENCILLA LA VELOCIDAD DE FLUTTER DADOS LOS PARÁMETROS ADIMENSIONALES QUE CONTROLAN LAS OSCILACIONES DEL PERFIL AERODINÁMICO Y QUE ESTÁN DEFINIDOS EN LAS TRANSPARENCIAS ANTERIORES . PUEDE COMPROBARSE QUE LOS VALORES CALCULADOS DE HALLAR LAS RAÍCES DEL POLINOMIO COINCIDEN CON LOS DE RESOLVER NUMÉRICAMENTE EL PROBLEMA USANDO EL MÉTODO DEL VORTEX-LATTICE.

A. Eigenvalues

```
% JM GORDILLO, ALGEBRAIC EQUATION TO SEEK FOR THE EIGENVALUES IN THE FLUTTER
   PROBLEM
%
clear all; close all; clc;
%
% DEFINITION OF THE DIMENSIONAL PARAMETERS AND VARIABLES
% a xalpha madim=m/(\rho \pi b^2) Iadim=Ie/(\rho \pi b^4) kadim=kalpha/(b^2
% kh) Fh=kh b^2/(m U^2_\infty)

syms a xalpha madim Iadim kadim Fh k A B C D Ck F G Gb

kcrit=zeros(1,8);
Fhcrit=zeros(1,8);

Ck=(i*k+0.0455)*(i*k+0.3)-0.165*i*k*(0.3+i*k)-0.335*i*k*(i*k+0.0455);
A=(i*k+0.0455)*(i*k+0.3)*(madim*(k^2-Fh)+k^2)-2*i*k*Ck;
B=(i*k+0.0455)*(i*k+0.3)*(Iadim*k^2-madim*kadim*Fh+(0.125+a^2)*k^2+(a-0.5)*i*k)
   +2*(0.5+a)*(1+i*k*(0.5-a))*Ck;
C=(i*k+0.0455)*(i*k+0.3)*(madim*k^2*xalpha-i*k-a*k^2)-2*(1+i*k*(0.5-a))*Ck;
D=(i*k+0.0455)*(i*k+0.3)*(madim*xalpha*k^2-a*k^2)+2*(0.5+a)*i*k*Ck;

F=A*B-C*D;

%PARAMETROS DEL CODIGO VLATTICE_LM_DINP
masa=1.5708;
kh=0.5674;
kalpha=0.09;
Ig=0.0355;
d=0.2;
Ie=Ig+masa*d*d;

%PARAMETROS ADIMENSIONALES DEL CODIGO DE CALCULO DE AUTOVALORES.
% FNUM=OMEGA^2

Fnum=0.5;
b=0.5;
rho=1;
ap=-0.4;
xalphap=0.4;
mp=masa/(rho*pi*b^2);
Ip=Ie/(rho*pi*b^4);
kp=kalpha/(kh*b^2);

true=1;

while true

    Fnum=Fnum-0.005

    G=expand(subs(F,[a xalpha madim Iadim kadim Fh],[ap xalphap mp Ip kp Fnum]))
       ;
    p=coeffs(G);
    p = fliplr(p);
    r=roots(p);
    s=double(r);
```

```
    plot(real(r),imag(r),'o')
    axis([-2 2 -0.05 0.5])
    grid on
    pause(0.01)
    hold on

    cont=0;
    for i=1:8
        if imag(s(i))<0
            cont=cont+1;
            kcrit(cont)=s(i);
            Fhcrit(cont)=Fnum;
            true=0;
        end
    end
end

for i=1:cont
    kcrit(i)
    Fhcrit(i)
    Uflutter=b/sqrt(Fhcrit(i))*sqrt(kh/masa)
end
```



$$\bar{\Omega}^{*2} = \frac{k_h b^2}{m U_\infty^{*2}} \Rightarrow U_\infty^* = b\sqrt{\frac{k_h}{m\bar{\Omega}^{*2}}} \quad (106)$$

whereas the flutter frequency is calculated from

$$\omega^*_{flutter} = \omega^*_r \frac{U^*_\infty}{b}\,, \quad (107)$$

Fcrit=

0.0200

Autovalor1 =

-0.5436 - 0.0061i

Fcrit =

0.0200

Autovalor2 =

0.5436 - 0.0061i

0.02=kh b ^2/(m U^2_\ infty,critico)
Omega*_r=0.5436

Comparar con los resultados del código numérico de VL