Discrete breather collisions in NLS and Klein–Gordon lattices

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Outline

1. Solitons and discrete breathers
   - Solitons
   - The DNLS equation
   - Klein–Gordon lattices

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    - Cubic DNLS
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4 Summary and Outlook

Integrable systems

- Solitons are spatially localized excitations that arise in continuum and integrable nonlinear systems.
- Example of integrable equations:
    - Korteweg–de Vries (KdV) equation (hydrodynamics):

$$u_t + 6uu_x + u_{xxx} = 0$$

    - Nonlinear Schrödinger (NLS) equation (BECs without trap, nonlinear optical fibers):

$$\mathrm{i}u_t + g|u|^2u + u_{xx} = 0$$

    - sine-Gordon equation (long Josephson junctions):

$$u_{tt} + \sin u - u_{xx} = 0$$

Integrable systems

- Soliton are considered quasi-particles → the function that describes them do not change after interaction.

- Inelastic collision → The solitons cross each other and a phase-shift appears.

Non-integrable systems

- In non-integrable systems, solitons do not behave as quasi-particles:
  - Their profile and velocity can change after interaction.
  - They could even get trapped after collision.
  - Example of non-integrable continuum system → ϕ^4 model:
$$u_{tt} + u^3 - u - u_{xx} = 0$$
- Nonlinear lattices are non-integrable (except for the Ablowitz–Ladik and Toda lattices).
- In this talk, we will consider two different kinds of lattices:
  - Generalized DNLS lattices:
$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} + f(|u_n|^2)u_n + C(u_{n+1} - u_{n-1} - 2u_n) = 0$$
  - Klein–Gordon lattices:
$$\frac{\mathrm{d}^2u_n}{\mathrm{d}t^2} + V'(u_n) - C(u_{n+1} - u_{n-1} - 2u_n) = 0$$

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The Discrete Nonlinear Schrödinger (DNLS) Equation

- In the mean-field approximation, the dynamics of a Bose-Einstein Condensate can be described by the Gross–Pitaevskii (GP) equation:

$$\mathrm{i}u_t + g|u|^2 u + V(x)u + u_{xx} = 0$$

- In the case that $V(x)$ is a deep periodic potential (optical lattice), the GP equation can be approximated by the cubic DNLS equation [Alfimov, Kevrekidis, Konotop and Salerno. PRE 66 (2002) 046608]:

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} + |u_n|^2 u_n + C(u_{n+1} - u_{n-1} - 2u_n) = 0$$

- Stationary discrete solitons or breathers are localized solutions of the form:

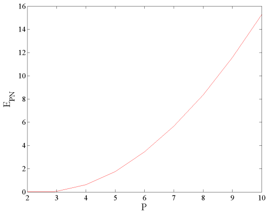
$$u_n(t) = v_n \exp(i\Lambda t), \qquad \Lambda = -\mu, \qquad \lim_{n\to\infty} |v_n| = 0$$

- Discrete moving breathers are generated by adding a momentum to a stationary breather:

$$u_n(0) = v_n \exp(iqn)$$

The Discrete Nonlinear Schrödinger (DNLS) Equation

- The DNLS equation possesses two conserved quantities:
  - Norm (also number of particles or power): $P = \sum_n |u_n|^2$
  - Hamiltonian: $H = -\sum_n \left[C|u_n - u_{n+1}|^2 + |u_n|^4/2\right]$
- Effects of non-integrability:
  - Existence of Peierls–Nabarro barrier (PNB):
    - Energy difference between on-site and inter-site solitons with the same norm.



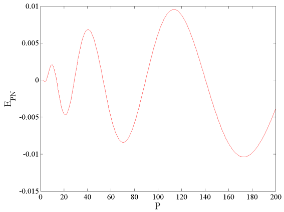
    - It is related with the minimum momentum needed for put breathers into movement.
    - It is unbounded (grows with P) $\longrightarrow$ Moving breathers exist only for small norms.
  - Breathers radiate when moving.
  - Collision scenario is complex (breathers are not quasi-particles).

The DNLS Equation with saturable nonlinearity

- The saturable DNLS equation (SDNLS) arises as trivial discretization of the Vinetskii-Kukhtarev model for photorefractive media:

$$\mathrm{i}\frac{\mathrm{d}u_n}{\mathrm{d}t} - \beta\frac{u_n}{1+|u_n|^2} + C(u_{n+1} - u_{n-1} - 2u_n) = 0$$

- One of the features of this equation is that the PNB is bounded ($\beta = 2$):



  - Moving breathers exist even for high norms.
  - There are values of the norm at which the PNB vanishes → Moving breathers without radiation can be found [Melvin, Champneys, Kevrekidis and Cuevas. PRL 97 (2006) 124101]

- For small norms, this equation can be approximated by the cubic DNLS

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Breathers in Klein–Gordon lattices

- Klein–Gordon chains are described by the equation:

$$\frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2} + V'(u_n) - C(u_{n+1} - u_{n-1} - 2u_n) = 0$$

- This equation possesses a conserved quantity, the Hamiltonian:

$$H = \sum_n \left[\frac{1}{2} \left(\frac{\mathrm{d}u_n}{\mathrm{d}t} \right)^2 + V(u_n) + C(u_n - u_{n+1})^2 \right]$$

- Discrete Klein–Gordon models describe the dynamics of crystals, biomolecules, pendulum and Josephson–junction arrays, ...
- Stationary breathers are solutions in the form:

$$u_n(t) = z_n^0 + 2\sum_k z_n^k \cos(k\omega_\mathrm{b} t), \qquad \lim_{n\to\infty} |z_n^k| = 0 \ \forall k$$

- They exist as long as no integer multiple of their frequency ω_b coincide with the frequency of a linear mode [MacKay and Aubry. Nonlinearity 7 (1994) 1263]

Breathers in Klein–Gordon lattices

- Moving breathers are also found by adding a momentum q:

$$u_n(0) = \left(z_n^0 + 2\sum_k z_n^k \right) \cos(qn)$$

$$\frac{\mathrm{d}u_n}{\mathrm{d}t}(0) = \left(z_n^0 + 2\sum_k z_n^k \right) \sin(qn)$$

- Stationary breathers are generic for all potential with convex parts.
- On the contrary, moving breathers exist only in some special potentials:
  - Morse potential: $V(u) = (\exp(-u) - 1)^2/2$
  - sine-Gordon potential: $V(u) = 1 - \cos(u)$
  - ϕ^4 (double well) potential: $V(u) = (u^2 - 1)^2/4$
- For breather frequencies close to the linear modes band, the Klein–Gordon equation can be approximated by the cubic DNLS.

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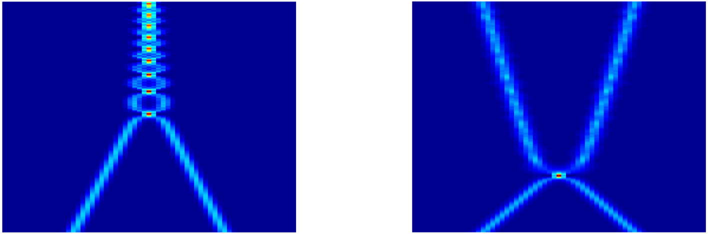
Collisions in cubic DNLS

- Breather collisions were considered in [Papacharalampous, Kevrekidis, Malomed and Frantzeskakis. PRE 68 (2003) 046604].
- We focus on incoming breathers with the same phase and velocity. Two main regimes are observed:
  - For small q: A bound state is formed.
  - For high q: Breathers are refracted.



- This behavior resembles the collision of two quasi-particles into an attractive effective potential. When the breather velocity is above the escape velocity, the bound state is not formed.

Breather trapping

Breather refraction

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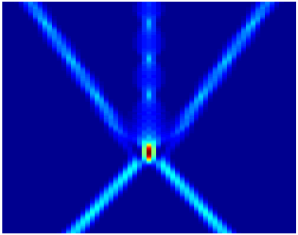
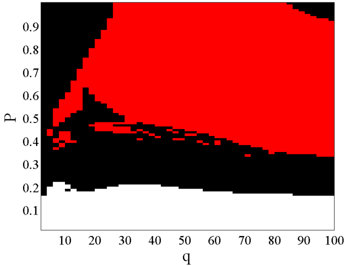
Collisions in saturable DNLS

- Breather collisions were considered in [Cuevas and Eilbeck. PLA 358 (2006) 15].
- For small norms, the outcome is similar to the cubic DNLS.
- For high norms, the scenario is as follows:
  - For small q, the breathers are trapped (bound state)
  - For intermediate q, breathers are refracted
  - For high q, a bound state is created apart from the refracted breathers. The energy of the bound state is smaller than that of the refracted breathers.





Breather refraction + trapping

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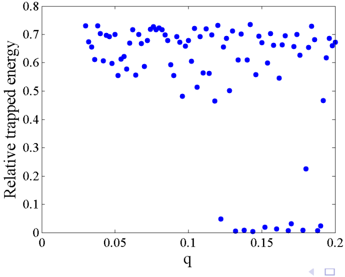
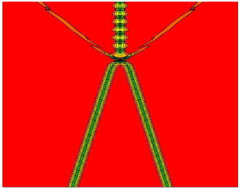
Collisions in Klein–Gordon lattices

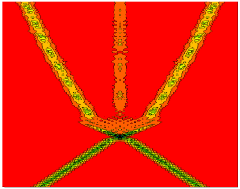
- We considered breather collisions in Klein–Gordon lattices with **Morse** potential [Álvarez, Romero, Cuevas and Archilla. PLA 372 (2008) 1256].
- For $\omega_b = 0.95$ (small nonlinearity), the behaviour is similar to the cubic DNLS equation.
- For $\omega_b = 0.8$ (high nonlinearity), two main regimes are observed:
  - Breathers can be refracted
  - A bound state is created apart from the refracted breathers.
- There are no critical values of q separating both regimes.
- The only *pseudo*-critical value of q indicates a threshold for a possible appearance of reflection without trapping.

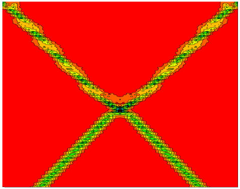


Collisions in Klein–Gordon lattices

- Examples of different regimes:







Breather refraction + high-energy trapping

Breather refraction + low-energy trapping

Breather refraction

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Conclusions

- The breather collision scenario in cubic DNLS, saturable DNLS and Klein–Gordon lattices has been reviewed.
- Three main regimes have been observed when two identical breathers collide:
  - Breather trapping (formation of a bound state)
  - Breather refraction
  - Mixed regime: Breather refraction + trapping
- Some of these features may be explained through energy balances

Outlook

- We are finishing the study of discrete breathers interaction in Klein–Gordon lattices with other potentials:
  - ϕ^4 hard potential
  - sine-Gordon potential
- We are also examining with more detail the effect of the relative phase on the collision.
- We still lack a clear explanation of the complex scenario, specially in the case of high nonlinearity.

Thanks for your attention

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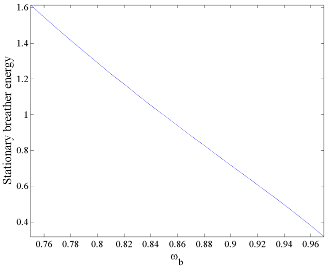
http://aleph.eii.us.es/LENCOS

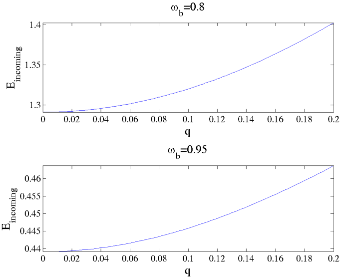
Collisions in Klein–Gordon lattices. Interpretation

- Roughly speaking, the moving breathers can be considered as quasiparticles.
- Neglecting the phonon radiation, it must be fulfilled that

$$2E_{\text{incoming}} = U_{\text{trapped}} + 2E_{\text{outgoing}}$$

- For a given C, the static breather energy has a maximum $\tilde{E}$ corresponding to the minimum frequency





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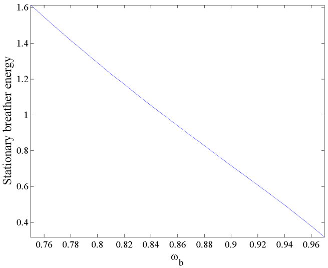

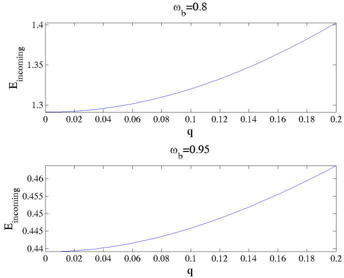

- It can be supposed that $U_{\text{trapped}} < \tilde{E}$. In this case, if $E_{\text{incoming}} > \tilde{E}/2 \rightarrow E_{\text{outgoing}} > 0 \rightarrow$ there is an exceeding energy.
- The last relation is fulfilled for $\omega_b = 0.8$ but not for $\omega_b = 0.95$.
- This analysis is not valid for the saturable DNLS as the energy of stationary breathers is not bounded.