Quantum Lattices

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Outline

- Introduction- Solitons and Breathers

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- The DNLS and A-L equations

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Solitons and Breathers in Lattices

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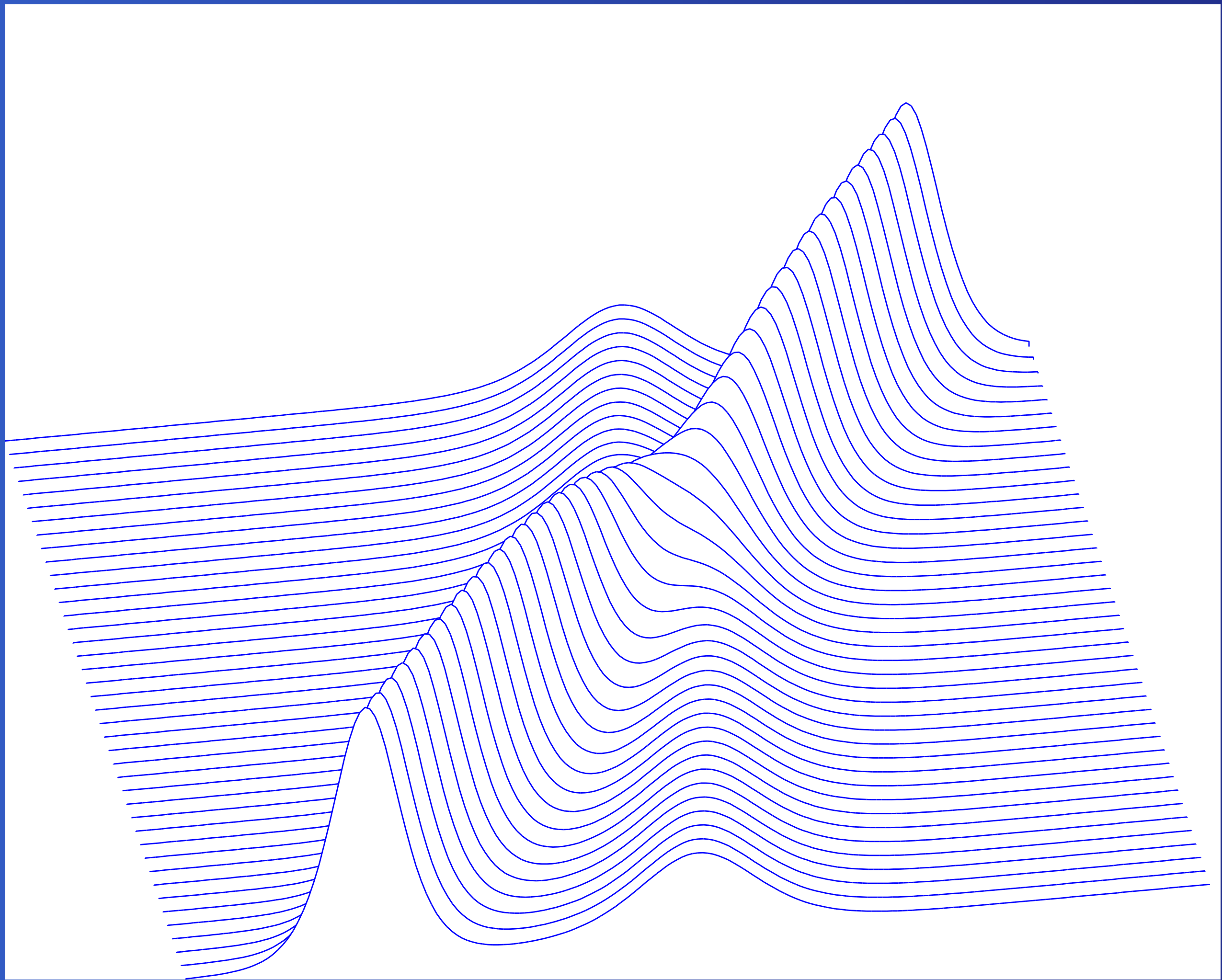
Solitons and Breathers in Lattices

- **Soliton.** Strongly localized package (lump) of energy, can move large distances with no distortion, very stable even under collisions or perturbations.
- **Breather.** A more complicated form of nonlinear wave which can often occur in discrete systems. It looks like a soliton modulated by an internal carrier wave.

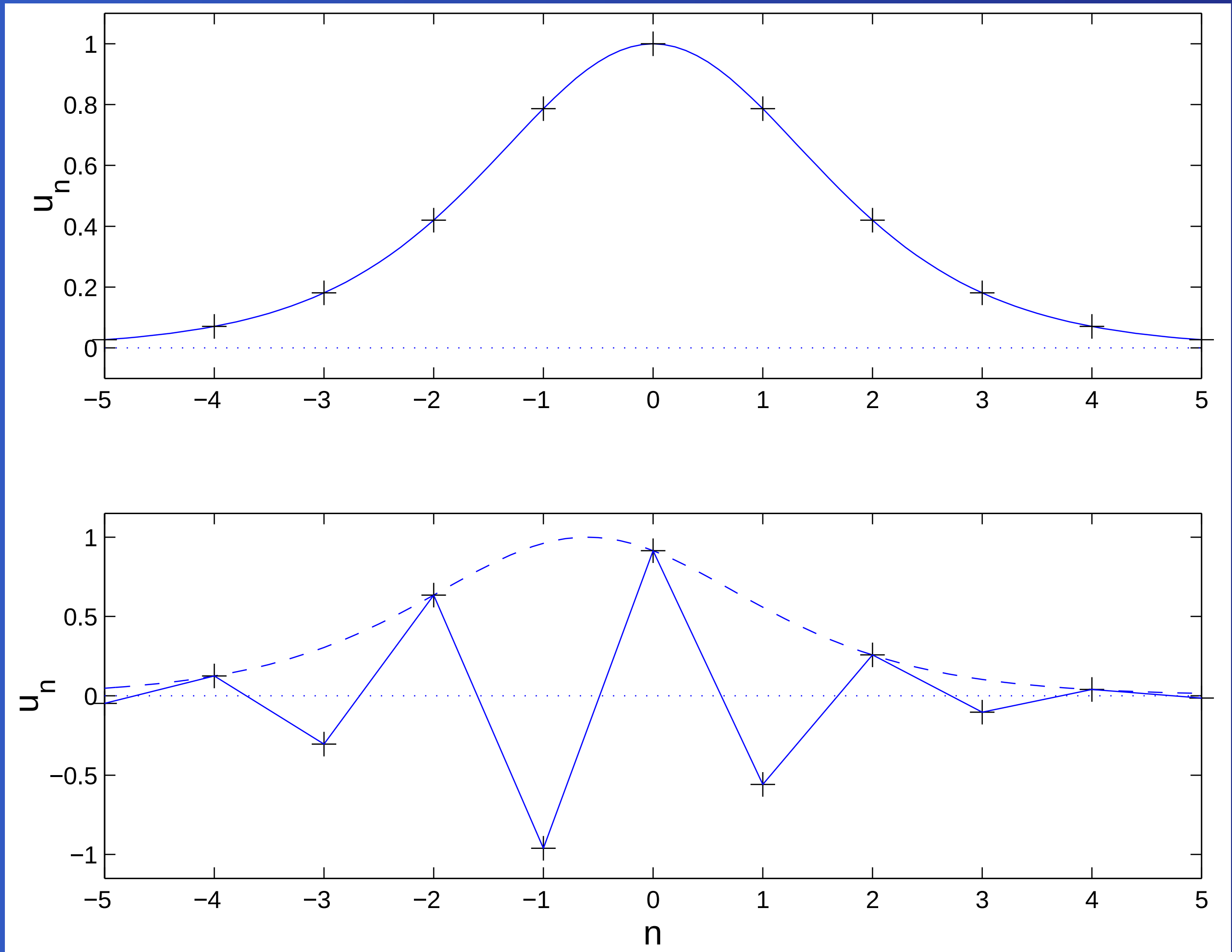
soliton collision

Start animation

soliton collision 2



Solitons and Breathers in Lattices



Breathers, DNLS

For simplicity we focus on two models, firstly the Discrete Nonlinear Schrödinger (DNLS) equation.

$$\mathrm{i}\frac{dA_j}{dt} + (A_{j-1} - 2A_j + A_{j+1}) + \gamma|A_j|^2 A_j = 0,$$

where $A_j(t)$ is the *complex* oscillator amplitude at the jth lattice site. DNLS Hamiltonian:

$$H = \sum_{j=1}^{f} \left[\frac{\gamma}{2}|A_j|^4 - A_j^*(A_{j-1} + A_{j+1})\right]$$

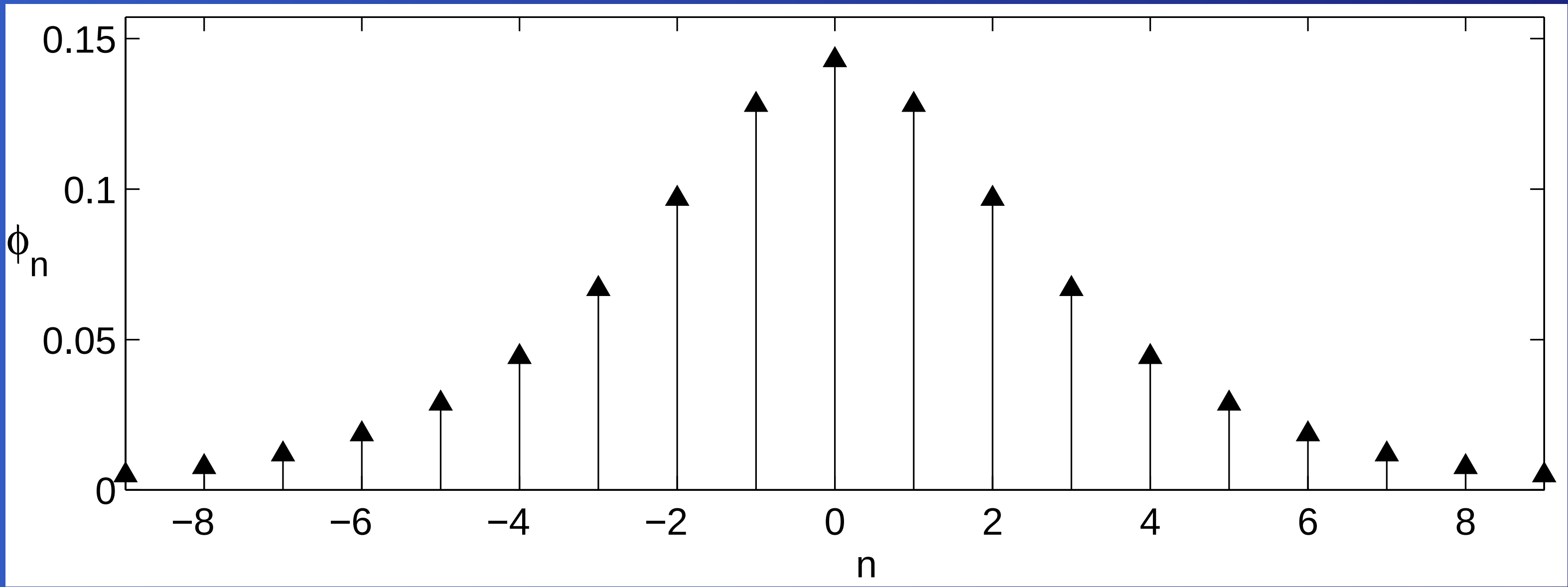
A-L equation

The second model we consider is the classical Ablowitz-Ladik system

$$\mathrm{i}\frac{dA_j}{dt} + (A_{j+1} - 2A_j + A_{j-1}) + \\ +\frac{1}{2}\gamma|A_j|^2(A_{j+1} + A_{j-1}) = 0$$

This is an *integrable* system.

Breathers, DNLS equation



This is a stationary breather on a larger lattice. The amplitude goes to zero exponentially as $|n| \to \infty$.

Breathers, DNLS equation

Simulations:

- Stationary breather
- Mobile breather
- Colliding breathers

Exact breather solutions?

Exact Breathers, DNLS equation?

In 1991, Henrik Feddersen (Springer Lect. Notes. in Phys., **393**, 159) made a numerical study of the DNLS equation using the ansatz

$$A_n(t) = \phi(n - ct)e^{i(kn-\omega t)}.$$

He found branches of localized solutions to high accuracy, but the *existence* of such solutions is still an open question.

Quantum breathers

Quantum DNLS (boson Hubbard) Hamiltonian in 1D, nearest neigbour interactions:

$$\hat{H} = -\frac{\gamma}{2}\sum_{j=1}^{f} b_j^\dagger b_j^\dagger b_j b_j - \sum_j b_j^\dagger b_{j+1}$$

$\hat{H}$ conserves the *number* of quanta

$$\hat{N} = \sum_{j=1}^{f} b_j^\dagger b_j \ ,$$

Quantum wavefunctions

The operators b_j, $b_j^\dagger$ acts on *number states*
$|\psi_n >= |n_1 > |n_2 > \ldots |n_f >= [n_1, n_2, \ldots, n_f]$,
where $N = \sum n_i$.
Example: [2,2,0,0,0,1] means 2 quanta on site 1, 2 quanta on site 2, 1 quanta on site 6, on a lattice with 6 sites.
Raising/Lowering operators satisfy
$b_j|n_j >= \sqrt{n_j}|n_j - 1 >$, $\quad b_j|0 >= 0$,
$b_j^\dagger|n_j >= \sqrt{n_j + 1}|n_j + 1 >$.
General wave function is $|\Psi_N >= \sum_n c_n|\psi_n >$.

Quantum Mechanics in Maple

[2,2,0,0,0,1] is represented in Maple as an undefined function `psi(2,2,0,0,0,1)`.

Then operator $b_i^\dagger$ are defined something like

```
bd:=proc(phi,i::nonnegint)
    ni:=op(i,phi);
    RETURN(sqrt(ni+1)*subsop(i=ni+1,phi)
end
```

$\hat{H}$ for QDNLS is defined along the following lines

```
sum('gamma/2*bd(bd(b(b(phi,i),i),i),i)
     +bd(b(phi,cyc(i+1)),i)
     +bd(b(phi,cyc(i-1)),i)', i=1..f)
```

Conserved number of quanta

We can block-diagonalize the Hamiltonian matrix $H = \left\langle \Psi \middle| \hat{H} \middle| \Psi \right\rangle$ as

$$H = \begin{pmatrix} H_1 & 0 & & \\ 0 & H_2 & 0 & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

where each H_N is the Hamiltonian for N quanta.

Example, $f = 2$, $N = 2$

$$|\Psi_2> = c_1[2,0] + c_2[1,1] + c_3[0,2]$$

$$\hat{H}|\Psi_2> = \Big[-\frac{\gamma}{2}\left(b_1^\dagger b_1^\dagger b_1 b_1 + b_2^\dagger b_2^\dagger b_2 b_2\right) - \\ - \left(b_1^\dagger b_2 + b_2^\dagger b_1\right)\Big]|\Psi_2> \\ = -\gamma c_1[2,0] - \gamma c_3[0,2] - \sqrt{2}c_1[1,1] - \\ - \sqrt{2}c_3[1,1] - \sqrt{2}c_2[2,0] - \sqrt{2}c_2[0,2]$$

Example, $f = 2$, $N = 2$ continued

Using $[2, 0]$,$[1, 1]$,$[0, 2]$ as basis vectors, we can write this in matrix (eigenvalue) form

$$H \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = - \begin{bmatrix} \gamma & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & \gamma \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = E \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

with eigenvalues

$$E = -\gamma, \quad \frac{-\gamma \pm \sqrt{\gamma^2 + 16}}{2}$$

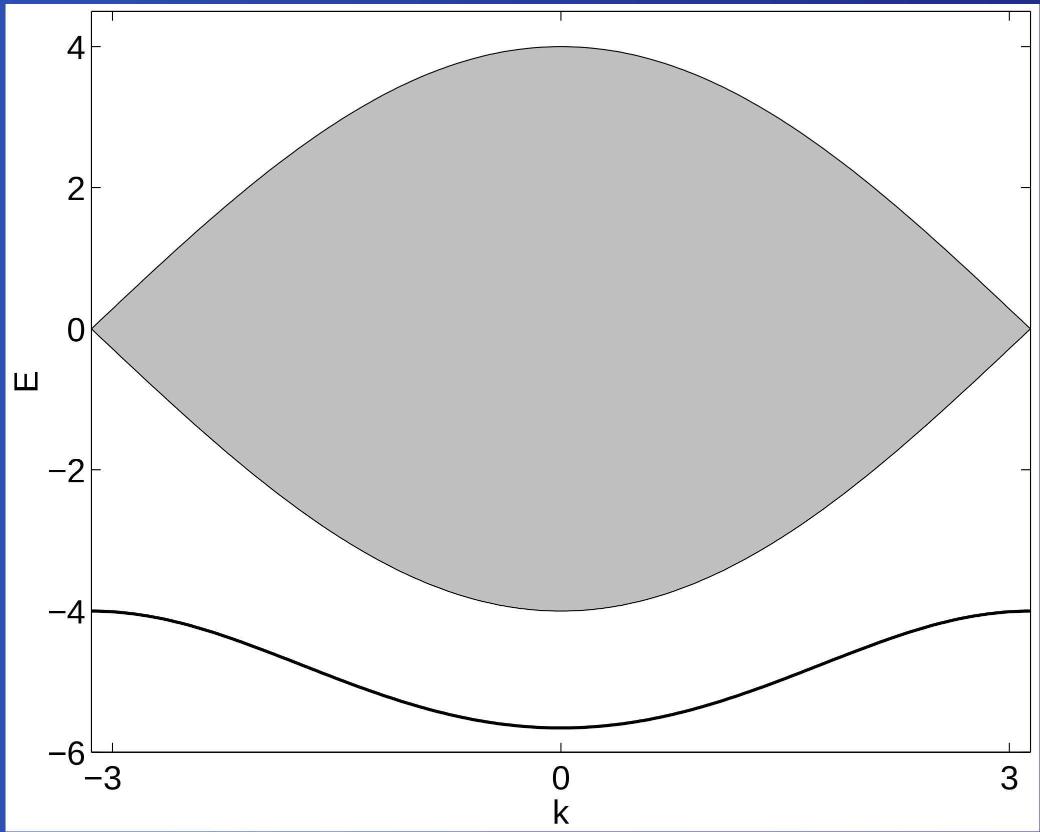
Rotational symmetry

If model is rotationally invariant (translations + periodic b.c.'s) we can further block-diagonalize H_N using eigenfunctions of the translation operator $\hat{T}$, giving states with fixed momentum k

$$H_N = \begin{pmatrix} H_{N,k_1} & 0 & & \\ 0 & H_{N,k_2} & 0 & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

N=2 Case In this case each H_{2,k_p} is tridiagonal.

1D Quantum Breather – 2 quanta



Eigenvalues $E(k)$ for QDNLS. The lower band is the “breather” band.

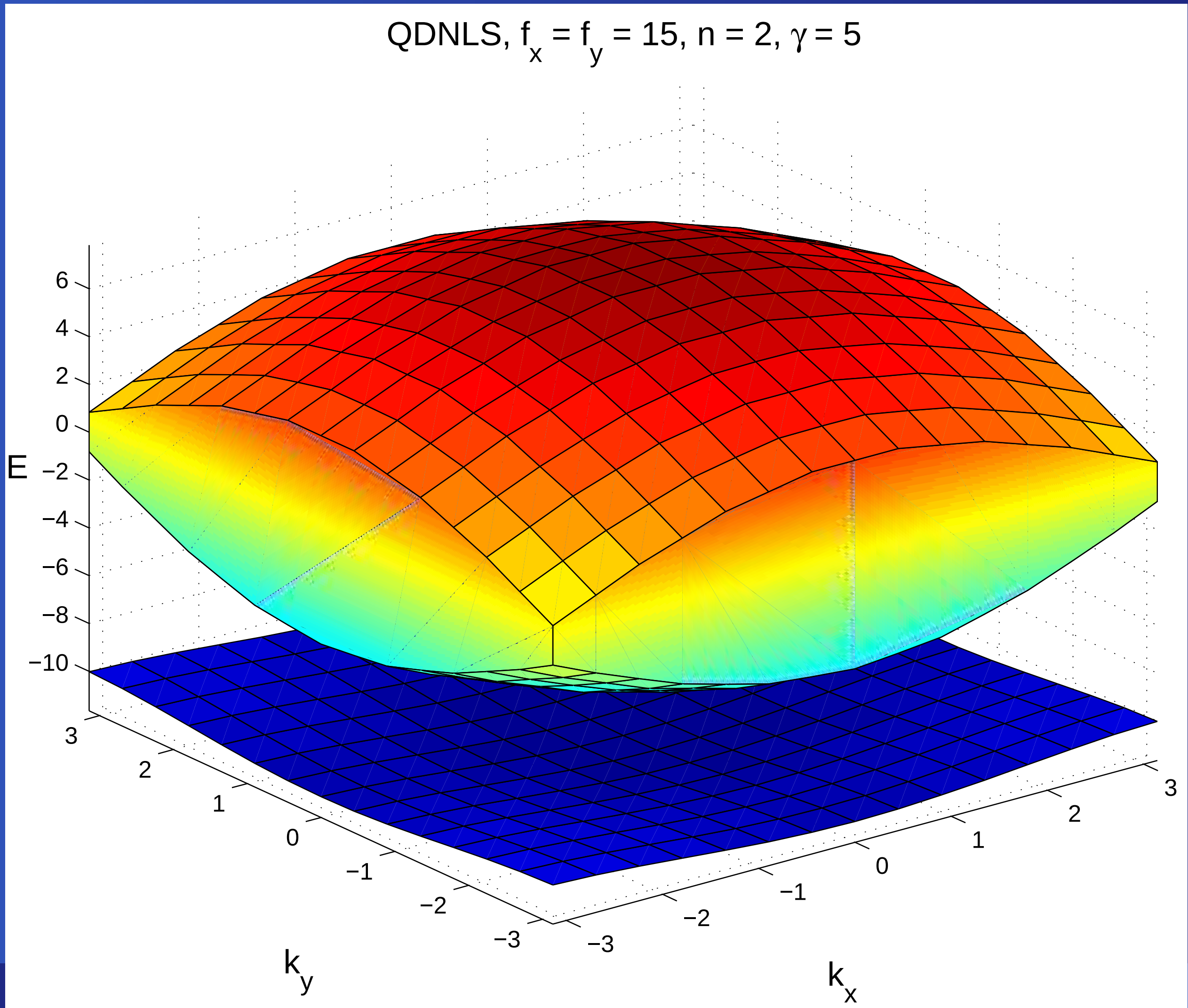
Quantum Breather? – 2 quanta

The “breather” band has wave function

$$|\Psi_n> = [2,0,0,\dots] + [0,2,0,\dots] + [0,0,2,\dots] + + \dots + O(1/\gamma)\,([1,1,0,\dots] + \dots)$$

So for large γ the quanta are *localized* (both on the same site), but occur at *all* sites with *equal* probability! Localized breathers in the *classical* sense are not eigenstates, but decay slowly.

2D Quan. Breather Bands – 2 quanta



Further reading

- D. B. Duncan, J. C. Eilbeck, H. Feddersen and J. A. D. Wattis, *Solitons on lattices*, Physica D **68** 1–11 (1993)
- A. C. Scott, J. C. Eilbeck and H. Gilhøj, *Quantum lattice solitons*, Physica D **78**, 194-213, (1994)
- A. C. Scott, *Nonlinear Science*, OUP, 1999 (2nd ed. 2003).