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Dead-time compensation of constrained linear systems with bounded disturbances: output feedback case

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Abstract: This study presents an analysis about state estimation and dead-time compensation effect over constrained control systems with bounded disturbances and dead-time. It is shown that input-to-state stability and constraint satisfaction can be guaranteed by using an equivalent dead-time free system with measurable states and modified disturbances. This result may be useful to simplify synthesis and analysis of a given constrained control strategy. A linear output feedback control scheme and a tube-based model predictive control strategy are used as motivating examples.

Nomenclature

For a given matrix $M \in \mathbb{V}^{n\times m}$ and a set $\mathbb{V} \subset \mathbb{R}^m$, $M\mathbb{V} \subset \mathbb{R}^n$ denotes the set $M\mathbb{V} = \{y = Mv,\ v \in \mathbb{V}\}$. Given two sets $\mathbb{U} \subset \mathbb{R}^n$ and $\mathbb{V} \subset \mathbb{R}^n$, the Minkowski sum is defined by $\mathbb{U} \oplus \mathbb{V} \triangleq \{u + v{:}u \in \mathbb{U}, v \in \mathbb{V}\}$ and the Pontryagin set difference is $\mathbb{U} \ominus \mathbb{V} \triangleq \{u{:}u \oplus \mathbb{V} \subseteq \mathbb{U}\}$. Operator z^{-1} denotes a discrete backward shift. The set of vertexes of a given compact set Γ is denoted by vert(Γ).

1 Introduction

In classical control techniques, constant input–output delay, also known as dead-time, imposes phase margin reduction which limits control performance because of the robustness specifications [1]. This undesired effect may be avoided using Smith's [2] original idea, by predicting delay effect, in order to remove dead-time from the control loop. In the last years, robust control of linear time-delay systems has been widely studied for the unconstrained case (e.g. [3] and references therein). However, deal with constrained dead-time systems remains an open problem, once most of the related works consider only input constraints [4–7].

In works such as [5, 7, 8], it is considered the problem of global stabilisation of neutrally stable systems with dead-time and input constraint, but disturbance effect and state constraint are not discussed. Other works, as [4, 6, 9], deal with the problem of anti-windup synthesis and uncertainties, but do not consider state constraints as well. On the other hand, state constraint is considered without taking disturbances into account in [10]. Furthermore, these works are focused on specific control strategies, so that a general robust dead-time compensation (DTC) scheme for constrained systems was not in their scope.

Disturbance and dead-time are important aspects in both theory and practice [11]. Active disturbance rejection control and disturbance observer strategies can deal with these kinds of issues, but it is not simple to guarantee robust constraint satisfaction. In general, constraint satisfaction is not discussed in these strategies. An exception is the model predictive control (MPC)-based scheme presented in [12], however constraint satisfaction is not guaranteed in the presence of plant-model mismatches. Other works discuss about the problem of state estimation in the presence of time-varying output delay [13], time-varying state delay [14] and constant dead-time [15], but none of them deals with constraint satisfaction.

It is well known that state estimation, robust stability and constraint satisfaction can be naturally considered by MPC strategies [16–21]. Except in [20], which considered a delay smaller than the sampling period, DTC was not considered as a key point. Recently, [17] considered variable dead-time as a source of uncertainty, but it was not considered other kind of disturbances. Actually, MPC schemes perform DTC intrinsically [22]. However, by using the augmented representation, a usual way to consider delay effect [23], the representation dimension may increase undesirably. In practice, DTC can be performed explicitly, outside the optimisation problem, in order to simplify the control algorithm [24] or to improve robustness [1]. As this kind of prediction is based on a nominal model, disturbance effect should be considered to guarantee robust stability and constraint satisfaction, as shown in [25].

In this paper, the results of Santos *et al.* [25] are generalised in order to control dead-time systems with non-measurable states. It is shown that robust stability and constraint satisfaction problems can be analysed from a dead-time free system with measurable states in the presence of modified disturbances. Moreover, a particular tube-based MPC is used to illustrate that the presented discussion may be employed to extend input-to-state stability (ISS) properties of a given dead-time free control strategy.

This paper is organised as follows: both problem statement and DTC background are discussed in Section 2; the main results, based on disturbance effect analysis, are presented in Section 3; and two strategies for constrained dead-time systems with non-measurable states are proposed in Section 4. Simulation examples are presented in Section 5 and the concluding remarks are discussed in Section 6.

2 Preliminaries

2.1 Problem statement

Consider that the plant can be described by the following uncertain time-invariant discrete-time linear system with dead-time

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k-d) + w(k) \\ y(k) &= Cx(k) + v(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is a vector of the current states, $u(k) \in \mathbb{R}^m$ is the current control input, $y(k) \in \mathbb{R}^p$ is the measured output, $w(k) \in \mathbb{R}^n$ is an unknown but bounded state disturbance, $v(k) \in \mathbb{R}^n$ is an unknown but bounded output disturbance, k denotes the current sampling instant and d represents the nominal dead-time. It is considered that may exist polytopic constraints on control input and state such as

$$u(k) \in \mathbb{U}, \quad x(k) \in \mathbb{X}$$

Moreover, it is assumed that: (a) the triple (A, B, C) is stabilisable and detectable, (b) disturbances lies in compact and convex polytopic sets given by

$$w(k) \in \mathbb{W}, \quad v(k) \in \mathbb{V}$$

which contains the origin. It is important to remark that dead-time estimation error can be described as an additive bounded disturbance if $u(k)$ is constrained by a compact set.

The main goal is to derive conditions in order to use a general dead-time free control law, based on a model without dead-time and a simple Luenberger observer, to regulate the constrained dead-time system with bounded disturbances, guaranteeing robust stability and constraint satisfaction.

2.2 Augmented state-space representation

In discrete-time state-space representation, dead-time effect can be translated from the control input to the state matrix by using an augmented representation [23]. In this case, the new state vector becomes

$$\xi(k) = [x(k)^{\mathrm{T}}\ u(k-d)^{\mathrm{T}}\ \cdots\ u(k-2)^{\mathrm{T}}\ u(k-1)^{\mathrm{T}}]$$

and the equivalent input–output representation has the following form

$$\begin{aligned} \xi(k+1) &= A_\xi \xi(k) + B_\xi u(k) + B_w w(k) \\ y(k) &= C_\xi \xi(k) + v(k) \end{aligned} \tag{2}$$

where $\xi(k) \in \mathbb{R}^{n+d\cdot m}$. This technique is useful since a given control strategy, for systems without dead-time, can be directly applied by using the augmented model. However, as the augmented model has $n+d\ m$ states, the representation order may increase undesirably, since it depends on both dead-time length and control dimension.

2.3 Explicit DTC: nominal case

Assuming measurable states, explicit DTC can be applied to linear state-space models as discussed in [24]. Since the effect of $u(k)$ appears after $x(k+d)$, a prediction for $x(k+d)$ at k is given by

$$\tilde{x}(k) \triangleq x(k+d|k) = A^d x(k) + \sum_{j=1}^{d} [A^{j-1} B u(k-j)] \tag{3}$$

Hence, a dead-time free nominal model at d steps ahead is given by

$$\tilde{x}(k+1) = A\tilde{x}(k) + Bu(k) \tag{4}$$

Thus, a control law $u(k) = \boldsymbol{\kappa}(\tilde{x}(k))$ is based on the model (4) without dead-time. In the absence of disturbances, $\tilde{x}(k) = x(k+d)$ and $\tilde{x}(k) \in \mathbb{R}^n$ as in the original representation. However, if $w(k) \neq 0$, then $\tilde{x}(k) \triangleq x(k+d|k) \neq x(k+d)$, which was not tackled in [24].

2.4 Luenberger observer for systems with dead-time

Since the states are not measurable, it is also necessary to consider a state estimation approach. Similarly to Mayne *et al.* [26], a Luenberger observer will be used to achieve implementation simplicity. Owing to dead-time effect, state estimation is handled by the following dynamics

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k-d) + L(y(k) - C\hat{x}(k)) \tag{5}$$

where $\hat{x}(k)$ is the estimation of $x(k)$.

As in the case without dead-time, estimation error, denoted by $\delta(k) \triangleq x(k) - \hat{x}(k)$, can be obtained from (1) and (5) as follows

$$\delta(k+1) = (A - LC)\delta(k) + w(k) - Lv(k) \tag{6}$$

As a consequence, if $A_L \triangleq A - LC$ has all eigenvalues strictly inside the unitary circle, $w(k)$ is inside a compact set $\mathbb{W}$ and $v(k)$ is inside a compact set $\mathbb{V}$, then $\delta(k)$ is also inside a compact set [27]. This set can be obtained as an invariant outer approximation of the minimal robust positively invariant set (mrpi) given by

$$\Delta = \bigoplus_{j=0}^{\infty} A_L^j [\mathbb{W} \oplus (-L\mathbb{V})]$$

if $\delta(0) \in \Delta$. An invariant outer approximation, $\overline{\Delta}$, described as follows

$$A_L\overline{\Delta} \oplus (\mathbb{W} \oplus -L\mathbb{V}) \subseteq \overline{\Delta}$$

with $\Delta \subseteq \overline{\Delta}$, can be computed by using specialised algorithms [28].

2.5 Continuous-time model and dead-time uncertainty

In general, additive uncertainty bounds are obtained experimentally as several sources of prediction mismatch are lumped together. However, some kinds of uncertainties, such as time-varying delay, are particularly important in order that they can be mathematically studied. This fact is illustrated by considering a continuous-time dead-time uncertainty, which may appear either because of a time-varying delay or from a simple dead-time estimation error.

Similarly to Lombardi *et al.* [17], consider the following continuous-time system

$$\dot{x}(t) = A_c x(t) + B_c u(t - L - \epsilon(t)) \tag{7}$$

where L is the nominal dead-time, $\epsilon(t)$ is the dead-time uncertainty, $x(t) \in \mathbb{R}^n$ represents the continuous-time state vector and $u(t) \in \mathbb{R}^m$ is the continuous-time control vector, obtained from a zero-order hold with with a sampling period T_s as follows

$$u(t) = u(k), \quad \forall t \in [kT_s, \ (k+1)T_s)$$

Without loss of generality, it is supposed that $\epsilon(t) > 0$ and $L = d \cdot T_s$. Moreover, for presentation simplicity, it is supposed that $\epsilon(t) < T_s$, but the most general case can be found in [17].

It is well known that in the nominal case ($\epsilon(t) = 0$), the model is given by

$$x(k+1) = Ax(k) + Bu(k-d)$$

where $A = e^{A_c T_s}$ and $B = \int_0^{T_s} e^{A_c(T_s - \theta)}\,d\theta B_c$. However, in the presence of dead-time uncertainty, the discrete-time system becomes

$$\begin{aligned} x(k+1) = {} & e^{A_c T_s} x(k) + \int_0^{\epsilon(t)} e^{A_c(T_s-\theta)}\,d\theta B_c u(k-d-1) \\ & + \int_{\epsilon(t)}^{T_s} e^{A_c(T_s-\theta)}\,d\theta B_c u(k-d) \\ = {} & e^{A_c T_s} x(k) + \int_0^{\epsilon(t)} e^{A_c(T_s-\theta)}\,d\theta B_c u(k-d-1) \\ & + \int_0^{T_s} e^{A_c(T_s-\theta)}\,d\theta B_c u(k-d) \\ & - \int_0^{\epsilon(t)} e^{A_c(T_s-\theta)}\,d\theta B_c u(k-d) \\ = {} & Ax(k) + Bu(k-d) \\ & + \underbrace{\int_0^{\epsilon(t)} e^{A_c(T_s-\theta)}\,d\theta B_c (u(k-d-1) - u(k-d))}_{w(k)} \end{aligned} \tag{8}$$

Therefore dead-time estimation error or time-varying delay should be treated as a bounded additive uncertainty since $u(k-d) \in \mathbb{U}$ and $u(k-d-1) \in \mathbb{U}$ with

$$w(k) = -\Gamma(\epsilon(t))[u(k-d) - u(k-d-1)]$$

where $\Gamma(\epsilon(t)) = \int_0^{\epsilon(t)} e^{A_c(T_s-\theta)}\,d\theta B_c$.

It is important to remark that the sampling period can be chosen by using any rule of thumb presented in control systems textbook as Åström and Wittenmark [29]. In practice, a smaller sampling period reduces inter-sampling interval, increases discrete dead-time length (d) and reduces the magnitude of the additive disturbance ($w(k)$). As a consequence, a smaller sampling period tends to improve control performance as the inter-sampling behaviour is reduced. However, the main advantage of DTC comes from the fact that controller design is performed by using a model without dead-time, so that T_s can be as small as necessary in order to achieve a desired performance, keeping control design simplicity.

3 Main results

The overall control system is depicted in Fig. 1, where state estimation and DTC block is computed as follows

$$\tilde{x}(k) = A^d \hat{x}(k) + \sum_{j=1}^{d} [A^{j-1} Bu(k-j)] \tag{9}$$

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k-d) + L(y(k) - C\hat{x}(k)) \tag{10}$$

Note that $\hat{x}(k)$ is obtained from (10) at $k-1$. In this case, $\tilde{x}(k) \triangleq \hat{x}(k+d|k)$ is the output of the overall system composed by a cascade connection of (1) and (9), (10) as depicted in Fig. 1.

As shown in the preliminary discussion, in presence of disturbances and/or estimation error $\tilde{x}(k) \neq x(k+d)$, but $u(k) = \kappa(\tilde{x}(k))$. Hence, disturbances and estimation error effect should be considered through an equivalent overall representation given by

$$\begin{aligned} \tilde{x}(k+1) &= A\tilde{x}(k) + Bu(k) + \tilde{w}(k) \\ y(k) &= C\tilde{x}(k-d) + Ce(k) + v(k) \\ e(k) &\triangleq x(k) - \tilde{x}(k-d) \end{aligned} \tag{11}$$

where $\tilde{w}(k)$ is a disturbance of whole system and $e(k)$ is the prediction error which should be related with $w(k)$, $v(k)$ and $\delta(k)$.

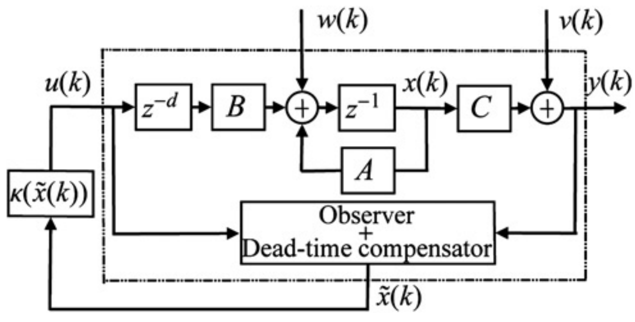

Fig. 1 *Overall control scheme*

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3.1 Overall disturbance analysis

The overall disturbance $\tilde{w}(k)$ can be computed from (11) and (9) as function of $\hat{x}(k)$ yielding

$$\begin{aligned}\tilde{w}(k) &= \tilde{x}(k+1) - A\tilde{x}(k) - Bu(k) \\ &= A^d\hat{x}(k+1) + \sum_{j=1}^{d}[A^{j-1}Bu(k-j+1)] \\ &\quad - A\left\{A^d\hat{x}(k) + \sum_{j=1}^{d}[A^{j-1}Bu(k-j)]\right\} - Bu(k) \\ &= A^d[\hat{x}(k+1) - A\hat{x}(k) - Bu(k-d)]\end{aligned} \tag{12}$$

Then, by using the Luenberger observer given by (10), expression (12) may be rewritten as following:

$$\tilde{w}(k) = A^d[L(y(k) - C\hat{x}(k))] = A^dL(C\delta(k) + v(k)) \tag{13}$$

This result is equivalent to the one presented in [18] for $d = 0$.

3.2 Prediction error

In the prediction error case, it is considered the deviation between the real state and its prediction, used for control purposes and given by

$$e(k) = x(k) - \tilde{x}(k-d) \tag{14}$$

By considering (9) at $k - d$, the following equation is obtained

$$\tilde{x}(k-d) = A^d\hat{x}(k-d) + \sum_{j=1}^{d}[A^{j-1}Bu(k-j-d)] \tag{15}$$

For the real system, $x(k)$ can be expressed as a function of $x(k - d)$, running (1) from $k - d$ until k recursively, which gives

$$\begin{aligned}x(k) &= A^dx(k-d) + \sum_{j=1}^{d}[A^{j-1}Bu(k-j-d)] \\ &\quad + \sum_{j=1}^{d}[A^{j-1}w(k-j)]\end{aligned} \tag{16}$$

Finally, replacing (15) and (16) in (14) yields

$$e(k) = A^d\delta(k-d) + \sum_{j=1}^{d}[A^{j-1}w(k-j)] \tag{17}$$

This expression relates the actual state value with the predicted (expected) one.

3.3 Bounding disturbance effect

As it was already pointed out, if the estimator is stable, it can be defined an overall disturbance set $\tilde{\mathbb{W}}$ such that

$$\tilde{w}(k) \in \tilde{\mathbb{W}}, \quad \forall v(k) \in \mathbb{V}, \quad \forall w(k) \in \mathbb{W}, \quad \forall k$$

In a similar way, a prediction error set $\mathbb{E}$ can be defined by the following statement

$$e(k) \in \mathbb{E}, \quad \forall v(k) \in \mathbb{V}, \quad \forall w(k) \in \mathbb{W}, \quad \forall k$$

By using (16) and (17) and the mrpi idea, briefly discussed in Section 2, outer bounds can be derived for prediction error and overall disturbances as follows

$$\tilde{w}(k) \in A^dL(C\overline{\Delta} \oplus \mathbb{V})$$

$$e(k) \in \mathbb{W} \oplus A\mathbb{W} \oplus \cdots \oplus A^{d-1}\mathbb{W} \oplus A^d\overline{\Delta}$$

where $\overline{\Delta}$ is an invariant outer approximation for $\mathbb{W} \oplus -L\mathbb{V}$. It is important to remark that the initial estimation assumption, $\hat{x}(0) - x(0) \in \overline{\Delta}$, is not a hard condition since it can be ensured by running the observer in an open-loop fashion before the controller startup.

Owing to the equivalence between the predicted representation and the real system, it is possible to guarantee robust stability of the real system by using a control strategy based in a dead-time free model. ISS stability theory [30, 31], which is briefly presented in the appendix, will be used to formalise this idea as presented in the following theorem.

Theorem 1:

(i) Let the system

$$\begin{aligned}x(k+1) &= Ax(k) + Bu(k-d) + w(k) \\ y(k) &= Cx(k) + v(k)\end{aligned}$$

with $w(k) \in \mathbb{W}$, be controlled by

$$u(k) = \kappa(\tilde{x}(k))$$

$$\tilde{x}(k) = A^d\hat{x}(k) + \sum_{j=1}^{d}[A^{j-1}Bu(k-j)]$$

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k-d) + L(y(k) - C\hat{x}(k))$$

and all eigenvalues of the matrix $(A - LC)$ are strictly inside the unitary circle.

(ii) Let the sets $\tilde{\mathbb{W}}$ and $\mathbb{E}$ be defined such that $\forall k > 0$, $[\tilde{x}(k+1) - (A\tilde{x}(k) + Bu(k))] \in \tilde{\mathbb{W}}$ and, $\forall k > d$, $(x(k) - \tilde{x}(k-d)) \in \mathbb{E}$.

(iii) Let the system

$$z(k+1) = Az(k) + B\kappa(z(k)) + w_z(k)$$

be ISS for all $w_z(k)$ satisfying $w_z(k) \in \tilde{\mathbb{W}}$.

Then:

(a) System (i) is ISS.
(b) If $\tilde{x}(k) \in \mathbb{X} \ominus \mathbb{E}$, $\forall k \geq 0$, then $x(k) \in \mathbb{X}$, $\forall k \geq d$.

Proof:

(a) For presentation simplicity, the proof for part 'a' is presented in the appendix.
(b) Owing to ISS guarantee, $\tilde{x}(k) \in \tilde{\mathbb{X}}$, implying that $x(k+d) \in \tilde{\mathbb{X}} \oplus \mathbb{E}$ since $x(k+d) = \tilde{x}(k) + e(k+d) \Rightarrow$

$x(k+d) \in \tilde{x}(k) \oplus \mathbb{E}$. In a similar way, as $x(k+d) - \tilde{x}(k) \in \mathbb{E}$, it can be directly concluded from Pontryagin set difference definition that if $\tilde{x}(k) \in \mathbb{X} \ominus \mathbb{E}$, then $x(k+d) \in \mathbb{X}$. □

This theorem has three interesting properties: (a) state and output constraints can be guaranteed by imposing tighter constraints over $\tilde{x}(k)$; (b) control constraints can be guaranteed by defining a suitable control law $\kappa(\cdot)$; and (c) ISS guarantee of a dead-time system with non-measurable states is converted into dead-time free problem with known states. Note that $z(k+1) = Az(k) + B\kappa(z(k)) + w_z(k)$ is an auxiliary dead-time free system used to define the stabilising control law $\kappa(\cdot)$ in order that if $w_z(k) = \tilde{w}(k)$, then $z(k) = \tilde{x}(k)$, for all $k \geq 0$.

4 Dead-time free strategies

In this section, DTC scheme will be applied to a linear controller and a tube-based MPC strategy by using the ideas of Theorem 1. As indicated in Section 2, the main idea is to derive a constrained control law for a system with dead-time and non-measurable states based on a dead-time free model.

Firstly, in order to ensure conditions (ii) of Theorem 1, it is used the idea of Mayne *et al.* [18], which was already revisited in Section 2. Thus, the bounds for $\tilde{w}$ and $e(k)$ are

$$\tilde{\mathbb{W}} = A^d L(C\overline{\Delta} \oplus \mathbb{V})$$

$$\mathbb{E} = \mathbb{W} \oplus A\mathbb{W} \oplus \cdots \oplus A^{d-1}\mathbb{W} \oplus A^d\overline{\Delta}$$

The next step is to define a control law $u(k) = \kappa(z(k))$ in order to guarantee condition (iii) of Theorem 1.

4.1 Linear controller

The first motivating example is a simple controller in the form $\kappa(z(k)) = Kz(k)$, which is based on a robust linear feedback gain presented in [32]. Without loss of generality, consider that the sets $\mathbb{U}$ and $\mathbb{X} \ominus \mathbb{E}$ are expressed as follows: $\mathbb{U} = \{u: |l_j u| < 1, j = 1, \ldots, n_{ru}\}$ and $\mathbb{X} \ominus \mathbb{E} = \{z: |h_i z| < 1, i = 1, \ldots, n_{rx}\}$. Constraint satisfaction and ISS problem is posed in terms of the following convex optimisation problem

$$\min_{Y,Q} - \log(\det(Q)) \tag{18}$$

s.t.

$$\begin{bmatrix} \lambda Q & * & * \\ 0 & 1-\lambda & * \\ AQ + BY & w_v & Q \end{bmatrix} > 0, \quad \forall w_v \in \text{vert}(\tilde{\mathbb{W}}) \tag{19}$$

$$\begin{bmatrix} 1 & * \\ Y' & l_j Q \end{bmatrix} > 0, \; j = 1, \ldots, n_{ru} \tag{20}$$

$$\begin{bmatrix} 1 & * \\ Qh_i & Q \end{bmatrix} > 0, \; i = 1, \ldots, n_{rx} \tag{21}$$

for a given $\lambda \geq 0$. If this optimisation problem is feasible, $\tilde{x}'Q^{-1}\tilde{x} < 1$ is a robust admissible invariant set for the system $z(k+1) = Az(k) + Bu(k) + \tilde{w}_z(k)$ controlled by $u(k) = YQ^{-1}\tilde{x}(k)$, under the constrains $x(k) \in \mathbb{X} \ominus \mathbb{E}$ and $u(k) \in \mathbb{U}$, and subject to the disturbance $\tilde{w}_z(k) \in \tilde{W}$.

The linear matrix inequality (LMI) (19) is used to guarantee robust invariance [32]. Control constraint satisfaction is ensured by the LMI (20) as in [33], state constraint satisfaction is ensured by the LMI (21) as in [33] and the criterion (18) is applied to obtain the largest ellipsoid (in volume) contained in the constraints polytope [33]. Note that other performance specifications could be added in the form of LMIs. Finally, from Theorem 1, it can be verified that original system (1) is ISS and guarantees robust constraint satisfaction.

4.2 Tube-based MPC

To obtain a larger domain of attraction and to improve performance, the proposed DTC scheme can be applied to robust MPCs strategies. Consider the strategy based on a dead-time free process proposed in [26]. For the dead-time case, it is just necessary to use a modified disturbance bound $\tilde{\mathbb{W}}$ instead of $\mathbb{W}$ and the tighter constraint $\mathbb{X} \ominus \mathbb{E}$ instead of $\mathbb{X}$. Thus, an output feedback tube-based MPC for dead-time system with tighter constraints, $\overline{\mathbb{X}} \triangleq \mathbb{X} \ominus \mathbb{E}$ and $\overline{\mathbb{U}} \triangleq \mathbb{U} \ominus K\mathcal{Z}$, may be given by

$$\min_{\bar{x}_0, \bar{u}_i} \quad V_N(\tilde{x}(k); \bar{x}, \bar{\mathbf{u}}) = \sum_{i=0}^{N-1} \ell(\bar{x}_i, \bar{u}_i) + V_f(\bar{x}_N) \tag{22}$$

s.t.

$$\begin{aligned} &\bar{x}_0 \in \tilde{x}(k) \oplus (-\mathcal{Z}) \\ &\bar{x}_{i+1} = A\bar{x}_i + Bu_i, \quad i = 0, \quad 1, \ldots, N-1 \\ &(\bar{x}_i, \bar{u}_i) \in \overline{\mathbb{X}} \times \overline{\mathbb{U}}, \quad i = 0, \quad 1, \ldots, N-1 \\ &\bar{x}_N \in X_f \end{aligned}$$

where (i) $\tilde{x}(k)$ is given by (9), (10); (ii) K and K_f are stabilising gains such that all the eigenvalues or $A + BK$ and $A + BK_f$ are strictly inside the unitary circle; (iii) $\mathcal{Z}$ is an outer approximation of an mrpi respecting $(A + BK)\mathcal{Z} \oplus \tilde{W} \subseteq \mathcal{Z}$; (iv) $\overline{\mathbb{X}}$ and $\overline{\mathbb{U}}$ are non-empty sets; (v) X_f is an admissible invariant set which fulfils $(A + BK_f)X_f \subset X_f$, $X_f \subset \overline{\mathbb{X}}$, $U_f \subset K_f\overline{\mathbb{U}}$; (vi) $V_f((A + BK_f)x) + \ell(x, K_f x) \leq V_f(x)$, $\forall x \in X_f$; and (vii) $u(k) = \kappa(\tilde{x}(k)) = \bar{u}_0^*(k) + K(\bar{x}_0^*(k) - \tilde{x}(k))$ with $\bar{u}_0^*(k)$ and $\bar{x}_0^*(k)$ being the arguments which minimise (22) at k.

The matrix K characterises the dynamics of the closed-loop system in the presence of disturbances. It can be defined in order to minimise the set $\mathcal{Z}$, ensuring condition (iv), by applying some convex optimisation techniques presented in [32]. The control gain K_f, the terminal set and the terminal cost can be obtained by considering the optimal unconstrained control law [26].

The proposed predictive controller requires the solution of a quadratic programming (QP) problem at each sampling time. Therefore in order to implement the controller, an algorithm to solve the QP problem within the sampling is compulsory. There exists QP solvers tailored to efficiently solve the QP problems derived from predictive controllers with a computation time that depends linearly on the prediction horizon [34]. Other possibility is the off-line solution of the QP problem by using the multi-parametric programme tools [35].

5 Simulation example

A double integrator system similar to the one presented in [18] is used to illustrate the effect of the proposed DTC in

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the two controllers of Section 4

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k-d) + w(k)$$
$$y(k) = [1 \ 1]x(k) + v(k)$$

where $w(k) \in \mathcal{W} \triangleq \{w{:}\|w\|_\infty \leq 0.1\}$, $v(k) \in \mathcal{V} \triangleq \{v{:}|v| \leq 0.05\}$ and the dead-time is set to $d = 2$ or $d = 3$ in order to show the difficulties associated to its effect. The state constraints are supposed to be $x(k) \in \mathbb{X} \triangleq \{x{:}x = [x_1 \ x_2]', -40 \leq x_1 \leq 5, -40 \leq x_2 \leq 5\}$ and the control constraints are $u \in \mathbb{U} \triangleq \{u{:}|u| \leq 5\}$. In the linear control strategy, it is used $\lambda = 1$. For the MPC case, it is used $\ell(\bar{x}_i, \bar{u}_i) = 0.5(x_i'Qx_i + u_i'Ru_i)$, $V_f(x_N) = 0.5x_N'Px_N$, where $Q = I$, $R = 0.01$ and K_f and P are stabilising elements related with the optimal unconstrained controller for (A, B, Q, R). Furthermore $K = [1 \ 1]$, $L = [-1 \ -1]'$ and $N = 20$.

Initial conditions are $x(0) = [10 \ \ -2.5]$ for the linear controller and $x(0) = [10 \ \ -5]$ for the predictive controller in order to explore its larger domain of attraction. Additive disturbances are the same in all simulations, but they were randomly set to be in their extremes. The first control action ($u(k) \neq 0$) is computed after $k = 2$ in order that state estimations can converge to their bounds. In this case, it is applied $u(j) = 0$, $j = -d, -d+1, \ldots, 1$.

Simulations for the linear and the predictive control strategies are shown in Figs. 2 and 3, respectively, where Ω represents a robust invariant region for the predicted estimation, $\tilde{x}(k)$. These figures are interesting to illustrate robust state constraint satisfaction, which is one of the major advantages of the proposed approach. The shadowed zone represents the set $\tilde{x}(k) \oplus \mathbb{E}$, which is a confident set for the future real state $x(k+d)$. Even though $x(k+d)$ is not known at k because of estate estimation and prediction, it always lies inside the shadowed zone, $x(k+d) \in \tilde{x}(k) \oplus \mathbb{E}$ as expected from Theorem 1 proof. This can be directly verified from Figs. 2 and 3. Hence, imposing tighter constraints over $\tilde{x}(k)$, namely $\mathbb{X} \ominus \mathbb{E}$, it is guaranteed that $\tilde{x}(k+d) \in \mathbb{X}$. Moreover, as $\tilde{x}(k)$ is inside its invariant region, limited by dotted line, then $x(k+d)$ should lies inside $\Omega \oplus \mathbb{E}$, represented by the solid line. In other words, constraints satisfaction for $\tilde{x}(k)$ is used to guarantee robust constraint satisfaction for $x(k+d)$.

This approach is useful because dead-time effect can be easily evaluated from the estimation error bound. It can be noticed in both strategies, a simple dead-time can reduce considerably the invariant region depicted by the dotted line. Indeed, as could be expected, a longer dead-time affects both prediction error and overall disturbance effect, reducing the invariant region. It is important to note that constraint satisfaction can be guaranteed from $k+d$. As a consequence, if the control algorithm starts at $k = 0$, state

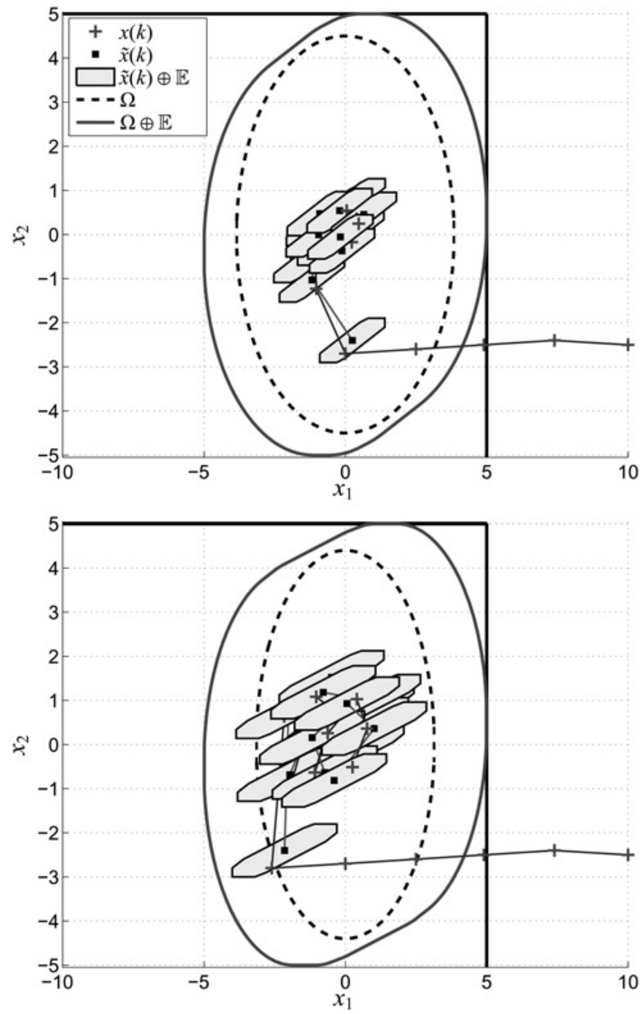

Fig. 2 *State evolution with a linear controller: d = 2 top and d = 3 bottom*

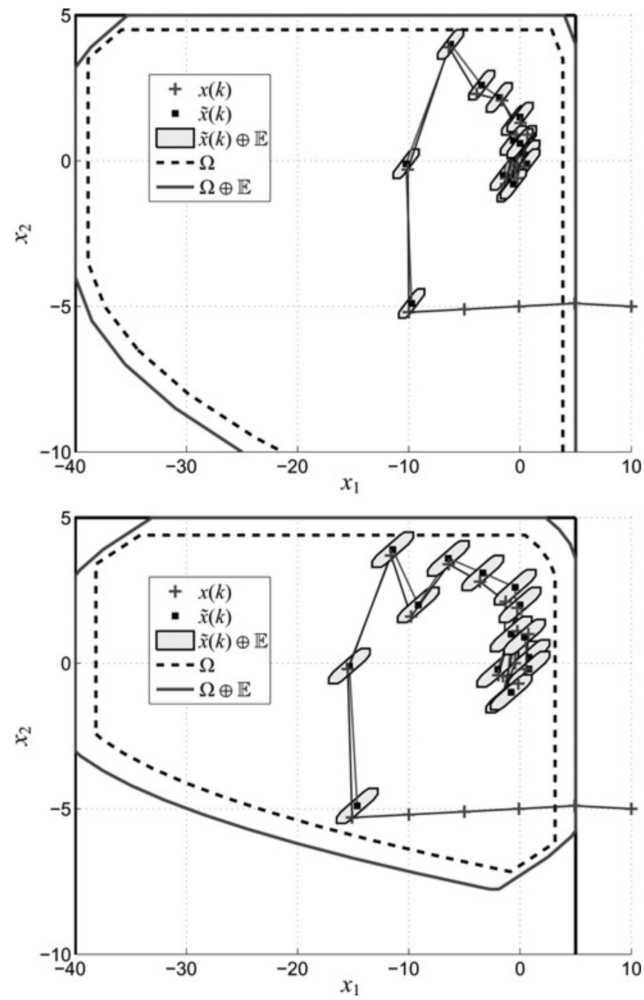

Fig. 3 *State evolution with a tube-base MPC: d = 2 top and d = 3 bottom*

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constraint satisfaction can be evaluated, but it cannot be forced because it depends on $u(-d), \ldots, u(-1)$ which were previously applied. This fact is a consequence of dead-time effect and cannot be avoided.

Another interesting issue is that the predictive scheme presents a significantly larger robust invariant set. Nevertheless, other schemes such as linear laws or anti-windup strategies are interesting alternatives because of their simplicity once it is possible to consider other convex specifications that are not in the scope of this work.

6 Conclusion

The problem of state estimation and DTC for constrained system was analysed by means of an equivalent dead-time free representation with measurable states. Compensation and state estimation effect were lumped together in a modified disturbance which is related to the real one in order to obtain a simplified equivalent control problem. This result was applied to a linear controller and a tube-based MPC, allowing the use of these strategies directly to control constrained dead-time systems. As a topic for future work, it would be interesting to apply a similar analysis in other kinds of uncertainty representation.

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9 Appendix

9.1 Appendix 1: ISS definition

The ISS definition, presented in [30], will be briefly revisited in this appendix. In this case, is necessary to recall some function definitions. A function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{KL}$-function if for a fixed $t \geq 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s, \cdot) \to 0$ as $t \to \infty$. For a bounded signal $w_z(k) \in \mathbb{W}_z$, its finite

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sequence is denoted by $\mathbf{w}_{\mathbf{z}[k-1]} \triangleq \{w_z(0), w_z(1), \ldots, w_z(k-1)\}$; and its standard l_∞ norm is represented by $\|\mathbf{w}_{\mathbf{z}[k-1]}\|_\infty \triangleq \sup_{k\geq 0}\{\|w_z(k)\|\}$.

It will be considered a system given by

$$z(k+1) = f(z(k), w_z(k)) \tag{23}$$

which can be used to represent a linear system controlled by a non-linear strategy $u(k) = \kappa(z(k))$ as

$$f(z(k), w_z(k)) = Az(k) + B\kappa(z(k)) + B_w w_z(k)$$

Note that state evolution until k depends only on $z(0)$ and $\mathbf{w}_{\mathbf{z}_{[k-1]}}$ because of causality. The solution of system (23) at sampling time k for the initial state $z(0)$ and a sequence of disturbances $\mathbf{w}_{\mathbf{z}_{[k-1]}}$ is denoted by $\phi(k, z(0), \mathbf{w}_{\mathbf{z}_{[k-1]}})$, where $\phi(0, z(0), \mathbf{w}_{\mathbf{z}_{[-1]}}) = z(0)$.

Definition 1: System (23) is (globally) input-to-state stable if there exist a $\mathcal{KL}$-function β and a $\mathcal{K}$-function γ, such that, for each sequence $\mathbf{w}$ and each initial condition $z(0)$, the following holds

$$\|\phi(k, z(0), \mathbf{w}_{\mathbf{z}_{[k-1]}})\| \leq \beta(\|z(0)\|, k) + \gamma(\|\mathbf{w}_{\mathbf{z}_{[k-1]}}\|_\infty)$$

for each $k \geq 0$.

In some cases, robustness can only be ensured in a neighbourhood of the origin and/or for small enough uncertainties. This problem can also be analysed within the ISS framework by means of the local ISS notion.

Definition 2: System (23) is locally input-to-state stable if there exist constants c_1 and c_2, a $\mathcal{KL}$-{function} β and a $\mathcal{K}$-{function} γ, such that, for each sequence $\mathbf{w}$ with $\|w_z(k)\| \leq c_1$ and each initial condition $z(0)$ with $\|z(0)\| \leq c_2$, the following holds

$$\|\phi(k, z(0), \mathbf{w}_{\mathbf{z}_{[k-1]}})\| \leq \beta(\|z(0)\|, k) + \gamma(\|\mathbf{w}_{\mathbf{z}_{[k-1]}}\|_\infty)$$

for each $k \geq 0$.

9.2 Appendix 2: ISS proof

In this section, the main goal is to prove that the original system is ISS because of stability conditions imposed to the following dead-time free auxiliary system

$$z(k+1) = Az(k) + B\kappa(z(k)) + w_z(k)$$

From the analysis of the controlled system it can be seen that the overall dynamics is given by

$$\delta(k+1) = A_L\delta(k) + w(k) - Lv(k)$$

$$\tilde{x}(k+1) = A\tilde{x}(k) + B\kappa(\tilde{x}(k)) + A^d LC\delta(k) + A^d Lv(k)$$

Given that the dynamics of $\delta(k)$ does not depend on $x(k)$ and it is nominally asymptotically stable, that is for $w(k) = 0$ and $v(k) = 0$, then this is ISS w.r.t. the exogenous signals $w(k)$ and $v(k)$ as in [30, Example 3.4] with

$$\|\delta(k)\| \leq \beta_1(\|\delta(0)\|, k) + \gamma_w(\|\mathbf{w}_{[k-1]}\|_\infty) + \gamma_v(\|\mathbf{v}_{[k-1]}\|_\infty)$$

where $\beta_1(\|\delta(0)\|, k) = c\rho^k(\|\delta(0)\|)$, $\gamma_w(\|\mathbf{w}_{[k-1]}\|_\infty) = \frac{c}{1-\rho}\|\mathbf{w}_{[k-1]}\|_\infty$, $\gamma_v(\|\mathbf{v}_{[k-1]}\|_\infty) = \frac{c\|L\|}{1-\rho}\|\mathbf{w}_{[k-1]}\|_\infty$ and $c > 0$ and $0 \leq \rho < 1$ are constants such that $\|A_L^k\| \leq c\rho^k$.

Then, there exists a $\mathcal{KL}$-function $\beta_\delta(r, k) = 3\beta_1(r, k)$, and a couple of $\mathcal{K}$-functions $\theta_\delta(r) = 3\gamma_w(r)$ and $\lambda_\delta(r) = 3\gamma_v(r)$ such that

$$\|\delta(k)\| \leq \max\{\beta_\delta(\|\delta(0)\|, k), \theta_\delta(\|\mathbf{w}_{[k-1]}\|_\infty), \lambda_\delta(\|\mathbf{v}_{[k-1]}\|_\infty)\}$$

On the other hand, the dynamics of $\tilde{x}(k)$ is ISS w.r.t $w_z(k) = A^d LC\delta(k) + A^d Lv(k)$. Thus, in a similar way as done before, there exists a $\mathcal{KL}$-function β_x, and a couple of $\mathcal{K}$-functions γ_x, θ_x and λ_x such that

$$\|\tilde{x}(k)\| \leq \max\{\beta_x(\|\tilde{x}(0)\|, k), \gamma_x(\|\delta_{[k-1]}\|_\infty), \theta_x \times (\|\mathbf{w}_{[k-1]}\|_\infty), \lambda_x(\|\mathbf{v}_{[k-1]}\|_\infty)\}$$

Then, in virtue of [30, Theorem 2], the overall system is ISS w.r.t the signals $w(k)$ and $v(k)$. Defining the extended estate $\pi(k) = [\delta(k)', \tilde{x}(k)']'$ and the extended exogenous input $w_\pi(k) = [w(k)', v(k)']'$, there exists a $\mathcal{KL}$-function β, and a $\mathcal{K}$-function θ such that

$$\|\pi(k)\| \leq \beta(\|\pi(0)\|, k) + \theta(\|\mathbf{w}_{\boldsymbol{\pi}[k-1]}\|_\infty) \tag{24}$$

Since $e(k) = x(k) - \tilde{x}(k-d) = A^d\delta(k-d) + \sum_{j=1}^{d} A^{j-1} \times w(k-j)$ (see (17)), we have that

$$x(k+d) = \tilde{x}(k) + A^d\delta(k) + \sum_{j=1}^{d} A^{j-1}w(k-j+d)$$

Then there exist positive constants α_1 and α_2, such that

$$\|x(k+d)\| \leq \alpha_1(\|\pi(k)\|) + \alpha_2(\|\mathbf{w}_{[k,k+d-1]}\|_\infty)$$

Finally, by using (24), it is obtained

$$\begin{aligned}\|x(k+d)\| &\leq \alpha_1(\beta(\|\pi(0)\|, k) + \theta(\|\mathbf{w}_{\boldsymbol{\pi}[k-1]}\|_\infty)) \\ &\quad + \alpha_2(\|\mathbf{w}_{[k,k+d-1]}\|_\infty) \\ &\leq \beta_2(\|\pi(0)\|, k) + \theta_2(\|\mathbf{w}_{\boldsymbol{\pi}[k-1]}\|_\infty) \\ &\quad + \alpha_2(\|\mathbf{w}_{[k,k+d-1]}\|_\infty) \\ &\leq \beta_2(\|\pi(0)\|, k) + \theta_3(\|\mathbf{w}_{\boldsymbol{\pi}[k+d-1]}\|_\infty)\end{aligned} \tag{25}$$

where β_2 and θ_3 are suitable $\mathcal{KL}$ and $\mathcal{K}$-functions, respectively. This result is important because it can be verified from (25) that, if $w(k)$ and $v(k)$ approach to zero, then the vector $x(k+d)$ also approaches to zero as $k \to \infty$.