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Brief paper

Piecewise affinity of min–max MPC with bounded additive uncertainties and a quadratic criterion ☆

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Abstract

This brief shows how a min–max MPC with bounded additive uncertainties and a quadratic cost function results in a piecewise affine and continuous control law. Proofs based on properties of the cost function and the optimization problem are given. The boundaries of the regions in which the state space can be partitioned are also treated. The results are illustrated by an example.

Keywords: Min–max MPC algorithm; Predictive control; Min–max optimization; Piecewise linear controllers; Uncertainty

1. Introduction

Model predictive control (MPC) is one of the control techniques able to cope with both model uncertainties and constraints in an explicit way. There are different approaches for modelling uncertainties. The approach considered here is that of bounded additive or global uncertainties (Camacho & Bordóns, 2004); this supposes that all uncertainties can be globalized in a single vector which is added to the 1-step ahead prediction equation. When bounded uncertainties are considered explicitly, it would seem that more robust control would be obtained if the controller minimized the objective function for the worst-case situation.

Min–max MPC (MMMPC) techniques have been used to explicitly consider the effect of the uncertainty on the control law (Campo & Morari, 1987; Casavola, Giannelli, & Mosca, 2000; Veres & Norton, 1993; Lee & Yu, 1997; Kim, Kwon, & Lee, 1998). However, all of these have a great computational burden in common which limits the range of processes to which they can be applied. When the cost function is based on 1 or ∞ norms the min–max problem can be efficiently solved using linear programming techniques (Allwright & Papavasilou, 1992). In other works (Kothare, Balakrishnan, & Morari, 1996; Lu & Arkun, 2000), the computational burden is lessened by minimizing an upper bound of the worst case instead of explicitly solving a min–max problem.

MMMPC controllers can be divided into two types: open-loop and closed-loop min–max predictive controllers. In the first type, predictions are computed in an open-loop manner (although the resulting controller is a feedback controller). These controllers are based on the solution of a single min–max problem optimizing a single control policy for all possible values of the uncertainty. Closed-loop min–max predictive controllers take into account that the control law is actually applied in a feedback manner when computing the predictions. These controllers employ different strategies such as nested min–max problems (Bemporad, Borrelli, & Morari, 2003; Lee & Yu, 1997), optimization of multiple control policies (Kerrigan & Maciejowski, 2004; Scokaert & Mayne, 1998), and, more recently, feasibility constraints (Sakizlis, Kakalis, Dua, & Pistikopoulos, 2004) when minimizing the nominal or expected cost. Open-loop MMMPC is known to be very conservative, whereas closed-loop MMMPC is known to suffer from a much greater computational burden.

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Bemporad et al. (2003) have shown that both open-loop or closed-loop MMMPC with ∞-norm (or 1-norm) have a piecewise affine (PWA) nature. This fact was deduced by the use of multiparametric programming and it allows explicit solutions of such control laws. In this brief we show that the constrained MMMPC control law with a quadratic objective function is also PWA and continuous. We provide proofs based mainly on the properties of the cost function and on the optimization problem. This result can be exploited to implement this type of control law to processes with fast dynamics. The results presented in the paper can be applied to open-loop prediction MMMPC or to MMMPC using a semi-feedback strategy (Mayne, 2001). In this, some kind of feedback is introduced into the predictions because the system is pre-controlled using an inner feedback gain. This technique (Rossiter, Kouvaritakis, & Rice, 1998) is known to reduce the conservatism of open-loop predictive controllers (Bemporad, 1998; Löfberg, 2003) without having to increase the computational burden.

The brief is organized as follows: Section 2 presents the MMMPC strategy, along with some easy properties. Sections 3 and 4 deal with the continuity and PWA nature of the control law. The boundaries of the regions are treated in Section 5. Finally, the results presented in this brief are illustrated with an example in Section 6.

2. Min–max MPC with bounded additive uncertainties

Consider the following state-space model with bounded additive uncertainties (Camacho & Bordóns, 2004):

$$x(t+1) = Ax(t) + Bu(t) + D\theta(t), \quad y(t) = Cx(t) \tag{1}$$

with $x(t) \in \mathbb{R}^{dim\,x}$, $u(t) \in \mathbb{R}^{dim\,u}$, $\theta(t) \in \{\theta \in \mathbb{R}^{dim\,\theta} : \|\theta\|_\infty \leqslant \theta_m\}$, $y(t) \in \mathbb{R}^{dim\,y}$. Consider a sequence $\mathbf{u} = [u(t) \ldots u(t+N_u-1)]^{\mathrm{T}}$ of values of the control signal over a control horizon N_u and $\boldsymbol{\theta} = [\theta(t+1) \ldots \theta(t+N)]^{\mathrm{T}}$ a sequence of future values of $\theta(t)$ over a prediction horizon N. Furthermore, let $J(\boldsymbol{\theta}, \mathbf{u}, x)$ be a quadratic performance index of the form:

$$\begin{aligned} J(\boldsymbol{\theta}, \mathbf{u}, x) = {} & x(t+N|t,\boldsymbol{\theta})^{\mathrm{T}} P x(t+N|t,\boldsymbol{\theta}) \\ & + \sum_{j=1}^{N-1} x(t+j|t,\boldsymbol{\theta})^{\mathrm{T}} Q_j x(t+j|t,\boldsymbol{\theta}) \\ & + \sum_{j=0}^{N_u-1} u(t+j)^{\mathrm{T}} L_j u(t+j), \end{aligned} \tag{2}$$

where $x(t+j|t,\boldsymbol{\theta})$ is the prediction of the state for $t+j$ made at t when the future values of the uncertainty are supposed to be given by the sequence $\boldsymbol{\theta}$. When $N_u < N$ it is assumed that the control signal is constant and equal to $u(N_u-1)$ for $j = N_u, \ldots, N$. On the other hand $P, Q_j \in \mathbb{R}^{dim\,x \times dim\,x}$, $L_j \in \mathbb{R}^{dim\,u \times dim\,u}$ are symmetric positive definite matrices used as weighting parameters.

At any time, the state x and the sequence $\mathbf{u}$ must satisfy a set of nc affine constraints, such that only the pairs

$$(\mathbf{u}, x) : R_i^{\mathrm{T}}\mathbf{u} + \Gamma_i^{\mathrm{T}}\boldsymbol{\theta} \leqslant g_i + F_i^{\mathrm{T}} x, \quad i = 1, \ldots, nc \;\; \forall \boldsymbol{\theta} \in \boldsymbol{\Theta} \tag{3}$$

are admissible, where $\boldsymbol{\Theta} = \{\boldsymbol{\theta} \in \mathbb{R}^{N\cdot dim\,\theta} : \|\boldsymbol{\theta}\|_\infty \leqslant \theta_m\}$, $R_i \in \mathbb{R}^{(N_u \cdot dim\,u)}$, $F_i \in \mathbb{R}^{dim\,x}$, $\Gamma_i \in \mathbb{R}^{(N\cdot dim\,\theta)}$ and $g_i \in \mathbb{R}$. These constraints may arise from operational constraints or be used to guarantee stability. Note that

$$\max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \Gamma_i^{\mathrm{T}} \boldsymbol{\theta} = \max_{\|\boldsymbol{\theta}\| \leqslant \theta_m} \Gamma_i^{\mathrm{T}} \boldsymbol{\theta} = \theta_m \|\Gamma_i\|_1,$$

where $\|\Gamma_i\|_1$ is the 1-norm of Γ_i, i.e., the sum of the absolute value of its components. Thus, the robust fulfillment of the constraints (3) is satisfied if and only if $R_i^{\mathrm{T}}\mathbf{u} + \theta_m\|\Gamma_i\|_1 \leqslant g_i + F_i^{\mathrm{T}} x$, $i = 1, \ldots, nc$ (Alamo, Muñoz de la Peña, Limón Marruedo, & Camacho, 2005a). Therefore, robust constraint satisfaction of (3) is guaranteed by considering the following set of affine constraints:

$$R\mathbf{u} \leqslant c_\theta + Fx, \tag{4}$$

where matrices $R \in \mathbb{R}^{nc \times (N_u \cdot dim\,u)}$ and $F \in \mathbb{R}^{nc \times dim\,x}$ are composed of the row vectors R_i^{T} and F_i^{T} and the ith component of vector $c_\theta \in \mathbb{R}^{nc}$ is given by $g_i - \theta_m \|\Gamma_i\|_1$.

MMMPC (Campo & Morari, 1987) is based on finding the control correction sequence $\mathbf{u}$ that minimizes $J(\boldsymbol{\theta}, \mathbf{u}, x)$ for the worst possible case of the predicted future evolution of the process state or output signal. This is accomplished by the solution of a min–max problem such as

$$\begin{aligned} \mathbf{u}^*(x) = \operatorname*{argmin}_{\mathbf{u} \in \mathbf{U}} \quad & J^*(\mathbf{u}, x) \\ \text{s.t.} \quad & R\mathbf{u} \leqslant c_\theta + Fx, \end{aligned} \tag{5}$$

with

$$J^*(\mathbf{u}, x) = \max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} J(\boldsymbol{\theta}, \mathbf{u}, x),$$

where $\mathbf{U} \subseteq \mathbb{R}^{N_u \cdot dim\,u}$ is compact. Of all possible values of x only those feasible are considered: that is, those belonging to

$$K^* \triangleq \{x \in \mathbb{R}^{dim\,x} : \exists \mathbf{u} \in \mathbf{U},\, R\mathbf{u} \leqslant c_\theta + Fx\}. \tag{6}$$

The solution of problem (5) is applied in a feedback manner using a receding horizon strategy (Camacho & Bordóns, 2004). Note that $J^*(\mathbf{u}, x)$ is the pointwise maximum of a set of an infinite number of quadratic cost functions of $\mathbf{u}$ and x. Thus, $J^*(\mathbf{u}, x)$ is a piecewise quadratic function of $\mathbf{u}$ and x. Note that a polytopic terminal constraint devised to provide robust stability can also be included within the constraints $R\mathbf{u} \leqslant c_\theta + Fx$ (see Mayne, Rawlings, Rao, & Scokaert, 2000 and references therein). Also a stabilizing terminal cost can also be considered via a proper choice for matrix P (Mayne et al., 2000).

Problem (5) is of the open-loop type. However, the results presented in this paper are also valid when using a semi-feedback approach (Mayne, 2001) in which the control input is given by $u(t) = -Kx(t) + v(t)$ where the feedback matrix K is chosen to achieve a certain desired property such as nominal stability or LQR optimality. The MMMPC controller will compute the optimal sequence of correction control inputs $v(t)$. Rewriting (1) as

$$x(t+1) = A_{CL}x(t) + Bv(t) + D\theta(t) \tag{7}$$

with $A_{CL} = (A - BK)$ it is clear that such semi-feedback MMMPC can be expressed as (5).

With prediction model (1) the prediction equation can be written (Camacho & Bordóns, 2004) as

$$\mathbf{x} = G_u\mathbf{u} + G_\theta\boldsymbol{\theta} + F_x x(t), \tag{8}$$

where $\mathbf{x} \in \mathbb{R}^{N\cdot dim\, x}$ are the predictions of process state over the prediction horizon, $G_u \in \mathbb{R}^{(N\cdot dim\, x)\times(N_u\cdot dim\, u)}$, $G_\theta \in \mathbb{R}^{(N\cdot dim\, x)\times(N\cdot dim\, x)}$ and $F_x \in \mathbb{R}^{(N\cdot dim\, x)\times dim\, x}$. Taking into account (8) the cost function can be rewritten as

$$\begin{aligned} J(\boldsymbol{\theta}, \mathbf{u}, x) = {} & \mathbf{u}^{\mathrm{T}} M_{uu}\mathbf{u} + \boldsymbol{\theta}^{\mathrm{T}} M_{\theta\theta}\boldsymbol{\theta} + 2\boldsymbol{\theta}^{\mathrm{T}} M_{\theta u}\mathbf{u} \\ & + 2x^{\mathrm{T}} M_{uf}^{\mathrm{T}}\mathbf{u} + 2x^{\mathrm{T}} M_{\theta f}^{\mathrm{T}}\boldsymbol{\theta} + x^{\mathrm{T}} M_{ff} x \end{aligned} \tag{9}$$

where $M_{uu} = G_u^{\mathrm{T}} Q G_u + L$, $M_{\theta\theta} = G_\theta^{\mathrm{T}} Q G_\theta$, $M_{ff} = F_x^{\mathrm{T}} Q F_x$, $M_{\theta u} = G_\theta^{\mathrm{T}} Q G_u$, $M_{uf} = G_u^{\mathrm{T}} Q F_x$, $M_{\theta f} = G_\theta^{\mathrm{T}} Q F_x$ and Q and L are symetric positive definite block diagonal matrices, i.e., $Q = \mathrm{diag}(Q_1, \ldots, Q_{N-1}, P)$ and $L = \mathrm{diag}(L_1, \ldots, L_{N_u})$. Note that by construction $M_{\theta\theta}$ and M_{ff} are Gram matrices and, therefore, at least positive semi-definite. On the other hand, M_{uu} is positive definite as $L > 0$. Thus, $J(\boldsymbol{\theta}, \mathbf{u}, x)$ is convex and continuous on $x, \boldsymbol{\theta}$, and strictly convex on $\mathbf{u}$. Moreover, $J^*(\mathbf{u}, x)$ is strictly convex on $\mathbf{u}$ and convex on x as it is the pointwise maximum of a set of an infinite number of (strictly) convex functions (Boyd & Vandenberghe, 2004). Thus, the solution to the min–max problem is unique. Furthermore, by the *gluing lemma* (Munkres, 2000), $J^*(\mathbf{u}, x)$ is also continuous. Also, due to convexity on $\boldsymbol{\theta}$, the maximum of J will be reached at least one (or more) of the vertices of the polytope $\boldsymbol{\Theta}$ (see Bazaraa & Shetty, 1979, Theorem 3.4.6). Thus, $J^*(\mathbf{u}, x)$ can be computed as the pointwise maximum of every $J(\boldsymbol{\theta}_i, \mathbf{u}, x)$ for a given$x$, i.e, $J^*(\mathbf{u}, x) = \max\{J(\boldsymbol{\theta}_1, \mathbf{u}, x), \ldots, J(\boldsymbol{\theta}_{2^N}, \mathbf{u}, x)\}$ where $\boldsymbol{\theta}_i \in \mathrm{vert}\{\boldsymbol{\Theta}\}$, and $\mathrm{vert}\{\boldsymbol{\Theta}\}$ is the set of vertices of $\boldsymbol{\Theta}$.

3. Continuity of the control law

The continuity of the control law will be shown here using well-known results from point-to-set map theory (Hogan, 1973) (see Appendix A). Problem (5) is of the form

$$v(x) = \min_{\mathbf{u}\in\Omega(x)} f(\mathbf{u}, x), \tag{10}$$

where $f : \mathbf{U} \times K^* \longrightarrow [-\infty, +\infty] \equiv J^*$ and $\Omega(x)$ is a point-to-set map defined by $\Omega(x) \triangleq \{\mathbf{u} \in \mathbf{U} : R\mathbf{u} \leqslant c_\theta + Fx\}$. On the other hand, the set of minimizers for x is given by the point-to-set map $M(x) = \{\mathbf{u} \in \Omega(x) : v(x) \geqslant f(\mathbf{u}, x)\}$. To prove the continuity of the control law, it has to be proven that for (5) $M(x)$ is continuous on K^*. First, it is necessary to prove that K^* is a connected set and that $\Omega(x)$ is continuous on K^*.

Proposition 1. *The feasible set K^* is convex and therefore connected.*

Proof. Let $x_1, x_2 \in K^*$ be two feasible states and $\mathbf{u}_1^*$,$\mathbf{u}_2^*$ the solutions of (5) for x_1, x_2. Let $\alpha \in [0, 1]$ and define $\mathbf{u}_\alpha^* \triangleq \alpha\mathbf{u}_1^* + (1-\alpha)\mathbf{u}_2^*$, $x_\alpha \triangleq \alpha x_1 + (1-\alpha)x_2$. By feasibility, $\mathbf{u}_1^*$, $\mathbf{u}_2^*$ satisfy the constraints, thus $R\mathbf{u}_1^* \leqslant c_\theta + Fx_1$ and $R\mathbf{u}_2^* \leqslant c_\theta + Fx_2$. Multiply by α and $(1-\alpha)$, respectively, and add to obtain $R\mathbf{u}_\alpha^* \leqslant c_\theta + Fx_\alpha$. Thus, $\mathbf{u}_\alpha^*$ is feasible, and therefore problem (5) has a solution for $x = x_\alpha$. This implies that K^* is convex and therefore connected. □

Proposition 2. *The point-to-set map $\Omega(x)$ is continuous on K^*.*

Proof. A point-to-set map is continuous if it is both open and closed (Hogan, 1973). From Theorem 12 (see Appendix A), Ω is closed if $\forall x \in K^*$ each inequality of $\Omega(x)$ is lower semi-continuous on $K^* \times \mathbf{U}$. The affine inequalities of $\Omega(x)$ are continuous (hence, they are lower and upper semi-continuous) on $K^* \times \mathbf{U}$, thus Ω is closed on K^* . On the other hand, $\Omega(x) \subset \overline{I(x)}$, $\forall x \in K^*$, where $\overline{I(x)}$ is the closure of $I(x) \triangleq \{\mathbf{u} \in \mathbf{U} : R\mathbf{u} < c_\theta + Fx\}$. Thus, from Theorem 13 (see Appendix A), Ω is also open on K^* and, therefore, continuous. □

Theorem 3. *The control law that arises from the solution of the min–max problem (5) is continuous on K^*.*

Proof. From Propositions 1 and 2, it is known that K^* is connected and $\Omega(x)$ continuous. Also from Section 2 it is known that the solution is unique (thus, $M(x)$ is single-valued) and that $J^*(\mathbf{u}, x)$ is continuous. Finally, by assumption, U is compact, thus, from Theorem 14 (see Appendix A), M is continuous on K^* as all these conditions hold for $\forall x \in K^*$. This implies that the control law is continuous. □

4. Piecewise affinity of the control law

In this section it is shown that the control law which arises from the solution of the min–max problem is PWA on process state. Note that for a given state x the maximum is attained at a set of vertices of $\boldsymbol{\Theta}$. These vertices are said to be active, and the subset of $\mathrm{vert}\{\boldsymbol{\Theta}\}$ that contains all the active vertices is the active vertex set, which is formally defined in the following.

Definition 4. Let $\mathbf{u}^*(x)$ be the solution of the min–max problem (5) for a given $x \in K^*$. Then the active vertex set $S_I(x)$ is defined as

$$S_I(x) = \{\boldsymbol{\theta}_i \in \mathrm{vert}\{\boldsymbol{\Theta}\} : J^*(\mathbf{u}^*(x), x) = J(\boldsymbol{\theta}_i, \mathbf{u}^*(x), x)\},$$

where $\boldsymbol{\theta}_i$ is the ith vertex of $\boldsymbol{\Theta}$.

Proposition 5. *For a given $x \in K^*$ with active vertex set $S_I(x)$, there exists $K_i \in \mathbb{R}^{N_u\times dim\, x}$ and $v_i \in \mathbb{R}^{N_u}$ such that the solution of (5) for x is $\mathbf{u}^*(x) = K_i x + v_i$.*

Proof. The definition of $S_I(x)$ implies that for a given $x \in K^*$ the original min–max problem is equivalent to a reduced min–max problem where only those vertices of $\boldsymbol{\Theta}$ contained in

$S_I(x)$ are taken into account,[1] (Alamo, Ramirez, & Camacho, 2005b) i.e.,

$$\mathbf{u}^*(x) = \underset{\mathbf{u}\in\mathbf{U}}{\operatorname{argmin}} \max_{\boldsymbol{\theta}_i\in S_I(x)} J(\boldsymbol{\theta}_i, \mathbf{u}, x) \quad \text{s.t.} \quad R\mathbf{u} \leqslant c_\theta + Fx. \tag{11}$$

Note that for x, all quadratic functions $J(\boldsymbol{\theta}_i, \mathbf{u}, x)$ in problem (11) have the same value at the solution as only active vertices are considered. Thus, the solution of the min–max problem for x is the same as that of the following problem:

$$\begin{aligned} \min_{\mathbf{u}\in\mathbf{U}} \quad & J(\boldsymbol{\theta}_p, \mathbf{u}, x) \\ \text{s.t.} \quad & J(\boldsymbol{\theta}_p, \mathbf{u}, x) = J(\boldsymbol{\theta}_i, \mathbf{u}, x) \quad \forall \boldsymbol{\theta}_i \in S_I(x) \\ & R\mathbf{u} \leqslant c_\theta + Fx, \end{aligned} \tag{12}$$

where $\boldsymbol{\theta}_p \in S_I(x)$ is an active vertex. The equality constraints in (12) can be written as

$$\begin{aligned} 2(\boldsymbol{\theta}_p^{\mathrm{T}} - \boldsymbol{\theta}_i^{\mathrm{T}}) M_{\theta u}\mathbf{u} = {} & (\boldsymbol{\theta}_i^{\mathrm{T}} M_{\theta\theta}\boldsymbol{\theta}_i - \boldsymbol{\theta}_p^{\mathrm{T}} M_{\theta\theta}\boldsymbol{\theta}_p) \\ & + 2x^{\mathrm{T}} M_{\theta f}^{\mathrm{T}}(\boldsymbol{\theta}_i - \boldsymbol{\theta}_p) \quad \forall \boldsymbol{\theta}_i \in S_I(x). \end{aligned} \tag{13}$$

Thus, all constraints in (12) are affine. Affine equality constraints can be easily removed from the problem (decreasing the number of free moves). Suppose that the equality constraints are grouped into matrix–vector form as $R_{JE}\mathbf{u} = b_{JE}(x)$ and that $\mathbf{u}_0$ is any control sequence that satisfies these constraints. Then let $W_{JE} \in \mathbb{R}^{N_u \times N_{ur}}$ be a matrix whose range is the nullspace of R_{JE}, where $N_{ur} = N_u - \operatorname{Rank}(R_{JE})$ (Boyd & Vandenberghe, 2004). Then, problem (12) is equivalent to the following QP problem:

$$V(x) = k_\theta(x) + \min_{\mathbf{u_r}\in\mathbf{U_r}} \tfrac{1}{2}\mathbf{u_r}^{\mathrm{T}} H \mathbf{u_r} + b_\theta^{\mathrm{T}}(x)\mathbf{u_r} \quad \text{s.t.} \quad R_r\mathbf{u_r} \leqslant c_{\theta r} + Fx, \tag{14}$$

where $H = H^{\mathrm{T}} = 2W_{JE}^{\mathrm{T}} M_{uu} W_{JE}$, $b_\theta^{\mathrm{T}}(x) = 2(\boldsymbol{\theta}_p^{\mathrm{T}} M_{\theta u} + x^{\mathrm{T}} M_{uf}^{\mathrm{T}}) W_{JE}$ and $k_\theta(x) = \boldsymbol{\theta}_p^{\mathrm{T}} M_{\theta\theta}\boldsymbol{\theta}_p + 2x^{\mathrm{T}} M_{\theta f}^{\mathrm{T}}\boldsymbol{\theta}_p + x^{\mathrm{T}} M_{ff} x + \mathbf{u}_0^{\mathrm{T}} M_{uu}\mathbf{u}_0 + (\boldsymbol{\theta}_p^{\mathrm{T}} M_{\theta u} + x^{\mathrm{T}} M_{uf}^{\mathrm{T}})\mathbf{u}_0$, $R_r = RW_{JE}$ and $c_{\theta r} = c_\theta + R\mathbf{u}_0$. Let z be $z \triangleq \mathbf{u_r} + H^{-1} b_\theta(x)$. Then problem (14) is equivalent to

$$V_z(x) = \min_z \tfrac{1}{2} z^{\mathrm{T}} H z \quad \text{s.t.} \quad R_r z \leqslant v_\theta + Sx \tag{15}$$

with $V_z(x) = V(x) - c_\theta + \frac{1}{2} b_\theta^{\mathrm{T}}(x) H^{-1} b_\theta(x)$, $v_\theta = c_{\theta r} + 2R_r H^{-1} W_{JE}^{\mathrm{T}} M_{\theta u}^{\mathrm{T}}\boldsymbol{\theta}$ and $S = 2R_r H^{-1} M_{uf} W_{JE} + F$. The solution of problem (15) is a PWA function of x as proven in the influential work of Bemporad, Morari, Dua, and Pistikopoulos (2002). Thus, $\mathbf{u}_r^*(x) = K_i x + v_i$, and $\mathbf{u}^*(x) = K_i x + v_i$. □

[1] This is more evident in the epigraphic form (Boyd & Vandenberghe, 2004) of the min–max problem

$$\begin{aligned} \min_{\mathbf{u},\gamma} \quad & \gamma \\ \text{s.t} \quad & J(\boldsymbol{\theta}_i, \mathbf{u}, x) \leqslant \gamma, \quad i = 1, \dots, 2^N \\ & R\mathbf{u} \leqslant c_\theta + Fx \end{aligned}$$

with $\gamma \in \mathbb{R}$. It is clear that this problem is equivalent to another in which nonactive constraints have been removed. Those vertices $\boldsymbol{\theta}_i$ for which the constraints $J(\boldsymbol{\theta}_i, \mathbf{u}, x) \leqslant \gamma$ become active are the active vertices.

In the following it will be shown that the solution of the min–max problem is PWA on x. If the active vertex set does not change in a neighborhood of x, the solution of the min–max problem will remain the same as an mp-QP and, therefore, the solution will be PWA in that neighborhood. This is the idea used in the next theorem.

Theorem 6. *Let* $\mathbf{u}^*(x)$ *be the solution of problem* (5) *where* $x \in K^*$. *Then,* $\mathbf{u}^*(x)$ *is a PWA function of* x.

Proof. Suppose that $\boldsymbol{\theta}_p \in \operatorname{vert}\{\boldsymbol{\Theta}\}$ is an active vertex for $x \in K^*$. From Definition 4 active vertices satisfy the following conditions:

- C1: $J(\boldsymbol{\theta}_p, \mathbf{u}^*(x), x) = J(\boldsymbol{\theta}_i, \mathbf{u}^*(x), x)\ \forall \boldsymbol{\theta}_i \in S_I(x)$, i.e., its cost is equal to that of the remaining active vertices. This implies that $J(\boldsymbol{\theta}_p, \mathbf{u}^*(x), x) = J^*(\mathbf{u}^*(x), x)$, that is, $J(\boldsymbol{\theta}_p, \mathbf{u}^*(x), x)$ is equal to the worst-case cost, which in turn implies the following condition.
- C2: $J(\boldsymbol{\theta}_j, \mathbf{u}^*(x), x) < J(\boldsymbol{\theta}_p, \mathbf{u}^*(x), x)\ \forall \boldsymbol{\theta}_j \in \operatorname{vert}\{\boldsymbol{\Theta}\}$ such that $\boldsymbol{\theta}_j \notin S_I(x)$, because nonactive vertices have lower cost than the active ones.

These conditions are used to define a region of the $(\mathbf{u}, x)$ space for the active set $S_I(x)$ in which all the pairs $(\mathbf{u}^*(x), x)$ with active vertex set $S_I(x)$ are included. Let Ω_I be the region of the $(\mathbf{u}, x)$-space defined by

$$\Omega_I = \left\{ (\mathbf{u}, x) : \begin{array}{c} R\mathbf{u} \leqslant c_\theta + Fx \\ J(\boldsymbol{\theta}_p, \mathbf{u}, x) = J(\boldsymbol{\theta}_i, \mathbf{u}, x)\ \forall \boldsymbol{\theta}_i \in S_I(x) \\ J(\boldsymbol{\theta}_j, \mathbf{u}, x) < J(\boldsymbol{\theta}_p, \mathbf{u}, x)\ \forall \boldsymbol{\theta}_j \notin S_I(x) \end{array} \right\}.$$

From the definition of Ω_I if $u^*(x^+)$ with $x^+ \in K^*$ has the active vertex set $S_I(x)$ then $(\mathbf{u}^*(x^+), x^+) \in \Omega_I$. Moreover, if $(\mathbf{u}^*(x^+), x^+) \in \Omega_I$ then $S_I(x^+) = S_I(x)$.

Taking into account that $\mathbf{u}^*(x)$ is continuous, a small bounded change δx in x will result in a bounded change $\delta\mathbf{u}$ of $\mathbf{u}^*(x)$. Then a neighborhood of x will always exist such that all pairs $(\mathbf{u}^*(\mathbf{x}^+), x^+)$ included in that neighborhood will also be in Ω_I. This implies that in the neighborhood $N_{CI}(x) = \{x^+ \in K^* : (x^+ = x + \delta x, \mathbf{u}^*(x^+) = \mathbf{u}^*(x) + \delta\mathbf{u}) \in \Omega_I\}$ the active vertex set of the solution does not change and the solution of the min–max problem is the same as that of (12), therefore it is PWA. Thus, for every $x \in K^*$ there exists a neighborhood of x in which the solution of (5) is PWA and therefore the control law is *piecewise affine*. □

In the following CR_A denotes the region of K^* in which the active vertex set is $A \subset \operatorname{vert}(\boldsymbol{\Theta})$, i.e, $CR_A \triangleq \{x \in K^* : S_I(x) = A \subset \operatorname{vert}(\boldsymbol{\Theta})\}$ and $\overline{CR_A}$ its closure on K^*.

4.1. Types of solutions

The active vertex set of each region may contain one or more vertices. Three different cases may occur:

(1) The solution is said to be type I when the active set has only one vertex, i.e., $S_I(x) = \{\theta_p\}$ with $x \in K^*$. In this case the equality constraints in problem (12) are removed.

(2) The solution is said to be type II when more than one vertex is contained in $S_I(x)$. In this case, the min–max problem is equivalent to problem (12).
(3) The solution is said to be type III when more than one vertex is contained in $S_I(x)$ but the min–max problem is also equivalent to another with $S_I(x)=\{\theta_p\}$. In this case problem, (12) can also be reduced removing the equality constraints.

The regions of K^* in which the solution is type I, II or III are said to be type I, II or III regions. Note that in types I and III the solution of (5) is equal to the minimizer of $J(\boldsymbol{\theta}_p,\mathbf{u},x)$, thus the minimizer of $J(\boldsymbol{\theta}_p,\mathbf{u},x)$ is also a minimizer of $J^*(\mathbf{u},x)$.

5. Boundaries between regions

In this section the transitions between regions due to different active vertex sets are analyzed.

Lemma 7. *Let $\overline{CR_A}$, $\overline{CR_B}$ be two adjacent regions of K^* in which the solution is known to have certain active vertex sets $A\subset \mathrm{vert}(\boldsymbol{\Theta})$ and $B\subset \mathrm{vert}(\boldsymbol{\Theta})$. Then, the common boundary between $\overline{CR_A}$ and $\overline{CR_B}$ is contained in a hyperplane.*

Proof. This follows from the continuity of the control law. Due to continuity, the solution of (5) in $\overline{CR_A}\cap\overline{CR_B}$ must have the same cost $J(\boldsymbol{\theta}_i,\mathbf{u},x)$ and $J(\boldsymbol{\theta}_j,\mathbf{u},x)$ $\forall\boldsymbol{\theta}_i\in A$, $\boldsymbol{\theta}_j\in B$ $\forall x\in\overline{CR_A}\cap\overline{CR_B}$, thus all x in the boundary must satisfy

$$J(\boldsymbol{\theta}_i,\mathbf{u}_1^*(x),x)=J(\boldsymbol{\theta}_j,\mathbf{u}_1^*(x),x),\quad \boldsymbol{\theta}_i\in A,\quad \boldsymbol{\theta}_j\in B \tag{16}$$

$\forall x\in\overline{CR_A}\cap\overline{CR_B}$ where $\mathbf{u}_1^*(x)$ is the solution of (5) in $\overline{CR_A}$. Note that $\mathbf{u}_1^*(x)=M_1(x)x+v_1(x)$, where the matrix $M_1(x)$ and vector $v_1(x)$ depend on $x\in K^*$. Thus, Eq. (16) can be rewritten as

$$\begin{aligned}&(\boldsymbol{\theta}_i^{\mathrm{T}}M_{\theta\theta}\boldsymbol{\theta}_i-\boldsymbol{\theta}_j^{\mathrm{T}}M_{\theta\theta}\boldsymbol{\theta}_j+2(\boldsymbol{\theta}_i^{\mathrm{T}}-\boldsymbol{\theta}_j^{\mathrm{T}})M_{\theta u}v_1(x))\\&\quad+2(\boldsymbol{\theta}_i^{\mathrm{T}}-\boldsymbol{\theta}_j^{\mathrm{T}})(M_{\theta u}M_1(x)+M_{\theta f})x=0\end{aligned} \tag{17}$$

which is the hyperplane that contains the boundary between $\overline{CR_A}$ and $\overline{CR_B}$. □

Proposition 8. *If region $\overline{CR_A}$ or $\overline{CR_B}$ is type* I *or* III, *then the part of the hyperplane* (17) *which is not the boundary between $\overline{CR_A}$ and $\overline{CR_B}$ can be removed using affine constraints on x.*

Proof. Suppose that $\overline{CR_A}$ is type I or III and that the solution of (5) $\forall x\in\overline{CR_A}$ is equal to $\mathbf{u}_i^*(x)$ the solution of (12) with $\boldsymbol{\theta}_p=\boldsymbol{\theta}_i\in A$. As $\boldsymbol{\theta}_i\in A$ all x in $\overline{CR_A}\cap\overline{CR_B}$ must satisfy that $J(\boldsymbol{\theta}_i,\mathbf{u}_i^*(x),x)\geqslant J(\boldsymbol{\theta}_j,\mathbf{u}_i^*(x),x)$ $\forall\boldsymbol{\theta}_j\in\mathrm{vert}(\boldsymbol{\Theta})$. Moreover, if these conditions are met $\mathbf{u}_i^*(x)$ is a minimizer of the strict convex $J^*(\mathbf{u},x)$. Thus, $\mathbf{u}_i^*(x)$ is the solution to (5) and the part of (17) which is not the boundary of $\overline{CR_A}$ and $\overline{CR_B}$ can be removed by imposing that

$$J(\boldsymbol{\theta}_i,\mathbf{u}_i^*(x),x)\geqslant J(\boldsymbol{\theta}_j,\mathbf{u}_i^*(x),x),\quad \boldsymbol{\theta}_i\in A\ \ \forall\boldsymbol{\theta}_j\in\mathrm{vert}(\boldsymbol{\Theta})$$

which taking into account that $\mathbf{u}_i^*(x)=M_i(x)x+v_i(x)$ $x\in K^*$ can be rewritten as

$$\begin{aligned}&-(\boldsymbol{\theta}_i^{\mathrm{T}}M_{\theta\theta}\boldsymbol{\theta}_i-\boldsymbol{\theta}_j^{\mathrm{T}}M_{\theta\theta}\boldsymbol{\theta}_j+2(\boldsymbol{\theta}_i^{\mathrm{T}}-\boldsymbol{\theta}_j^{\mathrm{T}})M_{\theta u}v_i(x))\\&\quad-2(\boldsymbol{\theta}_i^{\mathrm{T}}-\boldsymbol{\theta}_j^{\mathrm{T}})(M_{\theta u}M_i(x)+M_{\theta f})x\leqslant 0\end{aligned} \tag{18}$$

with $\boldsymbol{\theta}_i\in A$ $\forall\boldsymbol{\theta}_j\in\mathrm{vert}(\boldsymbol{\Theta})$, which are affine constraints on x. □

Remark 9. If no constraint of (5) is active along the boundary or no constraints are considered in (5) then the minimizer of $J(\boldsymbol{\theta}_i,\mathbf{u},x)$ can be computed easily as

$$\mathbf{u}_i^*(x)=-M_{uu}^{-1}M_{uf}x-M_{uu}^{-1}M_{\theta u}^{\mathrm{T}}\boldsymbol{\theta}_i \tag{19}$$

thus (17) and (18) can be rewritten as:

$$\begin{aligned}&(\boldsymbol{\theta}_i^{\mathrm{T}}M_{\theta\theta}\boldsymbol{\theta}_i-\boldsymbol{\theta}_j^{\mathrm{T}}M_{\theta\theta}\boldsymbol{\theta}_j)+2x^{\mathrm{T}}M_{\theta f}^{\mathrm{T}}(\boldsymbol{\theta}_i-\boldsymbol{\theta}_j)\\&\quad-2(\boldsymbol{\theta}_i^{\mathrm{T}}-\boldsymbol{\theta}_j^{\mathrm{T}})M_{\theta u}M_{uu}^{-1}(M_{\theta u}^{\mathrm{T}}\boldsymbol{\theta}_i+M_{uf}x)=0\end{aligned} \tag{20}$$

$$\begin{aligned}&-(\boldsymbol{\theta}_i^{\mathrm{T}}M_{\theta\theta}\boldsymbol{\theta}_i-\boldsymbol{\theta}_k^{\mathrm{T}}M_{\theta\theta}\boldsymbol{\theta}_k-2(\boldsymbol{\theta}_i^{\mathrm{T}}-\boldsymbol{\theta}_k^{\mathrm{T}})M_{\theta u}M_{uu}^{-1}M_{\boldsymbol{\theta}u}^{\mathrm{T}}\boldsymbol{\theta}_i)\\&\quad-2(\boldsymbol{\theta}_i^{\mathrm{T}}-\boldsymbol{\theta}_k^{\mathrm{T}})(M_{\theta f}-M_{\theta u}M_{uu}^{-1}M_{uf})x\leqslant 0\\&\quad\forall k\notin A\cup B.\end{aligned} \tag{21}$$

Proposition 10. *If no constraints are active along its boundaries* (*or the min–max problem is unconstrained*) *a region $\overline{CR_A}$ of type* I *or* III *cannot be adjacent to another region $\overline{CR_B}$ type* I *or* III.

Proof. Two adjacent regions share a common boundary in which the solution of the min–max problem is unique. Thus, due to continuity, the solutions that come from both QP problems associated to each region must be the same. Assume that in $\overline{CR_A}$ the solution $\mathbf{u}_i^*(x)$ is the unconstrained minimizer of $J(\boldsymbol{\theta}_i,\mathbf{u},x)$, that is (19). Furthermore, suppose that $\overline{CR_B}$ is type I or III, and that the solution $\mathbf{u}_j^*(x)$ is equal to the unconstrained minimizer of $J(\boldsymbol{\theta}_j,\mathbf{u},x)$. Thus, by continuity

$$\mathbf{u}_i^*(x)-\mathbf{u}_j^*(x)=0\quad\forall x\in\overline{CR_A}\cap\overline{CR_B}. \tag{22}$$

Taking into account (19) for both $\boldsymbol{\theta}_i$ and $\boldsymbol{\theta}_j$ this can be rewritten as

$$-M_{uu}^{-1}M_{\theta u}^{\mathrm{T}}(\boldsymbol{\theta}_i-\boldsymbol{\theta}_j)=0. \tag{23}$$

Eq. (23) cannot be fulfilled unless[2] $\boldsymbol{\theta}_i=\boldsymbol{\theta}_j$. Thus, it will never hold for different quadratic functions. This implies that $\overline{CR_A}\cap\overline{CR_B}=\emptyset$, thus $\overline{CR_A}$ and $\overline{CR_B}$ cannot be adjacent. □

6. An illustrative example

In order to illustrate these results consider a first order system, described with a CARIMA (Camacho & Bordóns, 2004) model:

$$y(t+1)=1.9y(t)-0.9y(t-1)+0.5\Delta u(t)+\theta(t) \tag{24}$$

[2] This is true because the matrices appearing in (23) are full rank.

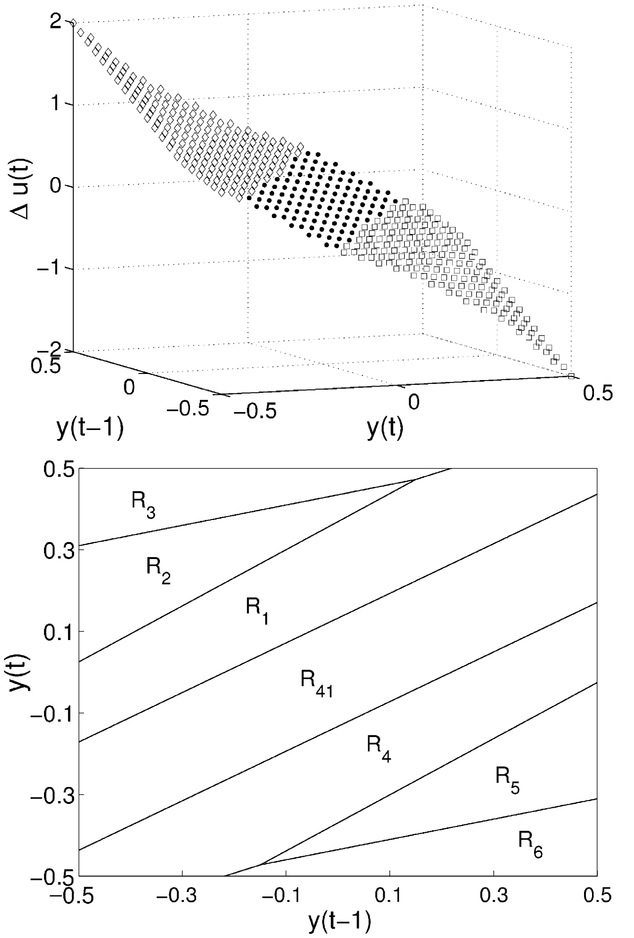

Fig. 1. Solution (top) and partition of the process state space (bottom) for the example.

with $-0.1 \leqslant \boldsymbol{\theta} \leqslant 0.1$ and $\Delta = 1 - z^{-1}$. Note that in this case $x(t) = [y(t)\ y(t-1)]^{\mathrm{T}}$. A constrained min–max MPC control law was calculated with the following parameters: $N_u = 2$, $N = 2$, and Q_1, P, L_1 and L_2 unit matrices of dimension 2. The following constraints were taken into account: $-2 \leqslant \Delta u(t) \leqslant 2$, $-0.5 \leqslant y(t) \leqslant 0.5$. Thus, in this case $K^* \triangleq \{x \in \mathbb{R}^2 : \exists \mathbf{u} \in \mathbb{R}^2,\ \|\mathbf{u}\|_\infty \leqslant 2,\ \|\mathbf{x}\|_\infty \leqslant 0.5\}$ with $\mathbf{u} = [\Delta u(t)\ \Delta u(t+1)]^{\mathrm{T}}$ and $\mathbf{x} = [x(t+1|t,\boldsymbol{\theta})\ x(t+2|t,\boldsymbol{\theta})]^{\mathrm{T}}$.

Fig. 1 (top) shows the solution for the min–max problem computed numerically. There are three regions due to different active sets. The lower region is a type I region, $CR_{S_{I1}}$, in which the active vertex set is $S_{I1}(x) = \{\boldsymbol{\theta}_1\}$, with $\boldsymbol{\theta}_1 = [0.1\,0.1]^{\mathrm{T}}$. This implies that in $CR_{S_{I1}}$ the solution of (5) is equal to the constrained minimizer of $J(\boldsymbol{\theta}_1, \mathbf{u}, x)$. In the upper region the active vertex set is $S_{I4}(x) = \{\boldsymbol{\theta}_4\}$, with $\boldsymbol{\theta}_4 = [-0.1\ -0.1]^{\mathrm{T}}$. Thus, $CR_{S_{I4}}$ is another type I region. On the other hand, in the central region the active vertex set is $S_{I41}(x) = \{\boldsymbol{\theta}_1, \boldsymbol{\theta}_4\}$, that is, $CR_{S_{I41}}$ is a type II region. Therefore in $CR_{S_{I41}}$ the solution of (5) is equal to the constrained minimizer of either $J(\boldsymbol{\theta}_1, \mathbf{u}, x)$ or $J(\boldsymbol{\theta}_4, \mathbf{u}, x)$ subject to the additional constraint $J(\boldsymbol{\theta}_1, \mathbf{u}, x) = J(\boldsymbol{\theta}_4, \mathbf{u}, x)$. The boundaries between each region can be computed as in (20) because the constraints are not active along the boundaries yielding,

$$
\begin{aligned}
\overline{CR_{S_{I1}}} &\triangleq \{x \in K^* : -0.22 + [-0.99\ \ 1.62]x \geqslant 0\},\\
\overline{CR_{S_{I4}}} &\triangleq \{x \in K^* : -0.22 + [0.99\ \ -1.62]x \geqslant 0\},\\
\overline{CR_{S_{I41}}} &\triangleq \{x \in K^* : -0.22 + [0.99\ \ -1.62]x \leqslant 0,\\
&\qquad -0.22 + [-0.99\ \ 1.62]x \leqslant 0\}.
\end{aligned}
$$

The multiparametric QP algorithm of Bemporad et al. (2002) has been applied to the constrained QP problems related to $CR_{S_{I1}}$, $CR_{S_{I4}}$ and $CR_{S_{I41}}$, yielding a Constrained min–max MPC controller in explicit form, in which $\Delta u(k)$ can be computed as

$$
\begin{aligned}
&v_4 x \ \text{if} \begin{bmatrix} 0.99 & -1.62 \\ -0.99 & 1.62 \end{bmatrix} x \leqslant \begin{bmatrix} 0.22 \\ 0.22 \end{bmatrix} \quad (\text{reg. R41}),\\
&c_1 + v_1 x \quad \text{if} \begin{bmatrix} 0.61 & -1 \\ -0.40 & 1 \\ -0.69 & 1 \\ 0.40 & -1 \end{bmatrix} x \leqslant \begin{bmatrix} -0.13 \\ 0.41 \\ 0.37 \\ 0.29 \end{bmatrix} \quad (\text{reg. R1}),\\
&c_2 + v_2 x \ \text{if} \begin{bmatrix} -0.62 & 1 \\ -0.25 & 1 \\ 0.25 & -1 \\ 0.69 & -1 \end{bmatrix} x \leqslant \begin{bmatrix} 0.93 \\ 0.43 \\ 0.64 \\ -0.37 \end{bmatrix} \quad (\text{reg. R2}),\\
&c_3 + v_3 x \ \text{if} \begin{bmatrix} 0.61 & -1 \\ -0.47 & 1 \\ 0 & 1 \\ 0.40 & -1 \\ 0.25 & -1 \end{bmatrix} x \leqslant \begin{bmatrix} -0.13 \\ 0.74 \\ 1.06 \\ -0.41 \\ -0.43 \end{bmatrix} \quad (\text{reg. R3}),\\
&-c_1 + v_1 x \ \text{if} \begin{bmatrix} -0.61 & 1 \\ 0.40 & -1 \\ 0.69 & -1 \\ -0.40 & 1 \end{bmatrix} x \leqslant \begin{bmatrix} -0.13 \\ 0.41 \\ 0.37 \\ 0.29 \end{bmatrix} \quad (\text{reg. R4}),\\
&-c_2 + v_2 x \ \text{if} \begin{bmatrix} 0.62 & -1 \\ -0.25 & 1 \\ 0.25 & -1 \\ -0.69 & 1 \end{bmatrix} x \leqslant \begin{bmatrix} 0.93 \\ 0.64 \\ 0.43 \\ -0.37 \end{bmatrix} \quad (\text{reg. R5}),\\
&-c_3 + v_3 x \ \text{if} \begin{bmatrix} -0.61 & 1 \\ 0.47 & -1 \\ 0 & -1 \\ -0.40 & 1 \\ -0.25 & 1 \end{bmatrix} x \leqslant \begin{bmatrix} -0.13 \\ 0.74 \\ 1.06 \\ -0.41 \\ -0.43 \end{bmatrix} \quad (\text{reg. R6}),
\end{aligned}
$$

with $c_1 = -0.14$, $v_1 = [0.89\ -1.53]$, $c_2 = 0.15$, $v_2 = [1.43\ -2.31]$, $c_3 = 0.8$, $v_3 = [1.8\ -3.8]$, $v_4 = [1.8\ -3.8]$. The state-space partition can be seen in Fig. 1 (bottom).

7. Conclusions

It has been shown that min–max MPC with bounded additive uncertainties and a quadratic criterion result in a PWA and continuous control law. This opens new possibilities for studying robustness and stability properties. However, many open questions remain to be addressed, the most interesting being how to compute the explicit form of the control law in an automated way.

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Appendix A. Known results in point to set maps

Definition 11. Let $\Omega : X \mapsto \mathbf{U}$ a point to set map. Then (Hogan, 1973)

(1) Ω is *open* at $x \in X$ if a sequence contained in X, $\{x_k\} \subset X$ such that $x_k \to x$ and $\mathbf{u} \in \Omega(x)$ imply the existence of an integer $m \in \mathscr{Z}$ and a sequence $\{\mathbf{u_k}\} \subset \mathbf{U}$ such that $\mathbf{u_k} \in \Omega(x)$ for $k \geqslant m$ and that $\mathbf{u_k} \to \mathbf{u}$.
(2) Ω is *closed* at $x \in X$ if a sequence contained in X, $\{x_k\} \subset X$ such that $x_k \to x$ and a sequence $\{\mathbf{u_k}\}$ such that $\mathbf{u_k} \in \Omega(x_k)$ and $\mathbf{u_k} \to \mathbf{u}$ imply that $\mathbf{u} \in \Omega(x)$.
(3) Ω is *continuous* at $x \in X$ if it is both open and closed at x.
(4) Ω is *open*, *closed or continuous* on X if it is open, closed or continuous $\forall x \in X$.

Let $P(x)$ be a point to set map determined by inequalities $P(x) \triangleq \{\mathbf{u} \in \mathbf{U} : g(x, \mathbf{u}) \leqslant 0\}$ where $g : X \times \mathbf{U} \mapsto [-\infty, +\infty]^m$. Furthermore, let $I(x) \triangleq \{\mathbf{u} \in \mathbf{U} : g(x, \mathbf{u}) < 0\}$.

Theorem 12 (*Theorem 10 of* Hogan, 1973). *If each component of g is lower semi-continuous on $x \times \mathbf{U}$ then P is closed at x.*

Theorem 13 (*Theorem 13 of* Hogan, 1973). *If each component of g is upper semi-continuous on $x \times I(x)$ and $P(x) \subset \overline{I(x)}$ (where $\overline{I(x)}$ is the closure of $I(x)$) then P is open at x.*

Suppose a constrained minimization problem as in (10), $\Omega : X \mapsto \mathbf{U}$ a point to set map and $M(x)$ the set of minimizers of (10) for $x \in X$.

Theorem 14 (*Corollary 8.2 of* Hogan, 1973). *Suppose Ω is continuous on X, X is connected, $\mathbf{U}$ is compact, f is continuous on $X \times \mathbf{U}$, and$M$ is single valued at x. Then M is continuous at x.*

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