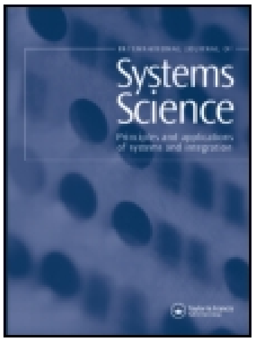


International Journal of Systems Science

ISSN: 0020-7721 (Print) 1464-5319 (Online) Journal homepage: https://www.tandfonline.com/loi/tsys20

Optimal MPC for tracking of constrained linear systems

A. Ferramosca , D. Limon , I. Alvarado , T. Alamo , F. Castaño & E.F. Camacho

International Journal of Systems Science
Vol. 42, No. 8, August 2011, 1265–1276

Taylor & Francis
Taylor & Francis Group

Optimal MPC for tracking of constrained linear systems

A. Ferramosca*, D. Limon, I. Alvarado, T. Alamo, F. Castaño and E.F. Camacho

Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla, Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n, Sevilla 41092, Spain

Model predictive control (MPC) is one of the few techniques which is able to handle constraints on both state and input of the plant. The admissible evolution and asymptotic convergence of the closed-loop system is ensured by means of suitable choice of the terminal cost and terminal constraint. However, most of the existing results on MPC are designed for a regulation problem. If the desired steady-state changes, the MPC controller must be redesigned to guarantee the feasibility of the optimisation problem, the admissible evolution as well as the asymptotic stability. Recently, a novel MPC has been proposed to ensure the feasibility of the optimisation problem, constraints satisfaction and asymptotic evolution of the system to any admissible target steady-state. A drawback of this controller is the loss of a desirable property of the MPC controllers: the local optimality property. In this article, a novel formulation of the MPC for tracking is proposed aimed to recover the optimality property maintaining all the properties of the original formulation.

Keywords: model predictive control; tracking; constrained linear systems; asymptotic stability; local optimality

1. Introduction

Model predictive control (MPC) is one of the most successful techniques of advanced control in the process industry. This is due to its control problem formulation, the natural usage of the model to predict the expected evolution of the plant, the optimal character of the solution and the explicit consideration of hard constraints in the optimisation problem. Thanks to the recent developments of the underlying theoretical framework, MPC has become a mature control technique capable to provide controllers ensuring stability, robustness, constraint satisfaction and tractable computation for linear and for non-linear systems (Camacho and Bordons 2004).

The control law is calculated by predicting the evolution of the system and computing the admissible sequence of control inputs which makes the system evolves satisfying the constraints. This problem can be posed as an optimisation problem. To obtain a feedback policy, the obtained sequence of control inputs is applied in a receding horizon manner, solving the optimisation problem at each sample time. Considering a suitable penalisation of the terminal state and an additional terminal constraint, asymptotic stability and constraints satisfaction of the closed-loop system can be proved (Mayne, Rawlings, Rao, and Scokaert 2000). Moreover, if the terminal cost is the infinite-horizon optimal cost of the unconstrained system, then the MPC control law results to be optimal in a neighbourhood of the steady-state. This property is the so-called local optimality property and allows to design finite horizon MPC controllers for constrained system with a local optimal closed-loop performance (Scokaert and Mayne 1998; Bemporad, Morari, Dua, and Pistikopoulos 2002; Hu and Linnemann 2002).

Most of the results on MPC consider the regulation problem, that is steering the system to a fixed steady-state (typically the origin), but when the target operating point changes, the feasibility of the controller may be lost and the controller fails to track the reference (Rossiter, Kouvaritakis, and Gossner 1996; Bemporad, Casavola, and Mosca 1997; Rao and Rawlings 1999; Pannocchia and Kerrigan 2005). This can be a consequence of one or both of the two following causes: (i) the terminal set shifted to the new operating point may not be an admissible invariant set, which means that the all time feasibility property may be lost and (ii) the terminal region at the new setpoint could be unreachable in N steps, which means that the optimisation problem is unfeasible, making necessary a re-calculation of an appropriate value of the prediction horizon to ensure feasibility. Therefore, this would require an on-line re-design of the controller for each set point, which can be computationally unaffordable.

For such cases, the steady-state target can be determined by solving an optimisation problem that determines the steady-state and input targets.

*Corresponding author. Email: ferramosca@cartuja.us.es

ISSN 0020–7721 print/ISSN 1464–5319 online

DOI: 10.1080/00207721.2011.588895
http://www.informaworld.com

This target calculation can be formulated as different mathematical programs for the cases of perfect target tracking or non-square systems (Muske 1997), or by solving a unique problem for both situations (Rao and Rawlings 1999). A switching strategy to recover feasibility has been proposed in Rossiter et al. (1996) and Chisci and Zappa (2003). This technique requires an auxiliary controller derived from the solution of an optimisation problem. In Bemporad et al. (1997), Angeli, Casavola, and Mosca (2000) and Ding, Xi, and Li (2004) the command governor approach is used considering predictions in the calculation of the reference at each sample time.

In Limon, Alvarado, Alamo, and Camacho (2008) a novel MPC for tracking is proposed, which is able to lead the system to any admissible set point in an admissible way. The main characteristics of this controller are: an artificial steady-state is considered as a decision variable, a cost that penalises the error with the artificial steady-state is minimised, an additional term that penalises the deviation between the artificial steady-state and the target steady-state is added to the cost function (the so-called *offset cost function*) and an invariant set for tracking is considered as extended terminal constraint. This controller ensures that under any change of the steady-state target, the closed-loop system maintains the feasibility of the controller and ensures the convergence to the target if admissible. The main drawback of the MPC for tracking is the loss of the optimality property due to the addition of the artificial steady-state together with the proposed cost function.

The aim of this article is to study the property of local optimality. The problem of optimality in MPC was previously addressed in Scokaert and Rawlings (1998) where the authors present a constrained LQR and prove that there exists a finite N such that the controller can be computed. In Bemporad et al. (2002) a critic value of N is presented. The problem of local optimality is studied in a more general formulation in Hu and Linnemann (2002) where it is proved that this property is ensured for all N and an optimality region is computed.

In this article, a novel formulation of the MPC for tracking is presented, considering a norm as offset cost function. It is proved that the new formulation recovers the local optimality property, and some conditions are given for defining when this property applies and characterising the regions into which this property can apply. If the offset cost function is chosen as a $\{1, \infty\}$-norm, the optimisation problem can be written as a QP by adding a new decision variable and a new constraint to the original optimisation problem.

This article is organised as follows. In the following section the constrained tracking problem is stated. In Section 3 the new MPC for tracking is presented and in Section 4 the property of local optimality is introduced and proved. Finally, an illustrative example is shown.

Notation: Vector (a, b) denotes $[a^T, b^T]^T$; for a given λ, $\lambda X = \{\lambda x : x \in X\}$; int$(X)$ denotes the interior of set X; a matrix T definite positive is denoted as $T > 0$ and $T > P$ denotes that $T - P > 0$. For a given symmetric matrix $P > 0$, $\|x\|_P$ denotes the weighted euclidean norm of x, i.e. $\|x\|_P = \sqrt{x^T P x}$. Matrix $\mathbf{0}_{n,m} \in \mathbb{R}^{n\times m}$ denotes a matrix of zeros. Consider $a \in \mathbb{R}^{n_a}$, $b \in \mathbb{R}^{n_b}$, and set $\Gamma \subset \mathbb{R}^{n_a+n_b}$, then projection operation is defined as $\text{Proj}_a(\Gamma) = \{a \in \mathbb{R}^{n_a} : \exists b \in \mathbb{R}^{n_b}, (a, b) \in \Gamma\}$. Vector $\mathbf{u}(p)$ is the sequence of control action $\mathbf{u}(p) = \{u(0; p), u(1; p), \ldots\}$, where p is a parameter. $\mathbf{u}^*(p)$ is the optimal sequence of control action.

2. Problem description

Let a discrete-time linear system be described by

$$\begin{aligned} x^+ &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ the current state of the system, $u \in \mathbb{R}^m$ the current input, $y \in \mathbb{R}^p$ the current output and x^+ the successor state. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$, respectively. The system is subject to hard constraints on state and control

$$(x(k), u(k)) \in \mathcal{Z} \tag{2}$$

for all $k \geq 0$. $\mathcal{Z} \subset \mathbb{R}^{n+m}$ is a compact convex polyhedron containing the origin in its interior.

Assumption 2.1: *The pair* (A,B) *is stabilisable.*

Under this assumption, the set of steady-states and inputs of the system (1) is a m-dimensional linear subspace of $\mathbb{R}^{n+m}$ (Alvarado 2007; Limon et al. 2008) given by

$$(x_s, u_s) = M_\theta \theta.$$

Every pair of steady-state and input $(x_s, u_s) \in \mathbb{R}^{n+m}$ is characterised by a given parameter $\theta \in \mathbb{R}^{n_\theta}$. This parameterisation allows us to characterise the subspace of steady-states and inputs by a minimal number of variables (θ), which simplifies further calculations necessary for the derivation of the proposed controller (Limon et al. 2008).

The problem we consider is the design of an MPC controller $u = k_N^O(x, \theta)$ to track a piece-wise constant sequence of set points given by $\theta(k)$ in such a way that the constraints are satisfied for all the time. The aim of the controller is to steer the system to a target set-point

θ in an admissible way, such that it is optimal in a neighbourhood of the target.

3. New formulation of the MPC for tracking

In Limon et al. (2008) a novel formulation of MPC for tracking to steer the system to an admissible set-point is presented. This controller copes with the tracking problem by (i) considering an artificial steady-state and input as decision variables, (ii) penalising the deviation of the predicted trajectory with the artificial steady conditions, (iii) adding a quadratic offset-cost function to penalise the deviation between the artificial and the target equilibrium point and (iv) considering an extended terminal constraint.

In this article a new controller is proposed considering a new offset cost function based on the infinity norm. This new offset cost is

$$V_O(\bar{\theta} - \theta) = \|T(\bar{\theta} - \theta)\|_\infty$$

with T invertible. $\bar{\theta}$ is the parameter that characterises the artificial steady-state and input (that is $(\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}$) and θ characterises the given target operation point. In this section it will be proved that this choice ensures the stability guarantee of the MPC for tracking inheriting its main properties, and providing a novel and significant property: the local optimality property.

The proposed cost function of the MPC is given by

$$V_N^O(x, \theta, \mathbf{u}, \bar{\theta}) = \sum_{i=0}^{N-1} \|x(i) - \bar{x}_s\|_Q^2 + \|u(i) - \bar{u}_s\|_R^2 + \|x(N) - \bar{x}_s\|_P^2 + \|T(\bar{\theta} - \theta)\|_\infty \quad (3)$$

and the controller is derived from the solution of the optimisation problem $P_N^O(x, \theta)$ given by

$$V_N^{O*}(x, \theta) = \min_{\mathbf{u}, \bar{\theta}} V_N^O(x, \theta, \mathbf{u}, \bar{\theta}), \quad (4a)$$

$$\text{s.t. } x(0) = x, \quad (4b)$$

$$x(j+1) = Ax(j) + Bu(j), \quad (4c)$$

$$(x(j), u(j)) \in \mathcal{Z}, \quad j = 0, \ldots, N-1, \quad (4d)$$

$$(\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}, \quad (4e)$$

$$(x(N), \bar{\theta}) \in \Omega_{t,K}^w. \quad (4f)$$

Considering the receding horizon policy, the control law is given by

$$\kappa_N^O(x, \theta) = u^*(0; x, \theta).$$

It can be easily shown that the feasibility region of $P_N^O(x, \theta)$ does not depend on the target operating point θ. Then there exists a polyhedral region $\mathcal{X}_N$ such that for all $x \in \mathcal{X}_N$, $P_N^O(x, \theta)$ is feasible. This is the set of initial states that can be admissibly steered to the projection of $\Omega_{t,K}^w$ onto x in N steps.

This optimisation problem is a convex mathematical programming problem that can be efficiently solved by specialised algorithms (Boyd and Vandenberghe 2006); fortunately this can be re-casted as a standard quadratic programming problem defining the following cost function:

$$V_N^O(x, \theta, \mathbf{u}, \bar{\theta}, \gamma) = \sum_{i=0}^{N-1} \|x(i) - \bar{x}_s\|_Q^2 + \|u(i) - \bar{u}_s\|_R^2 + \|x(N) - \bar{x}_s\|_P^2 + \gamma, \quad (5)$$

where an additional decision variable γ has been added. The optimisation problem $P_N^O(x, \theta)$ is then equivalent to the following quadratic programming problem

$$V_N^{O*}(x, \theta) = \min_{\mathbf{u}, \bar{\theta}, \gamma} V_N^O(x, \theta, \mathbf{u}, \bar{\theta}, \gamma), \quad (6a)$$

$$\text{s.t. } x(0) = x, \quad (6b)$$

$$x(j+1) = Ax(j) + Bu(j), \quad (6c)$$

$$(x(j), u(j)) \in \mathcal{Z}, \quad j = 0, \ldots, N-1, \quad (6d)$$

$$(\bar{x}_s, \bar{u}_s) = M_\theta \bar{\theta}, \quad (6e)$$

$$(x(N), \bar{\theta}) \in \Omega_{t,K}^w, \quad (6f)$$

$$\|T(\bar{\theta} - \theta)\|_\infty \le \gamma. \quad (6g)$$

Remark 1: The controller can be formulated using any norm $\|x\|_q = (\sum |x_i|^q)^{1/q}$ as offset cost function. If $q = \{1, \infty\}$, then the optimisation problem can be formulated as a quadratic programming. Nevertheless, the results of this article hold for any chosen norm.

The proposed controller has a set of parameters that should be appropriately chosen. These parameters are taken to satisfy the following assumption:

Assumption 3.1:

(1) *Let $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$ be positive definite matrices.*
(2) *Let T be a non-singular matrix.*
(3) *Let $K \in \mathbb{R}^{m \times n}$ be a stabilising control gain such that $(A + BK)$ is Hurwitz.*
(4) *Let $P \in \mathbb{R}^{n \times n}$ be a positive definite matrix such that*

$$(A+BK)'P(A+BK) - P = -(Q + K'RK),$$

(5) *Let* $\Omega^w_{t,K} \subseteq \mathbb{R}^{n+n_\theta}$ *be an admissible polyhedral invariant set for tracking for system* (1) *subject to* (2), *for the gain* K (*Limon et al.* 2008). *That is, for all* $(x,\theta) \in \Omega^w_{t,K}$ *we have that* $(x, Kx + L\theta) \in \mathcal{Z}$ *and* $((A+BK)x + BL\theta, \theta) \in \Omega^w_{t,K}$, *where* $L = [-K, I_m]M_\theta$.

In the following theorem, under the proposed conditions on the controller parameters, asymptotic stability and constraints satisfaction are proved for every admissible target steady-state θ satisfying

$$\theta \in \Theta = \{\theta : ([I_n, \mathbf{0}]M_\theta\theta, \theta) \in \Omega^w_{t,K},\ M_\theta\theta \in \mathcal{Z}\}.$$

It is worth remarking that this set is potentially the set of all admissible operating points (Limon et al. 2008).

The set of admissible steady-states and inputs contained in the invariant set for tracking $\Omega^w_{t,K}$ is given by

$$\mathcal{Z}_s = \{(x,u) = M_\theta\theta :\ (x,\theta) \in \Omega^w_{t,K}\}.$$

In what follows, notation $\Omega^w_{t,K}$ is used to refer to the invariant set for tracking in the augmented state (x, θ), while $\Omega_{t,K} = \mathrm{Proj}_x(\Omega^w_{t,K})$.

Theorem 3.2 Stability: *Consider that Assumptions* 2.1 *and* 3.1 *hold and that the target operation point is such that* $\theta \in \Theta$. *Then for any feasible initial state* $x_0 \in \mathcal{X}_N$, *the proposed MPC controller* $\kappa^O_N(x,\theta)$ *asymptotically steers the system to the target operating point fulfilling the constraints all the time.*

Proof: It is assumed that the Assumption 3.1 is satisfied.

The first part of the proof is devoted to prove the feasibility of the controlled system, that is, $x(k+1) \in \mathcal{X}_N$, for all $x(k) \in \mathcal{X}_N$, and θ. Consider the optimal solution of $P_N(x(k),\theta)$, then the successor state is $x(k+1) = Ax(k) + B\kappa^O_N(x,\theta)$. Define the following sequences:

$$\begin{aligned}\mathbf{u}(x(k+1),\theta) &\triangleq [u^*(1; x(k),\theta), \ldots, u^*(N-1; x(k),\theta),\\ &\qquad K(x^*(N; x(k),\theta) - \bar{x}^*_s(x(k),\theta))\\ &\qquad + \bar{u}^*_s(x(k),\theta)]\\ \bar{\theta}(x(k+1),\theta) &\triangleq \bar{\theta}^*(x(k),\theta).\end{aligned}$$

Then, $(\mathbf{u},\bar{\theta})$ is a feasible solution for the optimisation problem $P_N(x(k+1),\theta)$ due to the following Propositions.

- Since $x(0; x(k+1),\theta) = x^*(1; x(k),\theta)$, then $x(i; x(k+1),\theta) = x^*(i+1; x(k),\theta)$ and $u(i; x(k+1),\theta) = u^*(i+1; x(k),\theta)$ for $i = 0, 1, \ldots, N-1$; then $(x(i; x(k+1), \theta), u(i; x(k+1),\theta)) \in Z$ for $i = 0, 1, \ldots, N-1$.
- Since $(x(N-1; x(k+1),\theta), \bar{\theta}(x(k+1),\theta)) \in \Omega^w_{t,K}$, then the control action $u(N-1; x(k+1),\theta) = K(x(N-1;\ x(k+1),\ \theta) - \bar{x}_s) + \bar{u}_s$ is such that $(x(N-1;\ x(k+1),\ \theta),\ u(N-1;\ x(k+1),\ \theta)) \in Z$.
- From the invariance of $\Omega^w_{t,K}$, it derives that $(x(N; x(k+1),\theta), \bar{\theta}(x(k+1),\theta)) \in \Omega^w_{t,K}$.
- Feasibility of $\mathbf{u}^*(x(k),\theta)$ and admissibility of set $\Omega^w_{t,K}$ ensures the feasibility of $\mathbf{u}(x(k+1),\theta)$.

Convergence is derived proving that the optimal cost is a Lyapunov function. Consider the proposed feasible solution. Taking into account the properties of the feasible nominal trajectories for $x(k+1)$, the condition (4) of Assumption 3.1 and using standard procedures in MPC (Mayne et al. 2000) it is possible to obtain

$$\begin{aligned}&V^O_N(x(k+1),\theta; \mathbf{u},\bar{\theta}) - V^{O*}_N(x(k),\theta)\\ &\quad \le -\|x^*(0; x(k),\theta) - \bar{x}^*_s(x(k),\theta)\|^2_Q\\ &\qquad - \|u^*(0; x(k),\theta) - \bar{u}^*_s(x(k),\theta)\|^2_R\\ &\quad \le -\|x^*(0; x(k),\theta) - \bar{x}^*_s(x(k),\theta)\|^2_Q.\end{aligned}$$

By optimality, we have that $V^{O*}_N(x(k+1),\theta) \le V^O_N(x(k+1),\theta; \mathbf{u},\bar{\theta})$ and then

$$\begin{aligned}&V^{O*}_N(x(k+1),\theta) - V^{O*}_N(x(k),\theta)\\ &\quad \le -\|x^*(0; x(k),\theta) - \bar{x}^*_s(x(k),\theta)\|^2_Q.\end{aligned}$$

Taking into account that $Q > 0$, then

$$\lim_{k\to\infty} \|x^*(0; x(k),\theta) - \bar{x}^*_s(x(k),\theta)\|^2_Q = 0.$$

From Lemma 6.1 (appendix), we can deduce that if $\|x^*(x(k),\theta) - \bar{x}^*_s(x(k),\theta)\|$ tends to 0 then $\|\bar{x}^*_s(x(k),\theta) - x_s\|$ also tends to 0. Therefore, $\bar{x}^*_s(x(k),\theta)$ tends to x_s and then $x(k)$ tends to x_s. □

The proposed controller inherits the following properties from the MPC for tracking (Limon et al. 2008):

Remark 2 Controller properties:

(1) Since the optimisation problem is feasible for every value of θ, the proposed controller can be used to track a varying sequence of operating points $\theta(k)$.

(2) For any $\theta \in \Theta$ the system evolves to the target without offset. If $\theta \notin \Theta$, the controller steers the system to an admissible steady-state and input $(\bar{x}^*_s, \bar{u}^*_s) = M_\theta\bar{\theta}^*$ such that

$$\bar{\theta}^* = \arg\min_{\bar{\theta}\in\Theta} \|T(\bar{\theta} - \theta)\|_\infty$$

that is, the steady-state and input which minimises the offset cost function w.r.t. the target.

This property provides a rule for the selection of the matrix T since this characterises offset of the plant present in permanent regime when a non-admissible operating point is provided as target.

(3) The structure of the equivalent optimisation problem ensures that the proposed control law $\kappa_N^O(x,\theta)$ is a piecewise affine function of (x,θ) that can by explicitly calculated by means of the existing multiparametric programming tools (Bemporad et al. 2002).

Besides these properties, the main property of the proposed controller is its capability to guarantee the local optimality property. The following section is devoted to this topic.

4. Conditions for local optimality property

This section is devoted to analyse the optimality of the proposed controller. To this aim, consider an admissible control law $u=\kappa(x,\theta)$ and define a performance index given by

$$V_\infty(x,\theta,\kappa(\cdot,\theta)) = \sum_{i=0}^{\infty} \|x(i)-x_s\|_Q^2 + \|\kappa(x(i),\theta)-u_s\|_R^2, \tag{7}$$

where $x(i)=\phi(i;x;\kappa(\cdot,\theta))$ is calculated from the recursion $x(j+1)=Ax(j)+B\kappa(x(j),\theta)$ for $j=0,\ldots,i-1$ with $x(0)=x$, $(x(j),u(j))\in\mathcal{Z}$ and $(x_s,u_s)=M_\theta\theta$. Then the control law is said to be optimal if it is the one that minimises such performance index.

Model predictive controllers can be considered as suboptimal controllers since the cost function is only minimised for a finite prediction horizon. The standard MPC control law for regulating the system to the target θ, $\kappa_N^r(x,\theta)$, can be derived from the following optimisation problem $P_N^r(x,\theta)$

$$V_N^{r*}(x,\theta) = \min_{\mathbf{u},\bar{\theta}} \sum_{i=0}^{N-1} \|x(i)-\bar{x}_s\|_Q^2 + \|u(i)-\bar{u}_s\|_R^2 + \|x(N)-\bar{x}_s\|_P^2, \tag{8a}$$

$$\text{s.t. } x(0)=x, \tag{8b}$$

$$x(j+1)=Ax(j)+Bu(j), \tag{8c}$$

$$(x(j),u(j))\in\mathcal{Z}, \quad j=0,\ldots,N-1, \tag{8d}$$

$$(\bar{x}_s,\bar{u}_s)=M_\theta\bar{\theta}, \tag{8e}$$

$$(x(N),\bar{\theta})\in\Omega_{t,K}^w, \tag{8f}$$

$$\|T(\bar{\theta}-\theta)\|_\infty = 0. \tag{8g}$$

This optimisation problem is feasible in the polyhedral region $\mathcal{X}_N^r(\theta)$, where the control law is defined. Then the solution to the problem $P_N^r(x,\theta)$ with $N=\infty$ yields to the constrained LQR control law $\kappa_\infty(x,\theta)$ which is the admissible control law that minimises $V_\infty(x,\theta,\kappa(x))$, that is, the controller that provides the best performance according to the given quadratic index.

It would be desirable to calculate the optimal control law $\kappa_\infty(x,\theta)$, but its calculation may be computationally unaffordable. Fortunately the following lemma demonstrates that if the terminal cost function is the optimal cost of the unconstrained LQR, then the resulting finite horizon MPC equals the constrained LQR in a neighbourhood of the terminal region (Scokaert and Rawlings 1998; Bemporad et al. 2002; Hu and Linnemann 2002).

Lemma 4.1: *Consider that Assumptions* 2.1 *and* 3.1 *hold. Consider that the control gain K is equal to the unconstrained LQR gain K_{LQR} and define the set $\Upsilon_N(\theta)\subset\mathbb{R}^n$ as*

$$\Upsilon_N(\theta) = \{\bar{x}\in\mathbb{R}^n : (\phi(N;\bar{x},\kappa_\infty(\cdot,\theta),\theta)\in\Omega_{t,K}^w\}. \tag{9}$$

Then for all $x\in\Upsilon_N(\theta)$, $V_N^{r*}(x,\theta)=V_\infty(x,\theta)$ and $\kappa_N^r(x,\theta)=\kappa_\infty(x,\theta)$.

This lemma directly stems from Hu and Linnemann (2002, Theorem 2).

Unfortunately, the MPC for tracking proposed in Limon et al. (2008) does not guarantee that this remarkable property holds due to the artificial steady-state and input considered as decision variables. In this section we show that this optimality loss is derived from the quadratic nature of the offset cost function and the new controller can recover the local optimality property thanks to the considered ∞-norm offset cost function. In what follows, it is first proved that the MPC for tracking proposed in this article is equal to the MPC for regulation, and then that the MPC for regulation is optimal, in the sense that it ensures the property of local optimality.

Lemma 4.2: *Consider that Assumptions* 2.1 *and* 3.1 *hold. For all $x\in\mathcal{X}_N^r(\theta)$, there exists a matrix T such that the proposed MPC for tracking equals to the MPC for regulation, that is $\kappa_N^O(x,\theta)=\kappa_N^r(x,\theta)$ and $V_N^{O*}(x,\theta)=V_N^{r*}(x,\theta)$.*

Proof: Define the following optimisation problem $P_N^m(x,\theta;\alpha)$, given by

$$V_N^{m*}(x,\theta,\alpha) = \min_{\mathbf{u},\bar{\theta}} \sum_{i=0}^{N-1} \|x(i)-\bar{x}_s\|_Q^2 + \|u(i)-\bar{u}_s\|_R^2$$
$$+\|x(N)-\bar{x}_s\|_P^2 + \alpha\|\bar{\theta}-\theta\|_1,$$
$$\text{s.t. } x(0)=x,$$
$$x(j+1)=Ax(j)+Bu(j),$$
$$(x(j),u(j))\in\mathcal{Z},\quad j=0,\ldots,N-1,$$
$$(\bar{x}_s,\bar{u}_s)=M_\theta\bar{\theta},$$
$$(x(N),\bar{\theta})\in\Omega_{t,K}^w.$$

This optimisation problem $P_N^m(x,\theta;\alpha)$ is derived from the optimisation problem $P_N^r(x,\theta)$ with the last constraint posed as an exact penalty function (Luenberger 1984). Therefore, there exist a finite constant $\alpha>0$ such that $V_N^{m*}(x,\theta)=V_N^{r*}(x,\theta)$ (Luenberger 1984; Boyd and Vandenberghe 2006).

Assume that the matrix T in the optimisation problem $P_N^O(x,\theta)$ satisfies $\|T^{-1}\|_1\leq(\alpha n_\theta)^{-1}$. Then, using the properties of the norms, we have that

$$\|T(\bar{\theta}-\theta)\|_\infty \geq \frac{1}{\|T^{-1}\|_1 n_\theta}\|\bar{\theta}-\theta\|_1 \geq \alpha\|\bar{\theta}-\theta\|_1.$$

Consequently, we have that $V_N^O(x,\theta,\mathbf{u},\bar{\theta},\theta)\geq V_N^m(x,\theta,\mathbf{u},\bar{\theta},\theta)$ which implies that $V_N^{O*}(x,\theta)\geq V_N^{m*}(x,\theta)=V_N^{r*}(x,\theta)$.

On the other hand, for all $x\in\mathcal{X}_N^r(\theta)$ the optimisers of the optimisation problem $P_N^r(x,\theta)$ are a feasible solution of $P_N^O(x,\theta)$, and hence $V_N^{r*}(x,\theta)\geq V_N^{O*}(x,\theta)$.

Then, combining these results we have that for all $x\in\mathcal{X}_N^r(\theta)$,

$$V_N^{r*}(x,\theta)\geq V_N^{O*}(x,\theta)\geq V_N^{m*}(x,\theta)=V_N^{r*}(x,\theta)$$

and hence $V_N^{r*}(x,\theta)=V_N^{O*}(x,\theta)$. □

Remark 1: In virtue of the well-known result on the exact penalty functions (Luenberger 1984), the constant α can be chosen such that $\|\nu(x,\theta)\|_1\leq\alpha$, where $\nu(x,\theta)$ is the Lagrange multiplier of the equality constraint $\|T(\bar{\theta}-\theta)\|_\infty=0$ of the optimisation problem $P_N^r(x,\theta)$. Since the optimisation problem depends on the parameters (x,θ), the value of this Lagrange multiplier also depends on (x,θ).

Remark 2: The local optimality property can be ensured using any norm, thanks to the property of equivalence of the norms, that is $\exists c>0$ such that $\|x\|_q\geq c\|x\|_1$. Otherwise, the square of a norm cannot be used. With the $\|.\|_q^2$ norm, in fact, there will be always a local optimality gap for a finite value of α since $\|.\|_q^2$ is a (not exact) penalty function (Luenberger 1984). That gap can be reduced by means of a suitable penalisation of the offset cost function (Alvarado 2007).

Remark 3: In Ferramosca, Limon, Alvarado, Alamo, and Camacho (2009) the MPC for tracking is extended considering a general offset cost function. Convergence to a set-point which minimises the offset cost function is ensured considering only a convex, positive definite and subdifferential function. Moreover, the proposed MPC for tracking deals with the case that the target is inconsistent with the prediction model or the constraints. If this function can be bounded above by $c\|x\|_1$, then the results presented in this article can be applied.

Some questions arise from this result as how a suitable value of the parameter α can be determined for all possible set of parameters. Another issue is if there exists a region where local optimality property holds for a given value of α. The following section, in which these issues are analysed, constitutes a contribution of this article.

4.1. *Characterisation of the region of local optimality and calculation of T*

From the previously presented results, it can be seen that this issue can be studied by characterising the region where the norm of the Lagrange multiplier $\nu(x,\theta)$ is lower than or equal to α. Once this region is determined, the open questions on the local optimality can be answered. The characterisation of this region is done by means of results of multiparametric quadratic programming problems (Bemporad et al. 2002).

To this aim, first, note that the optimisation problem $P_N^r(x,\theta)$ is a multiparametric problem and the set of parameters (x,θ) such that $P_N^r(x,\theta)$ is feasible is given by $\Gamma=\{(x,\theta):x\in\mathcal{X}_N^r(\theta)\}$. It can be proved that this set is a polytope.

This optimisation problem can be casted as a multiparametric quadratic programming (mp-QP) problem (Bemporad et al. 2002) in the set of the parameters $(x,\theta)\in\Gamma$, which can be defined as

$$\begin{aligned}&\min_z \frac{1}{2}z'Hz\\ &\text{s.t. } Gz\leq W+S_1x+S_2\theta,\\ &\quad Fz=Y+T_1x+T_2\theta,\end{aligned}\tag{10}$$

where

$$z=\begin{bmatrix}u\\ \bar{\theta}\end{bmatrix}+J_1x+J_2\theta \tag{11}$$

with J_1 and J_2 suitable matrices. $Gz\leq W+S_1x+S_2\theta$ describes the restrictions (8b)–(8f), and $Fz=Y+T_1x+T_2\theta$ is the only equality constraint represented by Equation (8g). Note that $H > 0$, then the problem is strictly convex.

The Karush–Kuhn–Tucker (KKT) optimality conditions (Boyd and Vandenberghe 2006) for this problem are given by

$$Hz + G'\lambda + F'\nu = 0, \tag{12a}$$

$$\lambda(Gz - W - S_1x - S_2\theta) = 0, \tag{12b}$$

$$\lambda \geq 0, \tag{12c}$$

$$Gz - W - S_1x - S_2\theta \leq 0, \tag{12d}$$

$$Fz - Y - T_1x - T_2\theta = 0. \tag{12e}$$

Solving (12a) for z and substituting in the other equations, we obtain a new set of constraints for the Lagrange dual problem associated with the problem (10) which depends on $(\lambda, \nu, x, \theta)$. Then the following region

$$\Delta = \left\{ (\lambda,\nu,x,\theta) : \left| \begin{array}{l} \lambda'(GH^{-1}G'\lambda + GH^{-1}F'\nu + W \\ \quad +S_1x + S_2\theta) = 0 \qquad \lambda \geq 0 \\ -(GH^{-1}G'\lambda + GH^{-1}F'\nu + W \\ \quad + S_1x + S_2\theta) \leq 0 \\ FH^{-1}G'\lambda + FH^{-1}F'\nu + Y + T_1x \\ \quad +T_2\theta = 0 \end{array} \right. \right\} \tag{13}$$

defines the set of $(\lambda, \nu, x, \theta)$ which is solution of the KKT conditions. Thus, for any $(x, \theta) \in \mathrm{Proj}_{(x,\theta)}\Delta$, the solution of the KKT equations is $(\lambda(x,\theta), \nu(x,\theta))$ such that $(x, \theta, \lambda(x,\theta), \nu(x,\theta)) \in \Delta$. Note that $\mathrm{Proj}_{(x,\theta)}\Delta$ is the set of (x,θ) where a feasible solution exists and hence $\mathrm{Proj}_{(x,\theta)}\Delta = \Gamma$ and it is polytope (Boyd and Vandenberghe 2006).

Following the same arguments of Bemporad et al. (2002), the finite number of inequality constraints makes that there exists a finite combination of possible active constraints. Consider the jth combination and assume that $\breve{\lambda}^j$ and $\tilde{\lambda}^j$ denote the Lagrange multipliers vectors set of inactive and active inequality constraints, respectively. Let $\breve{G}^j$, $\breve{W}^j$, $\breve{S}^j_1$, $\breve{S}^j_2$, and $\tilde{G}^j$, $\tilde{W}^j$, $\tilde{S}^j_1$, $\tilde{S}^j_2$ be the corresponding matrices derived from a suitable partition of matrices G, W, S_1 and S_2 for the set of inactive and active constraints. In virtue of the complementary slackness condition, we have that $\breve{\lambda}^j = 0$ for inactive constraints and $\tilde{G}^jH^{-1}\tilde{G}'j\tilde{\lambda}^j + \tilde{G}^jH^{-1}F'\nu + \tilde{W}^j + \tilde{S}^j_1x + \tilde{S}^j_2\theta = 0$ for active constraints. Then, the jth combination of active constraints remains active for every $(x, \theta, \lambda, \nu)$ contained in the following polyhedral region:

$$\Delta_j = \left\{ (\lambda,\nu,x,\theta) : \lambda = (\tilde{\lambda}^j, \breve{\lambda}^j) \left| \begin{array}{l} \breve{\lambda}^j = 0 \\ \tilde{\lambda}^j \geq 0, \quad j = 1,\ldots,N \\ \breve{G}^jH^{-1}F'\nu + \breve{W}^j + \breve{S}_1{}^jx + \breve{S}_2{}^j\theta > 0 \\ \tilde{\lambda}^j = -(\tilde{G}^jH^{-1}\tilde{G}'^j)^{-1}(\tilde{G}^jH^{-1}F'\nu + \tilde{W}^j + \tilde{S}^j_1x + \tilde{S}^j_2\theta) \\ FH^{-1}\tilde{G}'j\tilde{\lambda}^j + FH^{-1}F'\nu + Y + T_1x + T_2\theta = 0 \end{array} \right. \right\}. \tag{14}$$

It is clear that the union of every region Δ_j of a possible combination of active constraints is such that $\Delta = \bigcup_j \Delta_j$ and hence Δ is a polygon.

Using these results, the maximum and the minimum value of $\|\nu(x,\theta)\|_1$ for all possible values of (x, θ) can be computed, that is, the values of $\alpha_{\min}$ and $\alpha_{\max}$ such that for all $(x,\theta) \in \Gamma$, $\alpha_{\min} \leq \|\nu(x,\theta)\|_1 \leq \alpha_{\max}$. These are calculated by solving the following optimisation problems:

$$\alpha_{\max} = \max_{(x,\theta,\lambda,\nu)\in\Delta} \|\nu\|_1 = \max_j \left(\sup_{(x,\theta,\lambda,\nu)\in\Delta_j} \|\nu\|_1 \right), \tag{15}$$

$$\alpha_{\min} = \min_{(x,\theta,\lambda,\nu)\in\Delta} \|\nu\|_1 = \min_j \left(\inf_{(x,\theta,\lambda,\nu)\in\Delta_j} \|\nu\|_1 \right). \tag{16}$$

It is remarkable that each supremum and infimum can be calculated by solving a set of linear programming (LP) problems in the closure of Δ_j. Besides, since the optimisation problem $P^r_N(x, \theta)$ is such that the solution of the KKT conditions is unique, then the value of $\alpha_{\max}$ is finite.

We are also interested in characterising the set of (x,θ), $\Gamma(\alpha)$, such that the norm of the associate Lagrange multiplier $\nu(x,\theta)$ is bounded by α, that is

$$\Gamma(\alpha) = \{(x,\theta) : \exists(\lambda,\nu) \text{ s.t. } (\lambda,\nu,x,\theta) \in \Delta \text{ and } \|\nu\|_1 \leq \alpha\}.$$

This region can be characterised by means of the polyhedral partition of Δ. Defining the set $\Gamma_j(\alpha) = \{(x, \theta) : \exists(\lambda, \nu) \text{ s.t. } (\lambda, \nu, x, \theta) \in \Delta_j \text{ and } \|\nu\|_1 \leq \alpha\}$, which is a polyhedron, it can be seen that $\Gamma(\alpha)$ is a polygon given by $\Gamma(\alpha) = \bigcup_j \Gamma_j(\alpha)$. Note that set $\Gamma(\alpha)$ is non-empty for $\alpha > \alpha_{\min}$. Moreover, if $\alpha_{\min} < \alpha_a \leq \alpha_b$, then for all $(x, \theta) \in \Gamma(\alpha_a)$, $\|\nu(x, \theta)\|_1 \leq \alpha_a \leq \alpha_b$ and hence $(x,\theta) \in \Gamma(\alpha_b)$. Therefore, $\Gamma(\alpha_a) \subseteq \Gamma(\alpha_b)$.

Resorting to the previously presented results, the following lemma can be derived.

Lemma 4.3: *Consider that Lemma* 4.2 *holds. Let* $\alpha_{\max}$ *and* $\alpha_{\min}$ *be the solutions of* (15) *and* (16), *respectively, then*:

- for all $\alpha > \alpha_{\min}$, there exists a polygon $\Gamma(\alpha)$ such that if $(x, \theta) \in \Gamma(\alpha)$ and $\|T^{-1}\|_1 \leq \frac{1}{\alpha n_\theta}$, then $V_N^r(x, \theta) = V_N^O(x, \theta)$;
- for all $\alpha_{\min} < \alpha_a \leq \alpha_b$, $\Gamma(\alpha_a) \subseteq \Gamma(\alpha_b)$, that is, $\Gamma(\alpha)$ grows monotonically with α;
- for all $\alpha \geq \alpha_{\max}$, $\Gamma(\alpha) = \mathrm{Proj}_{(x,\theta)}\Delta = \Gamma$.

In the following theorem, the property of local optimality for the MPC for tracking is stated.

Theorem 4.4 Local optimality: *Consider that Lemmas* 4.2 *and* 4.3 *hold. Define the following region*

$$\mathcal{W}(\alpha, \theta) = \{x \in \Upsilon_N(\theta) : (\phi(i; x, \kappa_N^O(\cdot, \theta)), \theta) \in \Gamma(\alpha), \quad \forall i \geq 0\}$$

and let the terminal control gain K be the one of the unconstrained LQR. Then

(1) *for all* $\alpha > \alpha_{\min}$, $\mathcal{W}(\alpha, \theta)$ *is a non-empty polygon and it is a positively invariant set of the controlled system*;
(2) *if* $\alpha_{\min} < \alpha_a \leq \alpha_b$, *then* $\mathcal{W}(\alpha_a, \theta) \subseteq \mathcal{W}(\alpha_b, \theta)$;
(3) *if* $\alpha > \alpha_{\min}$, $\|T^{-1}\|_1 \leq (\alpha n_\theta)^{-1}$ *and* $x(0)$ *and* θ *are such that* $x(0) \in \mathcal{X}_N^r(\theta)$, *then*

   (a) *there exists an instant* $\bar{k}$ *such that* $x(\bar{k}) \in \mathcal{W}(\alpha, \theta)$ *and* $\kappa_N^O(x(k), \theta) = \kappa_\infty(x(k), \theta)$ *for all* $k \geq \bar{k}$;
   (b) *if* $\alpha \geq \alpha_{\max}$ *then* $\kappa_N^O(x(k), \theta) = \kappa_N^r(x(k), \theta)$ *for all* $k \geq 0$ *and there exist an instant* $\bar{k}$ *such that* $x(\bar{k}) \in \Upsilon_N(\theta)$ *and* $\kappa_N^O(x(k), \theta) = \kappa_\infty(x(k), \theta)$ *for all* $k \geq \bar{k}$.

Proof:

- From Lemma 4.1 we have that set $\Upsilon_N(\theta)$ is an invariant set for the system controlled by $u = \kappa_N^r(x, \theta)$ and besides, $\kappa_N^r(x, \theta) = \kappa_\infty(x, \theta)$. Since the control law $\kappa_N^r(x, \theta)$ is a piece-wise affine (PWA) function of (x, θ), the controlled system is PWA and the region $\Upsilon_N(\theta)$ is a polygon (Kerrigan 2000).

  On the other hand, set $\Xi(\alpha, \theta) = \{x : (\phi(i; x, \kappa_N^O(\cdot, \theta)), \theta) \in \Gamma(\alpha), \quad \forall i \geq 0\}$ is the maximum invariant set for the controlled system contained in the set $\{x : (x, \theta) \in \Gamma(\alpha)\}$ and besides in virtue of Lemma 4.3 for all $x \in \Xi(\alpha, \theta)$, $\kappa_N^O(x, \theta) = \kappa_N^r(x, \theta)$. The PWA nature of the control law ensures that $\Xi(\alpha, \theta)$ is a polygon.

  Finally, noting that $\mathcal{W}(\alpha, \theta) = \Upsilon_N(\theta) \cap \Xi(\alpha, \theta)$, we infer that $\mathcal{W}(\alpha, \theta)$ is a positively invariant polygonal set for the system controlled by $\kappa_N^O(x, \theta)$ and for all $x \in \mathcal{W}(\alpha, \theta)$, $\kappa_N^O(x, \theta) = \kappa_N^r(x, \theta) = \kappa_\infty(x, \theta)$.
- Since $\alpha_a \leq \alpha_b$, $\Gamma(\alpha_a) \subseteq \Gamma(\alpha_b)$. In virtue of the monotonicity of the maximal invariant set, $\Xi(\alpha_a, \theta) \subseteq \Xi(\alpha_b, \theta)$ and this implies that $\mathcal{W}(\alpha_a, \theta) \subseteq \mathcal{W}(\alpha_b, \theta)$.
- If $x(0) \in \mathcal{X}_N^r(\theta)$, then the closed-loop system is asymptotically stable to $(x_s, u_s) = M_\theta \theta$. Given that $\mathcal{W}(\alpha, \theta)$ has a non-empty interior and $x_s \in \mathcal{W}(\alpha, \theta)$ for any $\alpha > \alpha_{\min}$, there exist a $\bar{k}$ when $x(\bar{k}) \in \mathcal{W}(\alpha, \theta)$. Due to the invariance of $\mathcal{W}(\alpha, \theta)$, $x(k) \in \mathcal{W}(\alpha, \theta)$ for all $k \geq \bar{k}$. Taking into account Lemmas 4.2 and 4.3, $k_N^O(x(k), \theta) = k_\infty(x(k), \theta)$.
- From Lemma 4.3, for all $\alpha \geq \alpha_{\max}$ $\Gamma(\alpha) = \Gamma$, $\Xi(\alpha, \theta) = \mathcal{X}_N^r(\theta)$ and then $\mathcal{W}(\alpha, \theta) = \Upsilon_N(\theta)$. The result is derived from the last proposition.

□

From this theorem it can be inferred that for every T such that $\|T^{-1}\|_1 \leq (\alpha_{\min} n_\theta)^{-1}$, the MPC for tracking is locally optimal in a certain region. In particular, the value of $\alpha_{\min}$ is interesting from a theoretical point of view, because it is the critical value from which there exists a region of local optimality. In order to ensure the local optimality property of the standard MPC, one would like to know the maximal region into which the local optimality applies. This region is given for any $\alpha \geq \alpha_{\max}$. Then, from a practical point of view it is interesting to know $\alpha_{\max}$, but this requires the calculation of the partition of the feasibility region of the mpQP and the solution of a number of LPs. In the following corollary a method is proposed to calculate a value of $\alpha \geq \alpha_{\min}$ for which the local optimality region is the invariant set for tracking, by means of a single LP.

Corollary 4.5: *Consider that hypotheses of Theorem* 4.4 *hold. Let* α_Ω *be the solution of the following LP optimisation problem*:

$$\alpha_\Omega = \max_{x,\theta} \|(FH^{-1}F')^{-1}(Y + T_1 x + T_2 \theta)\|_1, \quad \text{s.t. } (x, \theta) \in \Omega_{t,K}^w. \tag{17}$$

Assume that $\|T^{-1}\|_1 \leq (\alpha_\Omega n_\theta)^{-1}$ *then for all* $x(0) \in \mathcal{X}_N^r(\theta)$, *there exists an instant* $\bar{k}$ *such that* $V_N^{O*}(x(k), \theta) = V_\infty^*(x(k), \theta)$ *and* $\kappa_N^O(x(k), \theta) = \kappa_\infty(x(k), \theta)$, *for all* $k \geq \bar{k}$.

Proof: Assume that no inequality constraint is active, then the Lagrange multiplier λ is zero. In this case, the KKT conditions are

$$-GH^{-1}F'\nu - W - S_1x - S_2\theta < 0,$$
$$-FH^{-1}F'\nu - Y - T_1x - T_2\theta = 0.$$

For any $(x, \theta) \in \text{int}(\Omega^w_{t,K})$, the optimal control law is the one of the unconstrained LQR, that is $u = K_{\text{LQR}}(x - x_s) + u_s$, where $(x_s, u_s) = M_\theta \ \theta$, such that $(x, u) \in \text{int}(\mathcal{Z})$. This means that no inequality constraint is active. Considering that $u = K_{\text{LQR}}(x - x_s) + u_s$ is the optimal control law of the unconstrained LQR, then for any $(x, \theta) \in \text{int}(\Omega^w_{t,K})$, $k^O_N(x, \theta) = K_{\text{LQR}}(x - x_s) + u_s$ and $x \in \Upsilon_N(\theta)$. Furthermore, for any $(x(\bar{k}), \theta) \in \text{int}(\Omega^w_{t,K})$, $(x(k), \theta) \in \text{int}(\Omega^w_{t,K})$ for any $k \geq \bar{k}$.

Hence, for any $(x, \theta) \in \text{int}(\Omega^w_{t,K})$, $\lambda(x, \theta) = 0$, and then $\nu(x, \theta) = -[(FH^{-1}F')^{-1}(Y + T_1x + T_2\theta)]$. Moreover, $\|\nu(x, \theta)\|_1 \leq \alpha_\Omega$, for any $(x, \theta) \in \text{int}(\Omega^w_{t,K})$.

Taking into account all these facts, if $\alpha \geq \alpha_\Omega$, then for any $x(0) \in \mathcal{X}_N$, there exists a $\bar{k} > 0$ such that $(x(\bar{k}), \theta) \in \text{int}(\Omega^w_{t,K})$, and hence $k^O_N(x, \theta)$ is the optimal control law. □

5. Example

5.1. *The two tank process*

We considered two cascaded tanks system of the four tank process located at the *Departamento de Ingeniería de Sistemas y Automática* at the Engineering School of the University of Seville (Alvarado 2007). A scheme of the system is presented in Figure 1. The nonlinear model of the system is

$$\frac{\mathrm{d}h_1}{\mathrm{d}t} = -\frac{a_1}{A}\cdot\sqrt{2gh_1} + \frac{a_3}{A}\cdot\sqrt{2gh_3} + \frac{\gamma}{A}\cdot q,$$
$$\frac{\mathrm{d}h_3}{\mathrm{d}t} = -\frac{a_3}{A}\cdot\sqrt{2gh_3} + \frac{1-\gamma}{A}\cdot q,$$

where h_1 and h_3 are the levels of water in each tank and q is the inlet flow. The cross-section of the tanks is $A = 0.06\,\text{m}^2$, the cross-sections of the outlets are $a_1 = 6.7371e^{-4}\,\text{m}^2$ and $a_2 = 4.0423e^{-4}\,\text{m}^2$, and $\gamma = 0.4$ (Alvarado 2007).

Linearising the model in an operating point given by $h^0_1 = 0.68\,\text{m}$, $h^0_3 = 0.65\,\text{m}$ and $q^0 = 2\,\text{m}^3/\text{h}$, and defining the variables $x_i = h_i - h^o_i$ and $u = q - q^o$ where $i = 1, 3$ we have that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \begin{bmatrix} \frac{-1}{\tau_1} & \frac{1}{\tau_3} \\ 0 & \frac{-1}{\tau_3} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma}{A} \\ \frac{1-\gamma}{A} \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x,$$

where $\tau_i = \frac{A}{a_i}\sqrt{\frac{2h^0_i}{g}} \geq 0$, $i = 1, 3$, are the time constants of each tank.

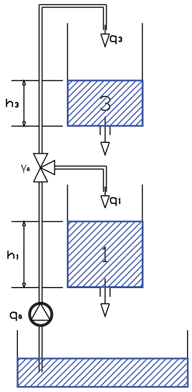


Figure 1. The two tanks system.

The system is constrained to $0.30 \leq x_1 \leq 1.36$, $0.30 \leq x_2 \leq 1.30$ and $0 \leq u \leq q_{\max}$, where $q_{\max} = 4$. The properties of the controller have been illustrated by means of simulations of the system.

In order to show the property of offset minimisation (Section 3, Remark 2) of the controller, the aim of the first test is to demonstrate the tracking property of the proposed controller. The offset cost function has been chosen as $V_O = \|T(\bar{\theta} - \theta)\|_\infty$. In the test, three references have been considered. The first reference and the second reference, $\text{Ref}_1 = (1.2, 1.17)$ and $\text{Ref}_2 = (0.4, 0.39)$, are admissible set-points. The third reference, $\text{Ref}_3 = (0.8, 1)$, is a not consistent operation point. The initial state is $x_0 = (0.32, 1.26)$. An MPC with $N = 3$ has been considered. The weighting matrices have been chosen as $Q = I_2$ and $R = 100 \times I_1$. Matrix P is the solution of the Riccati equation and $T = \alpha I_1$.

The maximal invariant set for tracking $\Omega_{t,K}$, the region of attraction $\mathcal{X}_3$, the set of equilibrium levels $\mathcal{X}_s = \text{Proj}_x(\mathcal{Z}_s)$ and the evolution of the levels for a given reference are shown in Figure 2(a). The time evolution of the system is shown in Figure 2(b). As it can be seen, since Ref_1 is an admissible set-point, the system reaches the first reference without any offset. At the sample times 500 the reference changes but the system still reaches the point without any offset, since Ref_2 is an admissible set-point. At time 1000 the reference changes, becoming a not consistent point. Note how the controller leads the system to the closest equilibrium point, in the sense that the offset cost function is minimised.

To illustrate the property of the local optimality, the proposed controller has been compared with the MPC for tracking with quadratic offset cost proposed in (Limon et al. 2008) and with the LQR. First, the difference between the MPC for tracking with quadratic offset cost V^*_N and the MPC for tracking with ∞-norm offset cost V^{O*}_N, with the MPC for regulation

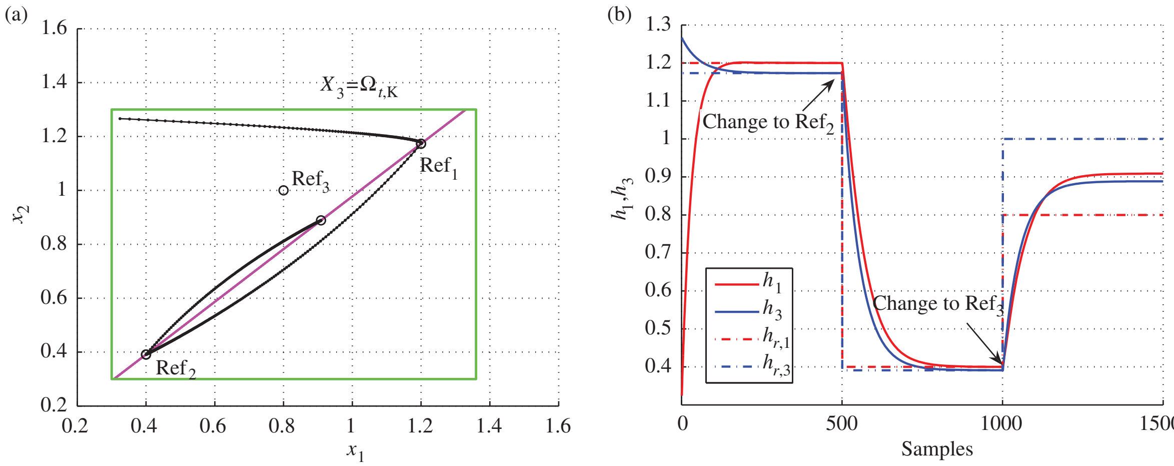
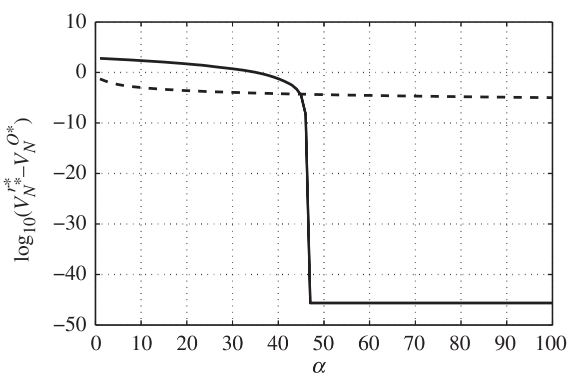


Figure 2. State-space and time evolutions. (a) Evolution of the levels and (b) Time evolution of the plant.

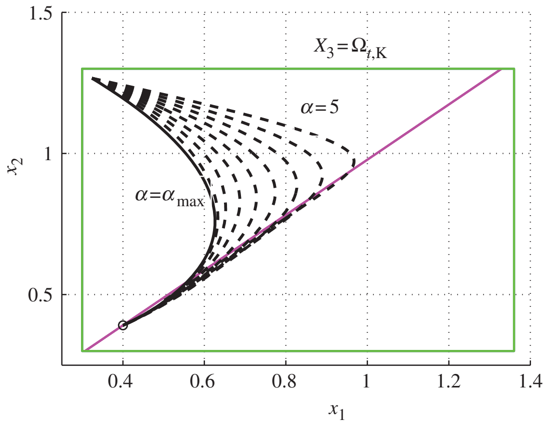


α	$V_N^{r*}-V_N^{O*}$
45	0.0111
46	2.51$e-4$
46.1	4.77$e-5$
46.17	4.09$e-7$
46.177	1.60$e-10$
46.1772	0
47	0
48	0

Figure 3. Difference between the regulation cost and the tracking cost versus α.

V_N^{r*} are illustrated, that are $V_N^{r*}-V_N^{O*}$ and $V_N^{r*}-V_N^{*}$. To this aim, the quadratic offset cost function has been chosen as $\|\bar{\theta}-\theta\|_{T_p}^2$ with $T_p=\alpha M_x^T P M_x$, where $M_x=[I_n, \mathbf{0}_{n\times m}]M_\theta$. The system has been considered to be steered to the point $x=(0.4, 0.39)$. In Figure 3 the value of $V_N^{r*}-V_N^{O*}$ versus α is plotted in solid line and the value of $V_N^{r*}-V_N^{*}$ versus α in dashed line. As it can be seen, $V_N^{r*}-V_N^{*}$ tends to zero asymptotically while $V_N^{r*}-V_N^{O*}$ drops to (practically) zero dramatically for a certain value of α. This result shows that the optimality gap can be made arbitrarily small by means of a suitable penalisation of the square of the 2 norm, and this value asymptotically converge to zero (Alvarado 2007), while in the case of the ∞-norm, the difference between the optimal value of the MPC for tracking cost function and the standard MPC for regulation cost function becomes zero. This shows the benefit of the new formulation of the MPC for tracking. Note how the value of $V_N^{r*}-V_N^{O*}$ drops to practically zero when $\alpha=47$. As we said, in Section 4, this happens because the value of α becomes greater than the value of the Lagrange multiplier of the equality constraint of the regulation problem V_N^{r*}. To point out this fact, consider that, for this example, the value of the Lagrange multiplier of the equality constraint of the regulation problem V_N^{r*}, is $\alpha_{\max}=46.1772$. The value of α_Ω, calculated by solving problem (17), is $\alpha_\Omega=46.1772$. This value is equal to the value of $\alpha_{\max}$ because the region Δ into which the value of $\alpha_{\max}$ can be evaluated, is exactly the invariant set for tracking $\Omega_{t,K}^{w}$. In the table, the value of $V_N^{r*}-V_N^{O*}$ in case of different values of the parameter α is presented. Note how the value seriously decrease when α becomes equal to $\alpha_{\max}$. So, using the procedure described in Section 4, we can determine the value of $\alpha_{\max}$ such that $V_N^{O*}(x, y_t)=V_N^{r*}(x, y_t)$.

To definitely prove the optimal performances ensured by the proposed controller, the optimal trajectories from the point $x_0=(0.32, 1.26)$ to the point $x=(0.4, 0.39)$ have been calculated, for a value of α that varies in the set $\alpha=\{5, 10, 15, 20, 25, 30, 35, 40, 45, \alpha_{\max}\}$. In Figure 4 the state-space trajectories



α	V_∞
5	203.6146
10	97.7560
15	63.0969
20	46.3738
25	36.9198
30	31.1948
35	27.7033
40	25.7250
45	24.9113
46.1772	24.8735

Figure 4. State-space trajectories and optimal cost for α varying.

and the values of the optimal cost V_∞ for α increasing are shown. See how the trajectories get better and how the value of the optimal cost decreases as the value of α increases. The optimal trajectory, in solid line, is the one for which $\alpha = \alpha_{\max}$. Notice that value of the optimal cost decreases from $V_\infty = 203.6146$ to $V_\infty = 24.8735$ when α reaches the value of $\alpha_{\max}$.

References

Alvarado, I. (2007), 'Model Predictive Control for Tracking Constrained Linear Systems', Ph.D. Thesis, University de Sevilla.

Angeli, D., Casavola, A., and Mosca, E. (2000), 'Predictive PI-control of Linear Plants Under Positional and Incremental Input Saturations', *Automatica*, 36, 1505–1516.

Bemporad, A., Casavola, A., and Mosca, E. (1997), 'Nonlinear Control of Constrained Linear Systems via Predictive Reference Management', *IEEE Transactions on Automatic Control*, 42, 340–349.

Bemporad, A., Morari, M., Dua, V., and Pistikopoulos, E. (2002), 'The Explicit Linear Quadratic Regulator for Constrained Systems', *Automatica*, 38, 3–20.

Boyd, S., and Vandenberghe, L. (2006), *Convex Optimization*, Cambridge: Cambridge University Press.

Camacho, E.F., and Bordons, C. (2004), *Model Predictive Control* (2nd ed.), London, UK: Springer-Verlag.

Chisci, L., and Zappa, G. (2003), 'Dual Mode Predictive Tracking of Piecewise Constant References for Constrained Linear Systems', *International Journal of Control*, 76, 61–72.

Ding, B., Xi, Y., and Li, S. (2004), 'A Synthesis Approach of On-line Constrained Robust Model Predictive Control', *AUT*, 40, 163–167.

Ferramosca, A., Limon, D., Alvarado, I., Alamo, T., and Camacho, E.F. (2009), 'MPC for Tracking with Optimal Closed-loop Performance', *Automatica*, 45, 1975–1978.

Hu, B., and Linnemann, A. (2002), 'Towards Infinite-horizon Optimality in Nonlinear Model Predictive Control', *IEEE Transactions on Automatic Control*, 47, 679–682.

Kerrigan, E.C. (2000), 'Robust Constraint Satisfaction: Invariant Sets and Predictive Control', Ph.D. Thesis, University of Cambridge.

Limon, D., Alvarado, I., Alamo, T., and Camacho, E.F. (2008), 'MPC for Tracking of Piece-wise Constant References for Constrained Linear Systems', *Automatica*, 44, 2382–2387.

Luenberger, D.E. (1984), *Linear and Nonlinear Programming*, Reading, Massachusets, USA: Addison-Wesley.

Mayne, D.Q., Rawlings, J.B., Rao, C.V., and Scokaert, P.O.M. (2000), 'Constrained Model Predictive Control: Stability and Optimality', *Automatica*, 36, 789–814.

Muske, K. (1997), 'Steady-state Target Optimization in Linear Model Predictive Control' in *Proceedings of the American Control Conference, ACC'97*, Albuquerque, New Mexizo, USA, June 4–6, 1997.

Pannocchia, G., and Kerrigan, E. (2005), 'Offset-free Reciding Horizon Control of Constrained Linear Systems', *AIChE Journal*, 51, 3134–3146.

Rao, C., and Rawlings, J. (1999), 'Steady States and Constraints in Model Predictive Control', *AIChE Journal*, 45, 1266–1278.

Rossiter, J., Kouvaritakis, B., and Gossner, J. (1996), 'Guaranteeing Feasibility in Constrained Stable Generalised Predictive Control', *IEEE Proceedings of Control theory Applications*, 143, 463–469.

Scokaert, P.O.M., and Mayne, D.Q. (1998), 'Min-max Feedback Model Predictive Control for Constrained Linear Systems', *IEEE Transactions on Automatic Control*, 43, 1136–1142.

Scokaert, P.O.M., and Rawlings, J.B. (1998), 'Constrained Linear Quadratic Regulator', *IEEE Transactions on Automatic Control*, 43, 1163–1169.

Appendix

Lemma A.1: *Let the Assumptions of Theorem* 3.2 *hold. Consider a desired steady-state* $(x_s, u_s) = M_\theta \theta$ *and assume that for a given state* x*, the optimal solution of* $P_N^O(x, \theta)$ *is such that* $\|x - \bar{x}_s^*(x,\theta)\|_Q = 0$ *(i.e.* $x = \bar{x}_s^*(x,\theta)$*), then* $\|x - x_s\|_Q = 0$.

Proof: The proof is obtained by contradiction. Consider $\bar{\theta}^* = \bar{\theta}^*(x, \theta)$ and $(\bar{x}_s^*, \bar{u}_s^*) = M_\theta \bar{\theta}^*$. Assume that $\bar{x}_s^* \neq x_s$.

By continuity we obtain that exists $\hat{\lambda} \in [0, 1)$ such that for every $\lambda \in [\hat{\lambda}, 1)$, $\tilde{\theta} = \lambda\bar{\theta}^* + (1-\lambda)\theta$ and $(\tilde{x}_s, \tilde{u}_s) = M_\theta \tilde{\theta}$, the state $\bar{x}_s^*$ is contained in the maximal admissible invariant set (denoted as $\Omega_\infty(\tilde{x}_s)$) for the nominal system controlled by $u = K(x - \tilde{x}_s) + \tilde{u}_s$ (Limon et al. 2008).

Defining as $\mathbf{u}$ the sequence of control actions derived from this control law, it is inferred that $(\mathbf{u}, \bar{x}_s^*, \tilde{\theta})$ is a feasible solution for $P_N(\bar{x}_s^*, \theta)$. Then, from Assumption 3.1,

$$V_N^{O*}(\bar{x}_s^*, \theta) \leq V_N^O(\bar{x}_s^*, \theta; \mathbf{u}, \tilde{\theta}) = \|\bar{x}_s^* - \tilde{x}_s\|_P^2 + \|T(\tilde{\theta} - \theta)\|_\infty.$$

Defining $M_x = [I_n, \; \mathbf{0}_{n\times m}]M_\theta$ and taking into account that $\bar{x}_s^* - \tilde{x}_s = (1-\lambda)M_x(\bar{\theta}^* - \theta)$ and $\tilde{\theta} - \theta = \lambda(\bar{\theta}^* - \theta)$, we have that

$$\|\bar{x}_s^* - \tilde{x}_s\|_P^2 + \|T(\tilde{\theta} - \theta)\|_\infty \leq (1-\lambda)^2\|M_x(\bar{\theta}^* - \theta)\|_P^2 + \lambda\|T(\bar{\theta}^* - \theta)\|_\infty.$$

Note that the rhs of this equation takes a value of $\|T(\bar{\theta}^* - \theta)\|_\infty$ for $\lambda = 1$, and besides, its derivative w.r.t. λ for $\lambda = 1$ is also equal to $\|T(\bar{\theta}^* - \theta)\|_\infty$ which is assumed to be strictly positive (namely $\bar{\theta}^* \neq \theta$). Then there exists a $\tilde{\lambda} \in (0, 1)$ such that for all $\lambda \in [\tilde{\lambda}, 1)$, we have that

$$V_N^{O*}(\bar{x}_s^*, \theta) \leq \|\bar{x}_s^* - \tilde{x}_s\|_P^2 + \|T(\tilde{\theta} - \theta)\|_\infty < \|T(\bar{\theta}^* - \theta)\|_\infty.$$

Since the optimal solution of $P_N(\bar{x}_s^*, \theta)$ is given by $\mathbf{u}^*(\bar{x}_s^*, \theta) = \{u_s^*, \ldots, u_s^*\}$ and the associated nominal state sequence is $\mathbf{x}^*(\bar{x}_s^*, \theta) = \{\bar{x}_s^*, \ldots, \bar{x}_s^*\}$, then the optimal cost is $V_N^{O*}(\bar{x}_s^*, \theta) = \|T(\bar{\theta}^* - \theta)\|_\infty$, yielding a contradiction and proving the lemma. □