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Online robust tube-based MPC for time-varying systems: a practical approach

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This article focuses on the design of a robust model predictive control law for constrained discrete-time time-varying systems with additive uncertainties. The proposed solution to the control problem is a tube-based MPC ensuring robustness and constraints fulfilment. Reachable sets are calculated online taking into account the system dynamics by means of an adaptive local control law and additive uncertainties. The proposed method represents a trade-off between small conservativeness and efficient real-time execution. This approach is applied to solve the trajectory tracking problem of a mobile robot. Simulation results provide a comparison between the tube-based MPC scheme and established motion control algorithms, showing the efficient execution and satisfactory behaviour of the proposed controller.

Keywords: robust control; model predictive control; reachable sets; LMI; invariant sets; mobile robots

1. Introduction

Model Predictive Control (MPC) constitutes a popular control technique to deal with constrained systems (Morari and Lee 1999; Mayne, Rawlings, Rao, and Scokaert 2000; Camacho and Bordons 2007). However, in order to tackle uncertainty, robust MPC formulations must be considered (Bemporad and Morari 1999).

The problem of robustness has been widely addressed in the context of MPC through different approaches. The first possible approach is to rely on the inherent robustness of deterministic MPC, which guarantees some degree of robustness due to the own feedback nature of predictive controllers (Limon, Alamo, and Camacho 2002). The drawback of deterministic MPC appears since the optimisation problem is only solved for a nominal system. However due to uncertainty the system can have different trajectories, and this can lead to constraints violation.

A popular technique to robustify MPC is the open-loop Min–Max MPC (Bemporad and Morari 1999). This strategy finds the value of the control signal by minimising the cost associated to the worst case of expected disturbances and uncertainty. The main drawback of open-loop Min–Max formulation is that the control action may be excessively conservative (Alamo, Ramirez, and Camacho 2005; Ramirez, Alamo, Camacho, and de la Peña 2006). In order to overcome the conservativeness, an attractive alternative is the feedback or closed-loop MPC (Scokaert and Mayne 1998). In the feedback MPC, the decision variable is a sequence of control laws that permits to reduce the spread of predicted trajectories resulting from uncertainty. However, the computation burden is often prohibitive since the related optimisation problem is non convex (Mayne et al. 2000). One interesting result concerning the problem of computational complexity of the feedback MPC is (Goulart, Kerrigan, and Maciejowski 2006).

An efficient technique for practical implementation of robust MPC is the tube-based MPC. In the pioneering work (Bertsekas and Rhodes 1971), the design of a robust control law ensuring hard constraints satisfaction is addressed by means of the computation of a sequence of state space regions, called *reachability tube*. The term tube-based refers to those control techniques whose objective is to maintain all the possible trajectories of an uncertain system inside a sequence of admissible regions by using set-theory related tools. Such approach has been widely employed to robustify MPC (Chisci, Rossiter, and Zappa 2001; Mayne and Langson 2001; Langson, Chryssochoos, Rakovic, and Mayne 2004; Mayne, Rakovic, Findeisen, and Allgower 2009; Limon, Alvarado, Alamo, and Camacho 2010; Trodden and Richards 2010).

Tube-based MPC approaches are motivated by the fact that the predicted evolution of a system obtained

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using a nominal model differs from the real evolution due to uncertainty. An MPC formulation that permits to consider this mismatch in the controller synthesis is the tube-based one, whose basis consists in computing the region around the nominal prediction that contains the state of the system under any possible uncertainties (Limon, Bravo, Alamo, and Camacho 2005). This region can be obtained in two different ways. One possibility is to calculate it at each step within the prediction horizon, this leads to a sequence of regions, $\{\mathcal{R}_i\}$, called reachable sets, that is, the smallest sets of states of the closed-loop uncertain system that ensure to contain the state at time i of any trajectory starting at the origin (Chisci et al. 2001). The second possibility is to determine a single region, which bounds the sequence of reachable sets, that is, $\mathcal{R}_i \subseteq \mathcal{R}_{i+1} \subseteq \mathcal{R}$, where usually $\mathcal{R}$ is a robust invariant set (Mayne and Langson 2001; Langson et al. 2004). The main difference between both approaches is that in the former case, online computation is slightly heavier since at each step within the prediction horizon is necessary to calculate the reachable sets. However, the latter case entails a higher degree of conservativeness, since the invariant set can be seen as the limit of the reachable sets and it contains every $\mathcal{R}_i$, for linear systems.

Tube-based MPC usually employs a pre-stabilising control policy (Chisci et al. 2001; Mayne and Langson 2001; Goulart et al. 2006). A local feedback gain compensates the mismatch between the nominal and real evolution of the system and a deterministic MPC controller is employed to stabilize the nominal system for which tighter constraint sets are considered.

This article proposes a robust tube-based MPC controller via reachable sets. Generally, reachable sets calculation is related to time-invariant systems, where a local feedback gain compensates the system realisation, leading to small conservativeness (Chisci et al. 2001). For the case of time-varying systems, the reachable sets depend on the realisation of the system dynamics, besides of on the uncertainty. Then, a proper time-varying control feedback could be designed to obtain smaller reachable sets and, then, to reduce the conservativeness. The main contribution of this work is that we calculate the reachable sets online compensating the current system dynamics by means of an adaptive local control law. Thanks to this approach, the conservativeness is reduced for the time-varying case and the feasible region is enlarged in comparison to similar approaches where reachable sets are solved offline (Langson et al. 2004; Limon et al. 2005; Gonzalez et al. 2011). Finally, Linear Matrix Inequalities (LMI) are employed to determine a Lyapunov function constituting the terminal cost for the MPC optimisation problem (Boyd, El Ghaoui, Feron, and Balakrishnan 1994; Kothare, Balakrishnan, and Morari 1996). Furthermore, a terminal robust invariant set is calculated through an adaptation of the ideas presented in the works of (Kolmanovsky and Gilbert 1998; Blanchini 1999). Several simulations related to the motion control of a mobile robot (where low computation burden is required) are presented to show the advantages of the proposed control scheme.

This article is organised as follows. Section 2 concerns with problem statement and control objectives. In Section 3, the online robust tube-based MPC control strategy is described. In Section 4, we demonstrate the performance of the proposal through two simulations related to the trajectory tracking problem of a mobile robot. Finally, Section 5 presents some conclusions and future research.

Notation: A *polyhedron* is the (convex) intersection of a finite number of open and/or closed half-spaces and a *polytope* is a closed and bounded polyhedron. Given two sets $X, Y \subseteq \mathbb{R}^n$, the Minkowski sum is defined by $X \oplus Y \triangleq (x + y \mid x \in X,\ y \in Y)$ and the Pontryagin set difference is $X \ominus Y \triangleq (x \mid x \oplus Y \subseteq X)$.

2. Problem statement

This article addresses the problem of designing a state feedback control for the following uncertain, linear, time-varying, discrete-time system

$$x_{k+1} = A_k^{\gamma} x_k + B u_k + w_k, \tag{1}$$

where $k \in \mathbb{Z}^+$ is the discrete-time, $x \in \mathbb{R}^n$ the state, $u \in \mathbb{R}^m$ the control input and w a bounded additive uncertainty, satisfying $w \in W$, with W a polytope in the state space $\mathbb{R}^n$ (Remark 1). The matrix $A^{\gamma} \in \mathbb{R}^{n \times n}$ depends on the time-varying parameter γ, the input matrix $B \in \mathbb{R}^{n \times m}$ is known and constant. For any admissible realisation of parameter $\gamma \in \Gamma$, a dynamic matrix A^{γ} is obtained. It follows that $A^{\gamma} \in \mathcal{A}$, with $\mathcal{A}$ represents the set of system matrices.

Assumption 2.1: *Assume that γ is known* a priori. *This implies that the system matrix A^{γ} is known at each sampling instant and at each step within the prediction horizon. The set Γ is a polytope in $\mathbb{R}^l$ and the set of system matrices $\mathcal{A}$ is also defined as a polytope in $\mathbb{R}^{n \times n}$. Note that this assumption makes sense for applications where system matrix can be known* a priori, *such as mobile robotics* (Gonzalez et al. 2010, 2011).

Remark 1: The set $W \subseteq \mathbb{R}^n$ represents the uncertainty affecting the state at each sampling instant. For instance, this set could bound the uncertainty in the robot location during the motion of such mobile robot (Thrun, Burgard, and Fox 2005).

The states and inputs are subject to the following constraints

$$x_k \in X, \quad u_k \in U, \tag{2}$$

where $X \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^m$ are polytopes that contain the origin.

We also define the nominal system as

$$\tilde{x}_{k+1} = A_k^{\gamma}\tilde{x}_k + Bg_k, \tag{3}$$

where $\tilde{x} \in \mathbb{R}^n$ is the nominal state, and $g \in \mathbb{R}^m$ is the control input for the nominal system. From now on, for notational convenience we use A_k to express A_k^{γ}.

Now, we calculate the difference between the real and the nominal system as (Chisci et al. 2001)

$$\bar{x}_k = x_k - \tilde{x}_k, \tag{4}$$

where $\bar{x} \in \mathbb{R}^n$ is the error state.

Then, the control objective is to compensate the mismatch between the real and the nominal state, and to steer the nominal system as close as possible to the reference without constraints violation. For that purpose, we consider the following control policy (Chisci et al. 2001; Goulart et al. 2006)

$$u_k = K_k\bar{x}_k + g_k, \tag{5}$$

where K_k is an adaptive local controller whose goal is to compensate the system realisation at each sampling instant (Section 3.1) and g_k deals with the nominal system (Section 3.4), once the uncertainty has been confined by means of the reachable sets calculation (Section 3.2).

Finally, the dynamics of the closed-loop error system is defined replacing (1) and (3) into (4), that is,

$$\bar{x}_{k+1} = x_{k+1} - \tilde{x}_{k+1} = \left(A_k + BK_k\right)\bar{x}_k + w_k. \tag{6}$$

The main features of the online robust tube-based MPC strategy are:

- Robustness: following the tube-based MPC policy, we take into account additive uncertainties and time-varying dynamics in the control design.
- Performance: an optimisation problem (QP) is solved at each sampling instant obtaining the proper control actions as a compromise between small deviations from the reference trajectory and suitable control actions.
- Input and state constraints fulfilment: this requirement is guaranteed by ensuring constraints satisfaction in the minimisation of the MPC control law.
- Asymptotic stability for the deterministic MPC: it is ensured through a quadratic Lyapunov function determined using LMI and a robust positively invariant set for the terminal region.
- Conservativeness reduction: system dynamics is taken into account by means of an adaptive local control law. This leads to reachable sets smaller than those obtained offline (Langson et al. 2004; Limon et al. 2005; Gonzalez et al. 2011), and hence to a wider feasible region.
- Real-time applicability: a standard nominal MPC is solved online to control the nominal system since the effect of the system uncertainties are already included in the restricted constraints. This fact implies that the robust tube-based MPC strategy fits properly to applications with fast dynamics and where high sampling frequencies are employed.

3. Online tube-based MPC

In this section, we present the online robust tube-based MPC approach (Figure 1). First, online computation is devoted to the calculation of the local feedback gain depending on the system realisation at each step within the prediction horizon (Section 3.1). Afterwards, reachable sets are calculated taking into account this feedback gain and additive uncertainty (Section 3.2). Then, a standard MPC problem with tighter constraints concerning the nominal system is solved (Section 3.4). Finally, LMI are employed to determine the terminal cost for the MPC optimisation problem and a terminal robust invariant set is calculated (Section 3.3).

The steps that will be explained subsequently to implement the online robust tube-based MPC approach are:

(1) Local control gain (Section 3.1). The local control law, K_k, that compensates the effect of the mismatch (4) has to be calculated. In this case, an adaptive local feedback control gain will be obtained. Such control strategy will be calculated solving an optimisation problem where constraints will be included in terms of LMI ensuring asymptotic stability and performance (Lyapunov function and LQR). In this way, such adaptive local control gain is solved *online* compensating the system realisation (time-varying) at each step within the prediction horizon

$$K_k = -(R + B^TPB)^{-1}B^TPA_k,$$

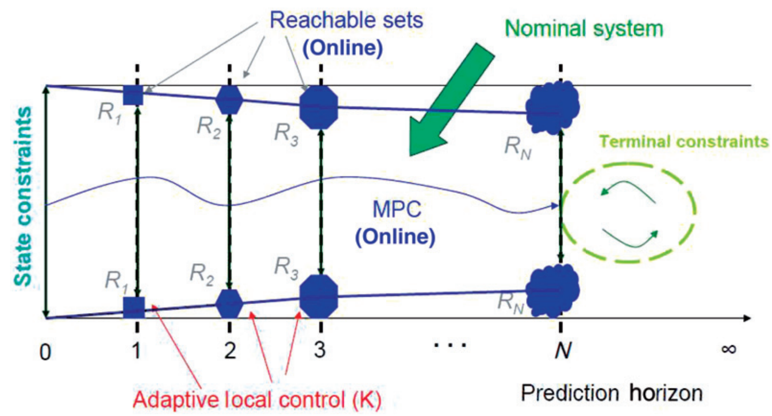

Figure 1. Robust tube-based MPC control strategy. Reachable sets are solved online depending on the current state realisation.

where K_k and A_k are the feedback gain and system matrix at each step within the prediction horizon, respectively, and matrix P, the positive definite matrix associated with a Lyapunov function, is obtained from an optimisation problem. The rest of the parameters are the input matrix, B, and matrix R which will be defined subsequently.

(2) Reachable sets (Section 3.2). Reachable sets representing a bound on the possible state of the mismatch closed-loop system (6) affected by uncertainty, are calculated *online* as

$$\mathscr{R}_{k+i+1} = (A_{k+i} + BK_{k+i})\mathscr{R}_{k+i} \oplus W \quad \forall i = 0, \ldots, N-1. \tag{7}$$

(3) Modification of state and input constraints (Section 3.4). In order to take into account the 'control effort' spent to compensate the system mismatch and the uncertainty, original state and input constraints are replaced with more restricted ones using the previously determined reachable sets

$$\tilde{X}_i = X \ominus \mathscr{R}_i \quad \forall i = 1, \ldots, N, \tag{8}$$

$$\tilde{U}_i = U \ominus K_i\mathscr{R}_i \quad \forall i = 0, \ldots, N-1. \tag{9}$$

Note that the modification of state and input constraints is carried out *online*.

(4) Terminal constraints for MPC (Section 3.3). Terminal cost and terminal invariant set ensuring asymptotic stability of the nominal predictive controller are calculated *offline*.

(5) A deterministic MPC strategy runs *online* for the nominal system with tighter constraints (8), (9) (Section 3.4).

Summing up, *online* computation is devoted to calculate the feedback gain K_k taking into account the system realisation at each step within the prediction horizon. Then, reachable sets are calculated considering the mismatch closed-loop system and the uncertainty. Finally, a deterministic MPC with tighter constraints is solved for the nominal system. Note that when reachable sets are calculated offline (Gonzalez et al. 2011; Limon et al. 2005), it entails a certain degree of conservativeness, since the information on the current dynamics is not used for computing the reachable sets. On the other hand, when reachable sets are replaced by an invariant set (Mayne and Langson 2001; Langson et al. 2004), it also entails a degree of conservativeness, since the invariant set contains all the reachable sets ($\mathscr{R}_i \subseteq \mathscr{R}_{i+1} \subseteq \mathscr{R}$). The main weakness of the proposed strategy is that computation time is slightly increased, due to the computation of (8), (9) online. For the proposed application (mobile robotics), admissible computation times are achieved (see the results presented in Section 4 where a sampling period of 0.35 [s] was employed).

3.1 *Online local compensation of system dynamics*

In this section the calculation of the local feedback control gain is addressed. Note that we are considering the closed-loop mismatch system (6) without additive uncertainty, since this term will be included in the reachable sets (Section 3.2). Hence, we are only interested in obtaining the proper local control gain, K_k, which compensates the mismatch system.

When dealing with the problem of determining asymptotically stable controllers, one classical way to proceed is to look for a Lyapunov function determined by a positive definite matrix $P > 0$, i.e. $V(\bar{x}) = \bar{x}^T P \bar{x}$, such that (Boyd, El Ghaoui, Feron, and

Balakrishnan 1994, Kothare, Balakrishnan, and Morari 1996)

$$V(\bar{x}_{k+1}) - V(\bar{x}_k) \leq 0, \quad \forall \bar{x} \neq 0. \tag{10}$$

Furthermore, in order to take into account the performance, the following quadratic objective is considered (Kothare et al. 1996)

$$V(\bar{x}_0) \geq \min_{u_{[0,\infty)}} \sum_{k=0}^{\infty} \bar{x}_k^T Q \bar{x}_k + u_k^T R u_k, \tag{11}$$

where $u_{[0,\infty)}$ denotes the infinite sequence of u_k, and $Q > 0$ and $R > 0$ are symmetric matrices weighting the state and input signals. Notice that $V(\cdot)$ is an upper bound of the optimal cost related to the LQR (Kothare et al. 1996).

Then, from previous requirements: asymptotic stability (10) and performance (11), the following optimisation problem can be proposed

$$\begin{array}{c} \min\limits_{P>0\ \kappa^j \forall \gamma^j} \ \mathrm{tr}(P) \\ \text{subject to } L^* \quad \forall \gamma^j, \end{array} \tag{12}$$

where $\mathrm{tr}(\cdot)$ means the trace of a matrix and L^* is defined by the following inequality

$$\begin{aligned} &\bar{x}^T (A^j + B\kappa^j)^T P (A^j + B\kappa^j) \bar{x} - \bar{x}^T P \bar{x} \\ &\quad \leq -\bar{x}^T (Q + (\kappa^j)^T R \kappa^j) \bar{x}, \quad \forall \bar{x} \in \mathbb{R}^n, \end{aligned} \tag{13}$$

where A^j refers to the j-th extreme realisation of the set $\mathscr{A}$, κ^j is the j-th feedback gain. Note that, all the required conditions are only imposed at the extremal values of the polytopic set Γ, i.e. at the n_γ vertices of Γ. Fulfilment of such conditions at the vertices yields the satisfaction at any point within Γ, as stated in Property 1 in (Gonzalez et al. 2010).

Previous optimisation problem can be solved using LMI formulation (Boyd et al. 1994; Gonzalez et al. 2010). Denoting $A_{cl}^j = A^j + B\kappa^j$, we get

$$\bar{x}^T ((A_{cl}^j)^T P A_{cl}^j) \bar{x} - \bar{x}^T P \bar{x} \leq -\bar{x}^T (Q + (\kappa^j)^T R \kappa^j) \bar{x}, \tag{14}$$

for every vertex γ^j of Γ, with $j = 1, \dots, n_\gamma$. Removing state variables from previous equation, it follows

$$(A_{cl}^j)^T P A_{cl}^j - P \leq -Q - (\kappa^j)^T R \kappa^j, \tag{15}$$

then, using the Schur complement (Boyd et al. 1994), the previous inequality becomes

$$\begin{bmatrix} P - Q - (\kappa^j)^T R \kappa^j & (A_{cl}^j)^T \\ A_{cl}^j & P^{-1} \end{bmatrix} \geq 0. \tag{16}$$

Operating previous LMI and replacing $S = P^{-1}$ and $Y^j = \kappa^j P^{-1}$, the linear matrix inequality to be fulfilled is given by

$$\begin{bmatrix} S & S(A^j)^T + (Y^j)^T B^T & SQ^{\frac{1}{2}} & (Y^j)^T R^{\frac{1}{2}} \\ A^j S + B Y^j & S & 0 & 0 \\ Q^{\frac{1}{2}} S & 0 & I & 0 \\ R^{\frac{1}{2}} Y^j & 0 & 0 & I \end{bmatrix} \geq 0. \tag{17}$$

This LMI is imposed for every vertex γ^j of Γ, with $j = 1, \dots, n_\gamma$. The convex optimisation problem to be solved is (Remark 2)

$$\begin{array}{c} \min\limits_{S>0,\ Y^j \forall \gamma^j} \ \mathrm{tr(P)} \\ \text{subject to (17)} \quad \forall \gamma^j, j = 1, \dots, n_\gamma, \end{array} \tag{18}$$

The solution of the optimisation problem determines the matrix P that guarantees the asymptotic stability of the mismatch system (without additive uncertainty) (Boyd et al. 1994; Kothare et al. 1996).

Remark 2: Matrix P is such that, for every possible realisation of the dynamics, there exists an appropriate feedback law such that the related quadratic function is a Lyapunov function. Then, once P is precalculated solving (18), we will see in the following how to choose online a particular feedback gain such that the quadratic function is decreasing. Note that an alternative approach could be to obtain matrix P online taking into account the actual system realisation. This would imply a conservativeness smaller than the one obtained here. However, it would require to solve an optimisation problem at each step within the prediction horizon. This would lead to increase the online computation burden with the risk of preventing to achieve one of our main purposes, which is the real-time applicability of the method to fast systems.

Once determined a positive definite matrix P related to the Lyapunov function V that ensures asymptotic stability, the next step is to calculate a local control law for the closed-loop system (6). Note that the adaptive local control law could be obtained as a convex combination of the previously determined gains (κ^j). However, this computation would require to solve online an LP problem for every step within the prediction horizon, see (Gonzalez et al. 2010).

Here, we formulate an explicit local feedback gain taking into account the online system realisation. In this way, computation burden is reduced and conservativeness would be similar to the time-invariant case. Furthermore, in order to ensure stability, we are explicitly considering the previous matrix P in the formulation of the feedback gain.

First, we follow an approach similar to the one presented in (18), but now, we consider an explicit solution for each sampling instant k. This leads to the following minimisation problem

$$\min_{\nu_k} \bar{x}_k^T Q \bar{x}_k + \nu_k^T R \nu_k + \left(A_k \bar{x}_k + B\nu_k\right)^T P\left(A_k \bar{x}_k + B\nu_k\right), \tag{19}$$

where $\nu_k = K_k \bar{x}_k$ is the local control law for the closed-loop mismatch system (6) without additive uncertainties. In this way, the convex cost function with respect to the control input ν is given by

$$\begin{aligned} f^* = {} & \bar{x}_k^T Q \bar{x}_k + \nu_k^T R \nu_k + \bar{x}_k^T A_k^T P A_k \bar{x}_k \\ & + \nu_k^T B^T P B \nu_k + 2\bar{x}_k^T A_k^T P B \nu_k. \end{aligned} \tag{20}$$

The optimal solution is analytically solved differentiating with respect to the control input ν_k and equating to zero, that is,

$$\frac{\partial f^*}{\partial \nu_k} = 2R\nu_k + 2B^T P B \nu_k + 2B^T P A_k \bar{x}_k = 0, \tag{21}$$

producing

$$\nu_k = -(R + B^T P B)^{-1} B^T P A_k \bar{x}_k. \tag{22}$$

Finally, the feedback control gain is defined as (Remark 3)

$$K_k = -(R + B^T P B)^{-1} B^T P A_k. \tag{23}$$

Remark 3: Note that the feedback gain (23) is obtained solving analytically the convex optimisation problem (19), which is related to a Lyapunov function where matrix P was calculated in (18). Therefore, the control gain (23) ensures asymptotic stability for the closed-loop mismatch system (6) without additive uncertainties. Recall that such uncertainties are bounded inside reachable sets (see the following section). The proof for a similar statement is found in (Kothare et al. 1996), see Theorem 3.

3.2 *Online reachable sets*

This section focuses on the reachable sets calculation. Consider the uncertain closed-loop system (6) whose evolution depends on the local control action ($K_k \bar{x}_k$) and the uncertainties. Then, the reachable set at first step within the prediction horizon is denoted as $\mathscr{R}_k = \{0\}$, where subscript k means current sampling instant. The rest of reachable sets are recursively calculated as (Remark 4)

$$\mathscr{R}_{k+i+1} \triangleq (A_{k+i} + BK_{k+i})\mathscr{R}_{k+i} \oplus W \quad \forall i = 0, \ldots, N-1, \tag{24}$$

where N is the prediction horizon.

Notice that $\mathscr{R}_{k+i+1}$ depends on the current system realisation, A_{k+i}, the control action, K_{k+i}, defined in (23) and the set of uncertainties, W.

Remark 4: Since reachable sets are calculated online, this means that large system uncertainty or large system complexity could lead to increase computation time. We remark that this is a mathematical problem related to the implementation of Minkowski sum algorithm and it is beyond the scope of this article. However, as we will see in Section 4, we achieve a low computation time for middle-size applications.

3.3 *Terminal cost and terminal invariant set*

A common approach to ensure the asymptotic stability of MPC consists in incorporating both a terminal cost (Υ) and a terminal constraint set (Ω) (Mayne et al. 2000). In this section, we are focusing in both issues related to the nominal system (3). Notice that, stability properties for analogous control strategies have been analyzed in the literature, see for instance (Limon et al. 2005).

The purpose of the terminal cost is to ensure closed-loop stability. To this end, it requires the use of a Lyapunov function with a stabilizing control law. In our case, a similar procedure to (18) has been followed for the nominal system (3), see (Gonzalez et al. 2011) for details.

On the other hand, the last element of the predicted state sequence must belong to an invariant set (Kolmanovsky and Gilbert 1998; Blanchini 1999). Once a stabilizing control law is given, we follow a similar approach to that found in (Kolmanovsky and Gilbert 1998; Blanchini 1999) to obtain the maximal robust invariant set for the uncertain system contained in the state constraint set. In addition, constraints on the input are considered. For that purpose, we employ the concept of one-step operator as

$$Q^j_{\mathscr{A}}(\Omega) = \{x \in X : \kappa^j x \in U, \quad (A^j + B\kappa^j)x + w \in \Omega \ \ \forall w \in W\}, \tag{25}$$

where A^j means the j-th extreme realisation of the set $\mathscr{A}$.

Note that the one-step operator is a standard tool for the invariant sets calculation through iterative procedures (Kolmanovsky and Gilbert 1998; Fiacchini 2010).

Then, given the one-step operator (25), the maximal robust invariant set for the uncertain system (1) is obtained by means of the following iterative procedure:

(1) Initialisation: $\Omega_0 = X \cap \{\omega \in \mathbb{R}^{n \times n} : \kappa^j \omega \in U \ \forall j = 1, \ldots, n_\gamma\}$.
(2) Iteration: $\Omega_{k+1} = \Omega_k \cap Q^j_{\mathscr{A}}(\Omega_k) \ \forall j = 1, \ldots, n_\gamma$.

(3) Termination condition: stop when $\Omega_{k+1} = \Omega_k$ or $\Omega_{k+1} = \emptyset$. Set $\Omega = \Omega_\infty = \Omega_{k+1}$.

An important issue when dealing with algorithmic procedure for computing the robust invariant sets is its finite determinedness, that is, the conditions under which the algorithm provides a solution after a finite number of iterations. Results regarding the problem of finite determination can be found in (Kolmanovsky and Gilbert 1998; Blanchini 1999). We have not proved finite determinedness for our case, since it is beyond the scope of this article.

3.4 *MPC Strategy*

Finally, once local feedback control law and reachable sets have been discussed, we focus on the online computation aspect related to MPC. Notice that, MPC policy deals with the nominal system, since the mismatch between the real system and the nominal one is compensated by the local control law in the reachable sets (24). For that reason, as explained in Figure 1, original constraints (2) must be replaced with more restricted ones. Following the ideas presented in (Chisci et al. 2001), imposing that $\tilde{x} \in \tilde{X}_i$, $\forall i = 1, \ldots, N$, where $\tilde{X}_i$ is defined as

$$\tilde{X}_{k+i} = X \ominus \mathscr{R}_{k+i} \quad \forall i = 0, \ldots, N, \tag{26}$$

the constraints satisfaction is ensured and feasibility is also preserved in the presence of uncertainties in the system (1). In addition, the input constraints are replaced by

$$\tilde{U}_{k+i} = U \ominus K_{k+i}\mathscr{R}_{k+i} \quad \forall i = 0, \ldots, N-1. \tag{27}$$

Assumption 3.1: *We assume that $\tilde{X}_{k+i}$ and $\tilde{U}_{k+i}$ are not empty sets.*

From previous discussion, the MPC optimisation problem to be solved at each sampling instant is given by

$$\min_{G_k \triangleq \{g_k, \ldots, g_{k+N-1}\}} J_N(\tilde{x}_k, G_k) = \sum_{i=0}^{N-1} \tilde{x}_{k+i|k}^T \Phi \tilde{x}_{k+i|k} + g_{k+i|k}^T \Lambda g_{k+i|k} + \Upsilon(\tilde{x}_{k+N|k}) \tag{28}$$

$$\text{subject to} \quad \tilde{x}_{k+i|k} \in \tilde{X}_i \quad \forall i = 1, \ldots, N, \tag{29}$$

$$g_{k+i|k} \in \tilde{U}_i \quad \forall i = 0, \ldots, N-1, \tag{30}$$

$$\tilde{x}_{k+N|k} \in \Omega \ominus \mathscr{R}_N, \tag{31}$$

where $\tilde{x}_{k+i|k}$ denotes the predicted state vector at time $k+i$, obtained by applying the input sequence $G_k \triangleq \{g_k, \ldots, g_{k+N-1}\}$ to model (3) starting from the state $\tilde{x}_k$. The terminal cost $\Upsilon(\cdot)$, and the terminal constraint set given by the region Ω were both calculated in previous section.

Notice that the MPC includes the new state and input constraints (26), (27). Finally, matrices Φ and Λ defined as $\Phi = \Phi^T \geq 0$ and $\Lambda = \Lambda^T > 0$, constitute tuning parameters for the MPC control law. Depending on their values, more attention will be given to the states or to the control signals.

4. Illustrative examples

The aim of this section is to validate the performance of the online robust tube-based MPC control law and to compare it with existing robot motion controllers. In this case, we have considered the trajectory tracking problem of a mobile robot in slip conditions.[1] The control objective is to steer a mobile robot such that it tracks a reference trajectory as close as possible along time. For comparison purposes, we have implemented a robust tube-based predictive controller based on reachable sets which are calculated offline. For that purpose, reachable sets include both a bounding set dealing with the system variation and a bounding set related to additive uncertainty, see (Gonzalez et al. 2011) for further details. Furthermore, the linear feedback controller presented in (Gonzalez et al. 2009) has also been implemented.

To test the online robust tube-based MPC control strategy, we consider the following linear, time-varying, discrete-time system, which corresponds to the trajectory tracking error model of a mobile robot considering longitudinal slip (Gonzalez et al. 2010, 2011)

$$e_{k+1} = A_k^\gamma e_k + B u_k + w_k, \tag{32}$$

where $e = [e_x \ e_y \ e_\theta]^T \in \mathbb{R}^3$ is the state (location error), $u = [u_1 \ u_2]^T \in \mathbb{R}^2$ is the control input, and w is a bounded additive uncertainty[2] satisfying $w \in W$, where W is a polytope in the state space $\mathbb{R}^3$. The matrices A^γ and B are defined as

$$A_k^\gamma = \begin{bmatrix} 1 & \varepsilon(k) & 0 \\ -\varepsilon(k) & 1 & \rho(k) \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} T_s & 0 \\ 0 & 0 \\ 0 & T_s \end{bmatrix}, \tag{33}$$

where $\varepsilon(k) = T_s(\frac{(1-\bar{i}_r)v_r^{ref}(k) - (1-\bar{i}_l)v_l^{ref}(k)}{b})$ and $\rho(k) = T_s(\frac{v_r^{ref}(k) + v_l^{ref}(k)}{2})$, T_s is the sampling period, $\bar{i}_r$ and $\bar{i}_l$ are the nominal slips, v_r^{ref} and v_l^{ref} are the reference linear velocities of right and left wheels, respectively, and b is the width of the robot. Note that system matrix A^{γ} depends on a parameter $\gamma = [v_r^{ref} \quad v_l^{ref}]^T \in \mathbb{R}^2$ which is a time-varying vector such that $\gamma_k \in \Gamma$, $\forall k \in \mathbb{Z}^+$, where $\Gamma \subseteq \mathbb{R}^2$ is a polytope (Assumption 2.1). For that reason, the minimisation problem formulated in term of LMI (18) has been solved for $n_\gamma = 2^2$ vertices.

Simulations have been carried out in Matlab® suite using the LMI toolbox (Gahinet, Nemirovski, Laub, and Chiali 2004) and MPT toolbox (Kvasnica, Grieder, and Baotić 2004). The reference trajectories have been calculated based on unicycle kinematics (Siegwart and Nourbakhsh 2004).

The parameters used for the simulations are: $T_s = 0.35$ [s], $b = 0.5$ [m], nominal slip is 0.10 (10 [%]), the uncertainty set is given by $W = \{w_1, w_2 \in \pm 0.0035\,[\text{m}], w_3 \in \pm 0.25\,[°]\}$. State constraints are $E = \{e_x, e_y \in \pm 0.5\,[\text{m}], e_\theta \in \pm 20\,[°]\}$, reference linear track velocities are restricted to $\{v_r^{ref}, v_l^{ref} \in [0.1, 1.4]\ [\text{m/s}]\}$, and real linear wheel velocities are restricted to $\{v_r, v_l \in [-2, 2]\ [\text{m/s}]\}$, $Q = \text{diag}([1 \ 1 \ 0.0001])$ and $R = I_2$, $\Phi = \text{diag}([1\ 1\ 0.0001])$ and $\Lambda = I_2$. The parameters of the linear feedback controllers are set to $\beta = 1$ and $\xi = 0.6$ in order to reach a soft overdamped closed-loop behaviour. The initial location of the mobile robot is always $[0\ 0\ 0]^T$.

4.1 *Simulation 1. Challenging trajectory*

First, a reference trajectory that comprises the full reference velocity range has been tested. In this case, the reference trajectory is always changing of direction and the reference velocities are always changing between the bounds and never become constant. The total travelled distance is close to 70 [m]. In this first experiment, a prediction horizon $N = 3$ was selected.

Figure 2a shows the reference trajectory and the followed trajectories using the three control strategies. In the figure, the robust tube-based MPC with online reachable sets calculation is denoted as 'Online MPC', the offline robust tube-based MPC approach as 'Offline MPC', and the linear feedback controller (LF) is labelled as 'Slip comp.'. In this case, it can be observed that the predictive controllers fix the reference. Figure 2b shows the simulated slip values. Notice that those slip values vary within the range previously defined. In this case, it has a mean value of 16 [%].

The errors between the reference trajectory and those steered by the compared controllers are displayed with respect to the travelled distance in Figure 3. Notice that, although a small random noise was added to the states, the online MPC controller achieves an almost zero error in the longitudinal and lateral directions, and in the robot orientation. The offline MPC controller also achieves an almost zero longitudinal error. However, it achieves a small oscillatory behaviour in lateral direction and in the orientation error. The LF controller obtains a greater error (maximum lateral error = −0.10 [m], maximum longitudinal error = 0.07 [m], and maximum orientation error = −3.16 [°].

Figure 4 displays the control inputs or linear velocities of the wheels. Notice the non-static reference velocities comprising all the range, that is, from 0.1 [m/s] to 1.4 [m/s]. In this figure, it is observed how controllers compensate the effect of longitudinal slip, that is, the motion controllers increase the set-points in order to compensate the loss of traction due to the slip effect.

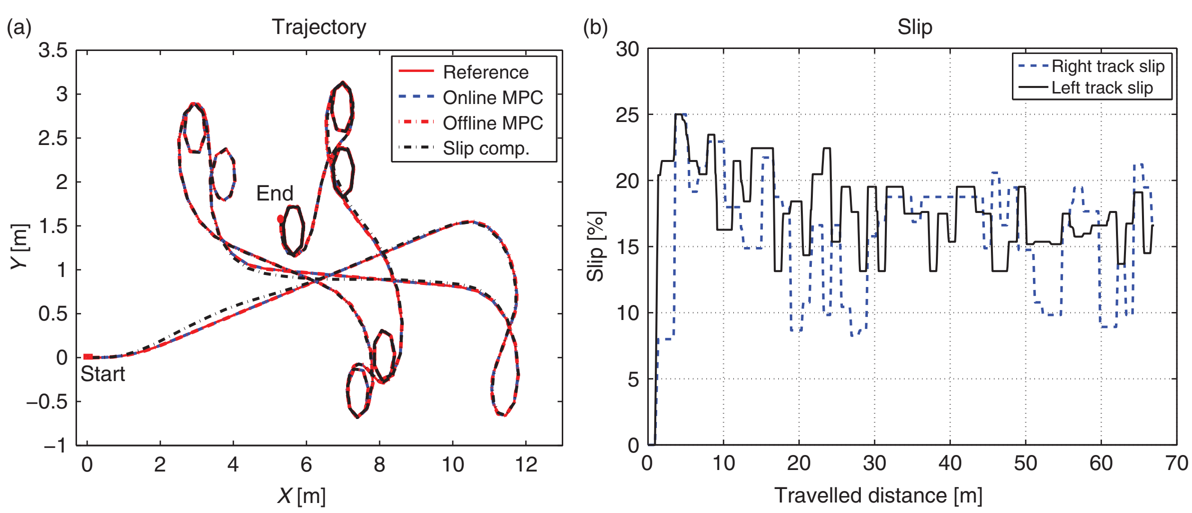

Figure 2. Simulation 1. Followed trajectories and simulated slip profile.

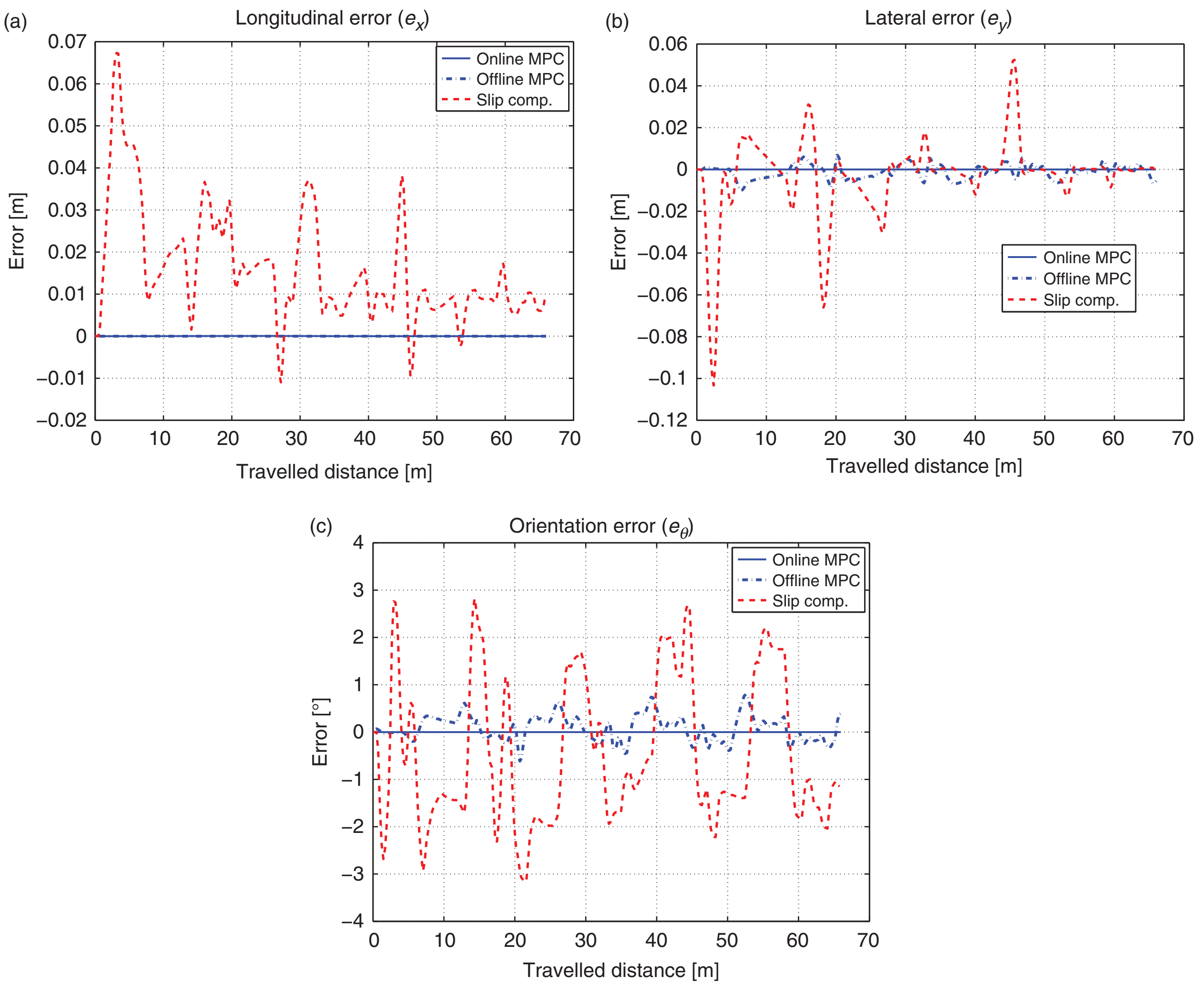

Figure 3. Simulation 1. Errors with respect to the reference trajectory. (a) Logitudinal error, (b) lateral error and (c) orientation error.

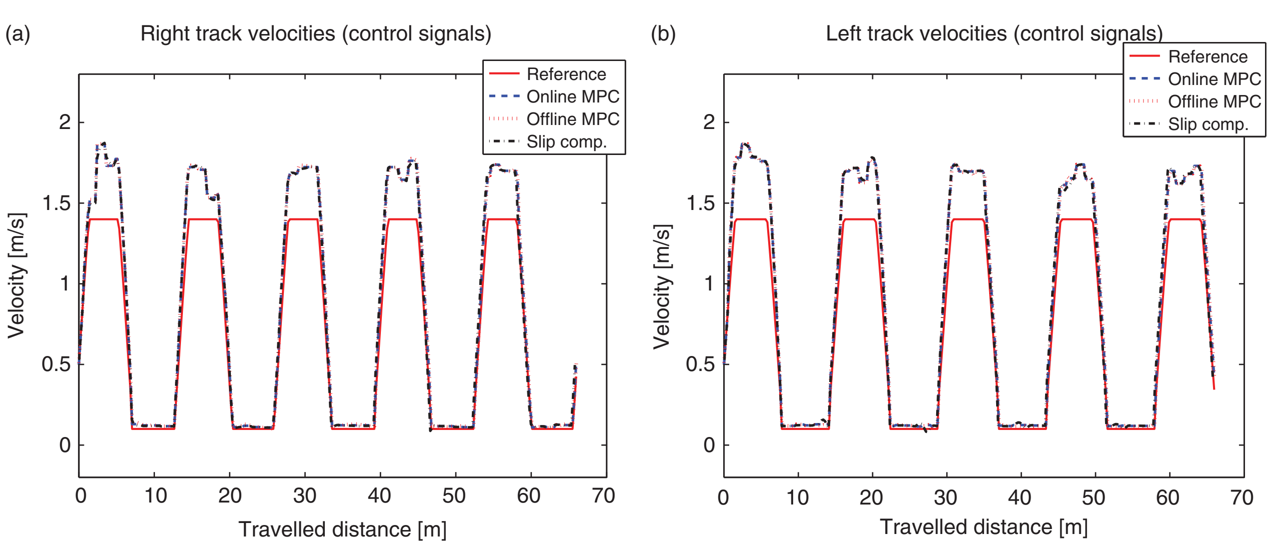

Figure 4. Simulation 1. Control signals to be sent to the wheels. (a) Right wheel and (b) left wheel.

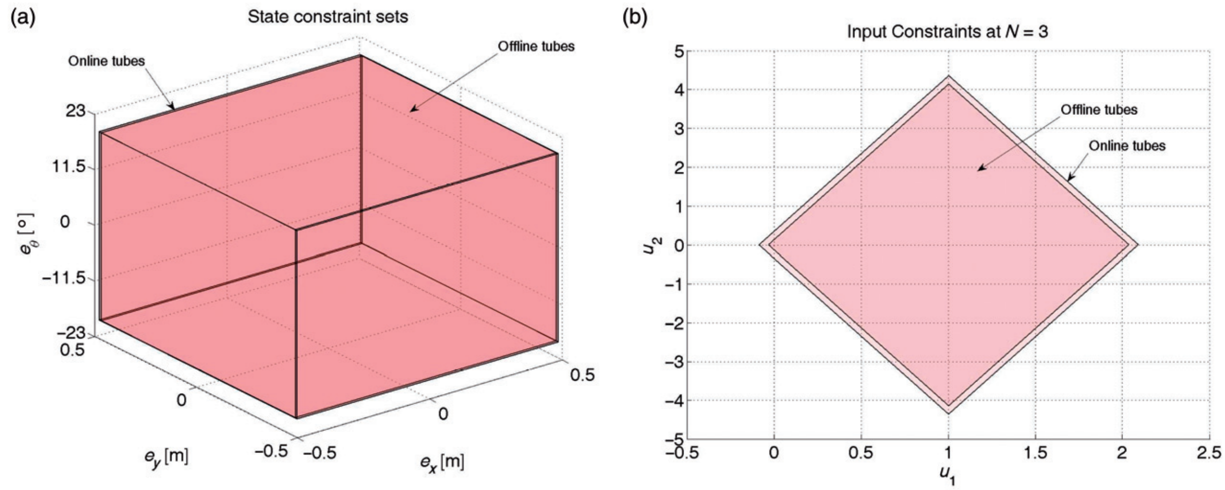

Figure 5. Simulation 1. Reduced constraints sets ($\tilde{E}_3$, $\tilde{U}_3$). (a) State constraints and (b) input constraints.

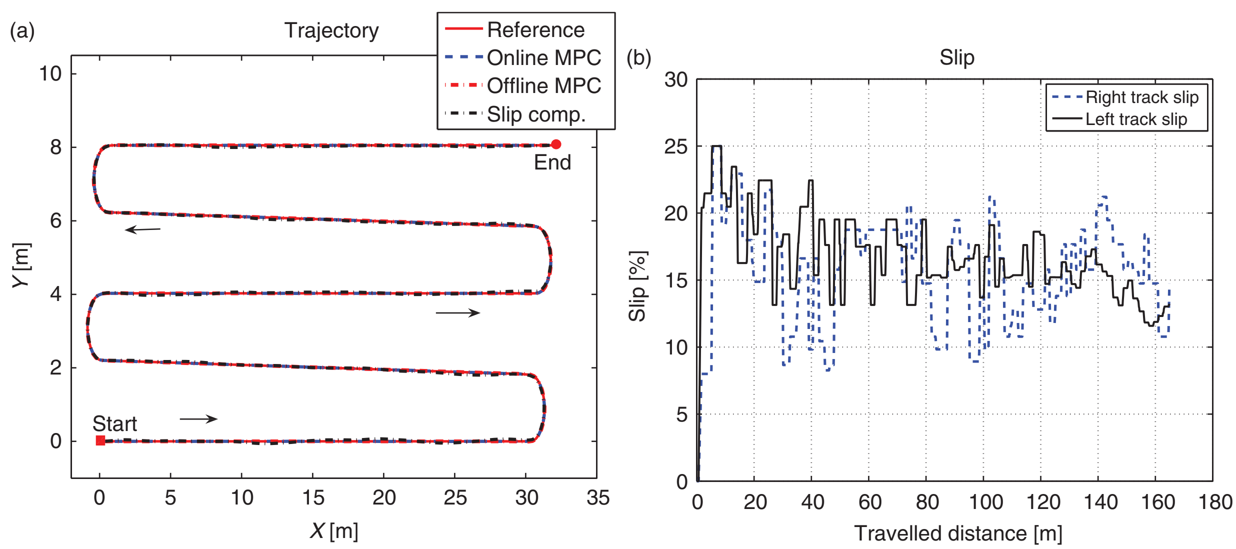

Figure 6. Simulation 2. Followed trajectories. (a) Trajectories and (b) slip profile.

Finally, Figure 5 shows the constraints sets for the last step within the prediction horizon for both robust tube-based strategies. It is possible to note that the state constraint set, $\bar{E}_3$ is slightly larger for the case of the online MPC than for the case of the offline predictive controller. The input constraint set, $\bar{U}_3$, obtained for the approach based on online reachable sets calculation is larger than the approach where reachable sets are solved offline. This implies that the domain of attraction for the nominal predictive controller is bigger than in the case in which reachable sets are solved offline. This fact explains why the proposed tube-based controller leads to a smaller conservativeness in comparison to similar approaches where reachable sets are calculated offline. Computation time is slightly higher in the case of the online MPC but it is always smaller than the sampling period.

4.2 *Simulation 2. Agriculture-like trajectory*

In this second simulation, an agriculture-like reference trajectory has been tested. Particularly, this reference trajectory looks like those trajectories employed in agricultural tasks. The total travelled distance is close to 160 [m]. In this case, the prediction horizon was $N = 5$.

Figure 6a shows the reference trajectory and the trajectories obtained using the compared controllers. As in previous simulation, it is possible to observe that the predictive controllers achieve the best results. In Figure 6b, the simulated slip values for each wheel are plotted. It has a mean value of 10 [%].

Figure 7 presents the errors. As in previous simulation, the online MPC controller achieves an almost zero error in the lateral and longitudinal directions and a small oscillatory orientation error. The offline MPC strategy achieves a small lateral, longitudinal and orientation errors. The LF controller obtains a maximum lateral error of −0.07 [m], a maximum longitudinal error of 0.09 [m], and a maximum orientation error of −2.61 [°]. Again, the online robust tube-based MPC strategy achieves the smallest errors.

The velocity profiles are shown in Figure 8. As in previous simulation, the motion controllers have to increase the control actions in order to compensate the negative slip effect.

Finally, Figure 9 shows the constraints sets $\bar{E}_5$ and $\bar{U}_5$. Again, online adaptive controller reduces the conservativeness of the offline tube-based MPC approach. In relation to this issue, the state and input constraint sets $\bar{E}_5$, $\bar{U}_5$, obtained for the approach based on online reachable sets calculation are larger than the approach where reachable sets were solved offline.

5. Conclusions

This article presents an online robust tube-based predictive control law for constrained time-varying systems with additive uncertainties. This is achieved via online calculation of an explicit local controller that compensates the system realisation at each step within the prediction horizon. Thanks to this approach, the conservativeness is reduced for the time-varying case and the feasible region is enlarged in comparison to other approaches, such as tube-based predictive control using an invariant set (Langson et al. 2004)

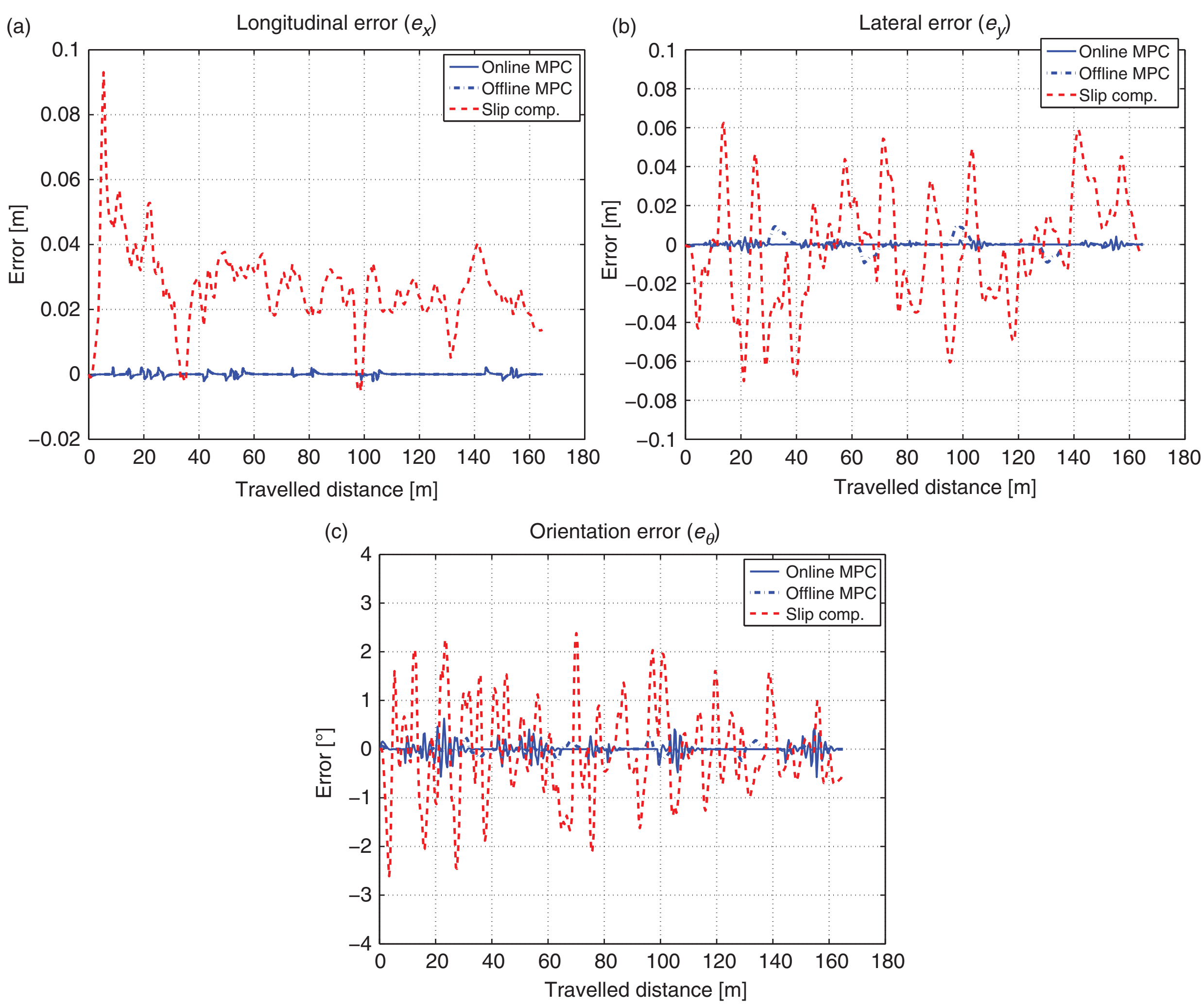

Figure 7. Simulation 2. Errors along the travelled distance. (a) Longitudinal error, (b) lateral error and (c) orientation error.

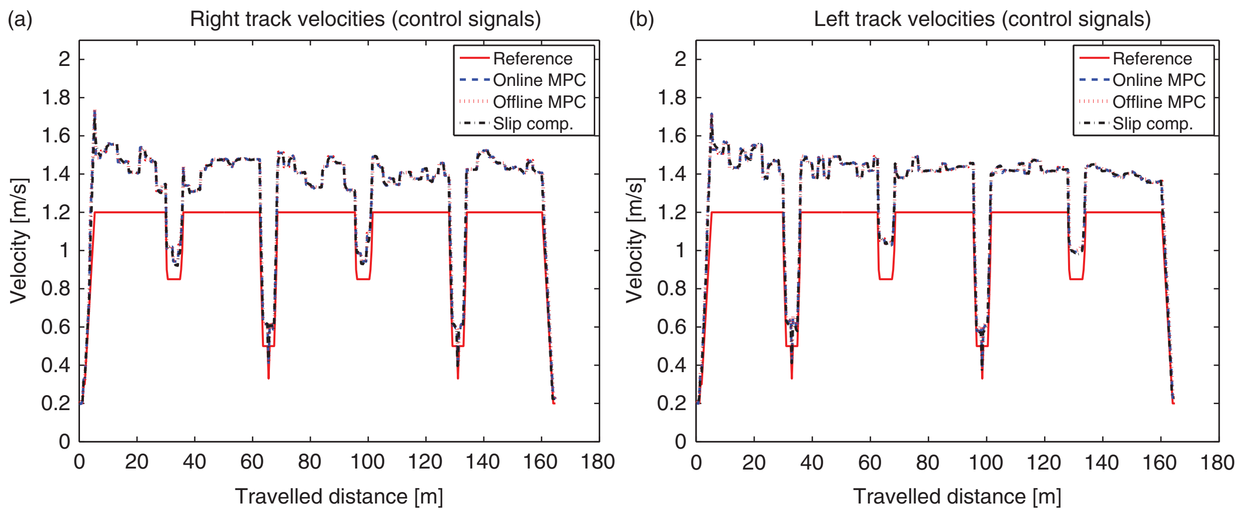

Figure 8. Simulation 2. Control signals to be sent to the tracks. (a) Right track and (b) left track.

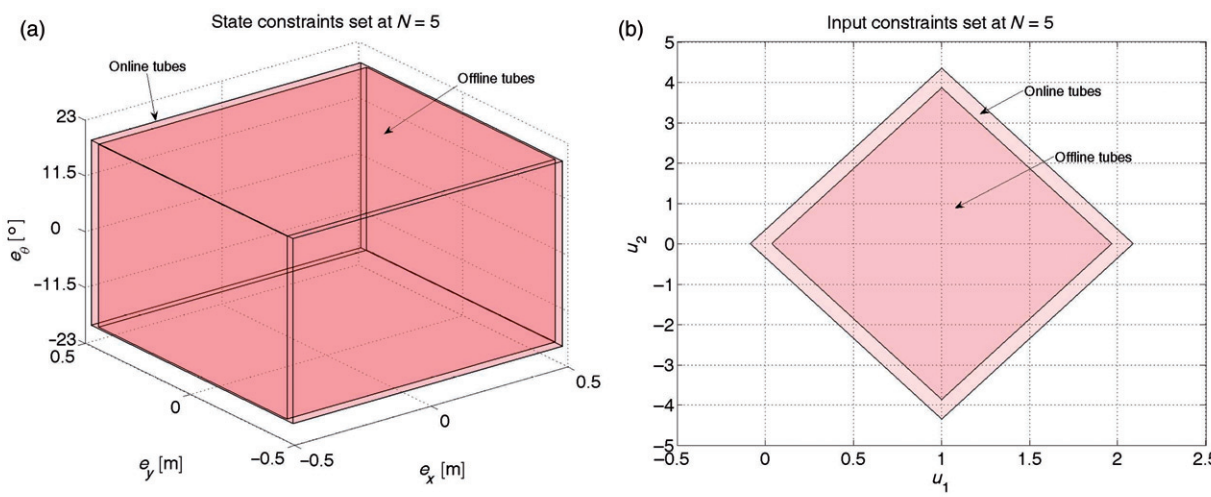

Figure 9. Simulation 2. Comparison of the reduced constraints sets ($\tilde{E}_5$, $\tilde{U}_5$). (a) State constraints and (b) input constraints.

or calculating offline reachable sets (Limon et al. 2005; Gonzalez et al. 2011).

This control strategy has been tested through the trajectory tracking problem of mobile robots subject to slip conditions and hard constraints. In this case, the control objective was to steer a mobile robot as close as possible to the reference trajectory at each sampling instant. The results have revealed the satisfactory behaviour of the control law compared to typical motion controllers.

Simulations have shown the achievement of the main goals of the proposed online tube-based predictive controller in relation to similar approaches: small conservativeness and efficient real-time execution. Following the proposed approach, small reachable sets are obtained leading to a larger feasible region in comparison to the offline MPC strategy. Furthermore, after many experiments, we have checked that the offline tube-based approach becomes unfeasible for a prediction horizon greater than 5. Following the proposed strategy (online tube-based MPC), we have obtained feasible solutions for $N > 10$. Regarding online computation, the proposed control law has been ensured for a small sampling period (0.35 [s]).

Notice that with the proposed approach, larger conservativeness reduction can be reached for those applications with more changing process dynamics and more restricted process constraints. Future works will include the test of the current control strategy through physical experiments. Another attractive future research line can be to replace polytopic structures by zonotopes to reduce further the online computation burden.

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Notes

1. Slip is a phenomenon that affects the motion of a mobile robot in off-road conditions. It induces traction/velocity loss during the robot motion, which can adversely influence the mobility and controllability (Wong 2001; Gonzalez et al. 2009). Slip can be present in lateral and longitudinal directions. Lateral slip is due to the deformation of the pneumatic tire surface and large centrifugal force. Longitudinal slip is mainly caused by the wheel-soil interaction, such as the sinkage effect. In this case, we only consider longitudinal slip which is a unavoidable phenomenon, even for mobile robots working at low velocities.
2. Notice that, the additive term W includes the deviation between the non-linear continuous-time model and the linear discrete-time model (Gonzalez et al. 2011). Furthermore, we enlarge it to take into account the effects of the noise in the slip measurements and the uncertainty in the estimation of the robot location.

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