NANZAN-TR-2010-05

# Robust Semidefinite Programming Problems with General Nonlinear Parameter Dependence: Approaches Using the DC-Representations

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March 2011

Technical Report of the Nanzan Academic Society
Information Sciences and Engineering

# Robust Semidefinite Programming Problems with General Nonlinear Parameter Dependence: Approaches Using the DC-Representations*

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Two conservative approaches are proposed to a semidefinite programming problem nonlinearly dependent on uncertain parameters. These approaches are applicable to general nonlinear parameter dependence not necessarily polynomial or rational. They are based on a mild assumption that the parameter dependence is expressed as the difference of two convex functions. The first approach uses constant bounds on the parameter dependence. Optimization of the bounds is reduced to convex nonsmooth minimization in general. The second approach uses parameter-dependent bounds for a less conservative result. Optimization of the bounds is immediate when the gravity center is computable for the parameter set. Numerical examples are presented for illustration of the approaches.

**Keywords:** robust semidefinite programming, nonlinear parameter dependence, DC-representation, convexity, conjugate functions, linear matrix inequalities, conservatism.

## 1. Introduction

A robust semidefinite programming problem (robust SDP problem) is an important optimization problem, to which many control problems are reduced (Ben-Tal & Nemirovski, 2001; Scherer, 2006; Ben-Tal, El Ghaoui, & Nemirovski, 2009). There, a linear objective function is to be optimized under a linear matrix inequality (LMI) constraint dependent on uncertain parameters. When the parameter dependence is affine, this problem is exactly solvable: the parameter-dependent constraint is replaced by the constraints corresponding to the vertices of the parameter set and the resulting problem is solved with the standard interior-point method. Note however that the problem is NP-hard even in this simple case (Nemirovskii, 1993). When the parameter dependence is not affine, a conservative approach is usually taken: the parameter-dependent constraint is replaced by its sufficient condition expressed by a finite number of parameter-independent LMI constraints. Considerable efforts have been paid in the case of polynomial or rational parameter dependence (*e.g.*, Ben-Tal & Nemirovski, 1998; El Ghaoui, Oustry, & Lebret, 1998; Bliman, 2004; Chesi, Garulli, Tesi, & Vicino, 2005; Scherer, 2005; Scherer & Hol, 2006; Peaucelle & Sato, 2009; Oishi, 2009). There, conservatism can be reduced to any desired degree. On the other hand, the case of general nonlinear parameter dependence has not been considered so much except for a few works (Papachristodoulou & Prajna, 2005; Chesi & Hung, 2008; Oishi & Fujioka, 2010).

*March 30, 2011

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In this paper, we consider a robust SDP problem with general nonlinear parameter dependence and show the usefulness of the, so-called, DC-representations, which will be explained later. Specifically, we consider minimization of a linear objective function subject to an LMI constraint affinely dependent on $q$ continuous nonlinear functions $a_j(\theta)$, $j = 1, 2, \ldots, q$, where $\theta$ is a $p$-dimensional uncertain parameter. This constraint has to be satisfied for all $\theta$ in the set $\Theta$, which is a $p$-dimensional closed convex polytope (*i.e.*, bounded polyhedron) having a nonempty interior. (See Section 2.1 for the explicit form of the problem.) One possible approach to this problem is to obtain upper and lower bounds, $\underline{a}_j \leq a_j(\theta) \leq \overline{a}_j$ ($\theta \in \Theta$), for each $j = 1, 2, \ldots, q$ and to replace $a_j(\theta)$ in the original LMI constraint either by $\underline{a}_j$ or $\overline{a}_j$. The resulting $2^q$ LMIs constitute a sufficient condition for the original LMI constraint to be satisfied. Hence, we can minimize the objective function subject to these LMIs and regard the obtained solution as an approximate solution of the original problem. Now, we have two issues in this approach. First, the upper and lower bounds on $a_j(\theta)$ can be difficult to find. Second, this approach can be considerably conservative because it uses constant bounds and neglects how $a_j(\theta)$ depends on $\theta$. (See Section 5 for an example.)

In this paper, we would like to address these two issues using the DC-representations. Here, the letters "DC" mean the difference of two convex functions. We assume that each of the nonlinear functions $a_j(\theta)$ is expressed as the difference of two convex functions. Then, we can compute upper and lower bounds on $a_j(\theta)$ with this representation. Furthermore, we can obtain parameter-dependent bounds incorporating the parameter dependence of $a_j(\theta)$. The constant bounds as well as the parameter-dependent bounds form parameterized families and they can be optimized in some specific sense. Between these two types of bounds, the parameter-dependent bounds look better because they can incorporate the parameter dependence of $a_j(\theta)$ and they have computational advantage in their optimization. The notion of the DC-representation has been utilized in the field of optimization (Tuy, 1998; Horst & Thoai, 1999). Its application in control was considered by Tuan, Apkarian, Hosoe, and Tuy (2000), Tuan, Hosoe, and Tuy (2000), Bravo, Alamo, Fiacchini, and Camacho (2007), and Alamo, Bravo, Redondo, and Camacho (2008).

The organization of the paper is as follows. In Section 2, the problem is presented and the notion of DC-representation is introduced. In Section 3, an approach with the constant bounds is presented. In Section 4, an approach with the parameter-dependent bounds is presented and its advantage is discussed. Section 5 gives numerical examples and Section 6 concludes the paper.

The following notation is used. The symbol $\mathbb{R}^n$ stands for the set of $n$-dimensional real column vectors. The symbol $^{\mathrm{T}}$ expresses the transpose of a vector. The symbol $I$ denotes the identity matrix of appropriate size. For a symmetric matrix $A$, the inequality $A \succeq O$ means that $A$ is positive semidefinite. For a closed convex polytope $\Theta$, the symbol $\mathrm{ver}\,\Theta$ indicates the set of the vertices of $\Theta$. The number of the vertices is expressed by $|\,\mathrm{ver}\,\Theta|$.

## 2. Preparations

### 2.1. Problem to be considered

The problem to be considered in this paper is the following:

$$\begin{aligned} P: \quad & \text{minimize} \quad c^{\mathrm{T}} x \\ & \text{subject to} \quad F_0(x) + \sum_{i=1}^{p} \theta_i F_i(x) + \sum_{j=1}^{q} a_j(\theta) F_{p+j}(x) \succeq O \quad (\forall \theta \in \Theta). \end{aligned} \tag{1}$$

Here, $x \in \mathbb{R}^n$ is a design variable; $c \in \mathbb{R}^n$ is a given vector; $F_i(x)$, $i = 0, 1, \ldots, p+q$, are affine functions of $x$ whose values are $m \times m$ symmetric matrices; $\theta = (\theta_1 \ \ \theta_2 \ \ \cdots \ \ \theta_p)^{\mathrm{T}}$ is an uncertain parameter taking the value in the parameter set $\Theta$; $\Theta$ is a closed convex polytope in $\mathbb{R}^p$ having a nonempty interior; $a_j(\theta)$, $j = 1, 2, \ldots, q$, are continuous nonlinear functions of $\theta \in \Theta$. This problem is difficult to solve because the constraint (1) has to be satisfied for all $\theta$ in $\Theta$. If the nonlinear functions $a_j(\theta)$ are polynomial or rational, a sufficient condition can be obtained for (1) and can be used for approximate solution of the problem $P$ (El Ghaoui, *et al.*, 1998; Bliman, 2004; Chesi, *et al.*, 2005; Scherer, 2005; Scherer & Hol, 2006; Peaucelle & Sato, 2009; Oishi, 2009). Hence, we are interested in the case that $a_j(\theta)$ are not polynomial or rational though not restricted to the case.

As discussed in the introduction, a conservative approach is possible to the problem $P$ if upper and lower bounds are available on $a_j(\theta)$, $j = 1, 2, \ldots, q$. To have these bounds, we use the DC-representation of $a_j(\theta)$, which is explained next.

### 2.2. DC-representation

Let $\Theta$ be a closed convex polytope in $\mathbb{R}^p$ having a nonempty interior. A continuous function $a(\theta)$ on $\Theta$ is said to have a *DC-representation* if it is expressed as the difference of two convex functions, that is,

$$a(\theta) = b(\theta) - c(\theta) \quad (\theta \in \Theta),$$

where $b(\theta)$ and $c(\theta)$ are convex functions on $\Theta$.

**Example 1.** Suppose that $a(\theta) = \sinh\theta$ and $\Theta = [-1, \ 1]$. Its DC-representation is given by $b(\theta) = \mathrm{e}^{\theta}/2$ and $c(\theta) = \mathrm{e}^{-\theta}/2$.

Suppose that $a(\theta) = \sin\theta$ and $\Theta = [-2, \ 2]$. The second derivative of $\sin\theta$ is larger than or equal to $-1$ on $\Theta$, which implies the convexity of $\sin\theta + \theta^2/2$ on $\Theta$. Hence, $b(\theta) = \sin\theta + \theta^2/2$ and $c(\theta) = \theta^2/2$ give the DC-representation. ◇

Polynomials and rational functions have DC-representations if they are continuous on $\Theta$. Any function continuous on $\Theta$ can be approximated to any precision by a function possessing a DC-representation. Moreover, the second example above shows that an explicit DC-representation can be obtained for a given function if a lower bound is known on its second derivative (or the Hessian matrix in the case of $p > 1$). Computation of such a bound is discussed by Adjiman and Floudas

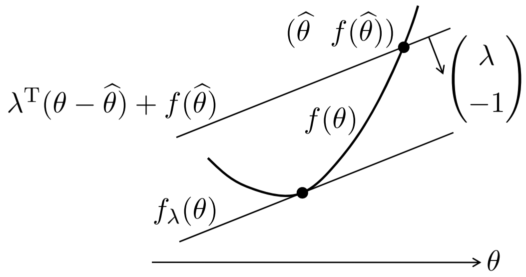



Figure 1. The graphs of $f(\theta)$ and $f_\lambda(\theta)$

(1996). It is hence reasonably mild to assume the nonlinear functions $a_j(\theta)$ in the problem $P$ to have known DC-representations. In the following sections, we are to consider the problem $P$ under this assumption.

## 3. Approach with constant bounds

We assume in the problem $P$ that the nonlinear function $a_j(\theta)$ has a known DC-representation

$$a_j(\theta) = b_j(\theta) - c_j(\theta) \quad (\theta \in \Theta)$$

for each $j = 1, 2, \ldots, q$, where $b_j(\theta)$ and $c_j(\theta)$ are convex functions on $\Theta$. We consider in this section how to obtain upper and lower bounds on $a_j(\theta)$ using its DC-representation. Once such bounds are obtained, we can form a sufficient condition for the constraint (1) of the problem $P$ and obtain an approximate solution of $P$.

For preparation, let us consider in general a convex function $f(\theta)$ on $\Theta$. For any $\lambda \in \mathbb{R}^p$, we define an affine function in $\theta$ by

$$f_\lambda(\theta) := \lambda^{\mathrm{T}}\theta - \max_{\widehat{\theta}\in\Theta}\bigl[\lambda^{\mathrm{T}}\widehat{\theta} - f(\widehat{\theta})\bigr]. \tag{2}$$

Rewriting it as

$$f_\lambda(\theta) = \min_{\widehat{\theta}\in\Theta}\bigl[\lambda^{\mathrm{T}}(\theta - \widehat{\theta}) + f(\widehat{\theta})\bigr], \tag{3}$$

we can see that the graph of this function forms the hyperplane having the normal vector $(\lambda^{\mathrm{T}}\ \ -1)^{\mathrm{T}}$ and supporting the graph of $f(\theta)$ from below (Figure 1). It is also seen that $f_\lambda(\theta) \le f(\theta)$ for any $\theta \in \Theta$ because the square bracket in (3) is equal to $f(\theta)$ with $\widehat{\theta} = \theta$. The function $f_\lambda(\theta)$ is related to the conjugate function of $f(\theta)$, which will be used later in Proposition 8.

The next proposition gives the desired bounds on $a_j(\theta)$.

**Proposition 2.** *For each $j = 1, 2, \ldots, q$, set the values*

$$\underline{a}_{j,\lambda} := \min_{\theta\in\operatorname{ver}\Theta}\bigl[b_{j,\lambda}(\theta) - c_j(\theta)\bigr], \quad \overline{a}_{j,\mu} := \max_{\theta\in\operatorname{ver}\Theta}\bigl[b_j(\theta) - c_{j,\mu}(\theta)\bigr] \tag{4}$$

*with $\lambda$ and $\mu$ being any vectors in $\mathbb{R}^p$. Then, they satisfy $\underline{a}_{j,\lambda} \le a_j(\theta) \le \overline{a}_{j,\mu}$ $(\theta \in \Theta)$.*

*Proof.* The relation $b_{j,\lambda}(\theta) \le b_j(\theta)$ implies $a_j(\theta) \ge b_{j,\lambda}(\theta) - c_j(\theta) \ge \min_{\theta\in\Theta}[b_{j,\lambda}(\theta) - c_j(\theta)]$ for any $\theta \in \Theta$. The function $b_{j,\lambda}(\theta) - c_j(\theta)$ is concave because $b_{j,\lambda}(\theta)$ is affine and $-c_j(\theta)$ is concave. This function hence attains the minimum at a vertex of $\Theta$. This implies one of the desired inequalities, $\underline{a}_{j,\lambda} \le a_j(\theta)$. The remaining inequality $a_j(\theta) \le \overline{a}_{j,\mu}$ follows by similar reasoning. □

Computation of the bounds $\underline{a}_{j,\lambda}$ and $\overline{a}_{j,\mu}$ is carried out as follows. For given $\lambda$ and $\mu$, we first obtain explicit forms of $b_{j,\lambda}(\theta)$ and $c_{j,\mu}(\theta)$ following the definition (2). This can be performed through maximization of the concave functions $\lambda^{\mathrm{T}}\widehat{\theta} - b_j(\widehat{\theta})$ and $\mu^{\mathrm{T}}\widehat{\theta} - c_j(\widehat{\theta})$, though it requires a numerical technique[1] in general (*e.g.*, Hiriart-Urruty & Lemaréchal, 1993; Bertsekas, 1999). Then, we carry out minimization and maximization in (4) to obtain the bounds. This step requires only comparison of a finite number of values.

Proposition 2 allows us to construct a sufficient condition for the constraint (1) of the problem $P$. Let us choose the vectors $\lambda(j)$ and $\mu(j)$ arbitrarily from $\mathbb{R}^p$ for each $j = 1, 2, \ldots, q$ and consider the following condition on $x \in \mathbb{R}^n$:

$$F_0(x) + \sum_{i=1}^{p} \theta_i F_i(x) + \sum_{j=1}^{q} \alpha_j F_{p+j}(x) \succeq O$$
$$(\forall \alpha_j \in \{\underline{a}_{j,\lambda(j)},\ \overline{a}_{j,\mu(j)}\},\ j = 1, 2, \ldots, q;\ \forall\theta \in \mathrm{ver}\,\Theta). \tag{5}$$

Here, all the $2^q$ assignments are considered for $\alpha_j$, $j = 1, 2, \ldots, q$, and all the vertices of $\Theta$ are considered for $\theta$.

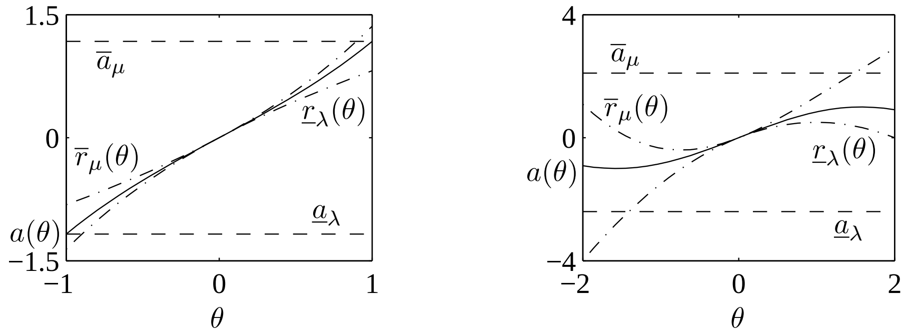
**Proposition 3.** *Let $\lambda(j)$ and $\mu(j)$ be some vectors in $\mathbb{R}^p$ for each $j = 1, 2, \ldots, q$. Then, for a given $x \in \mathbb{R}^n$, the condition* (1) *is satisfied if the condition* (5) *is satisfied.*

*Proof.* Proposition 2 assures $\underline{a}_{j,\lambda(j)} \le a_j(\theta) \le \overline{a}_{j,\mu(j)}$ $(\theta \in \Theta)$ for each $j = 1, 2, \ldots, q$. Since the condition (5) is convex in $\theta_i$ and $\alpha_j$, it implies the condition (1). □

We now have the following approach to the problem $P$. We replace the constraint (1) with its sufficient condition (5) to have a standard SDP problem solvable with the interior-point method. The optimal solution of this modified problem can be used as an approximate solution of $P$, though it attains, in general, a larger value than the minimum of $P$. The condition (5) consists of $2^q|\,\mathrm{ver}\,\Theta|$ LMIs with $|\,\mathrm{ver}\,\Theta|$ meaning the number of the vertices of $\Theta$. Although this number becomes large for a large $q$, it may be reasonable for the NP-hard problem $P$.

The choice of the vectors $\lambda(j)$ and $\mu(j)$ is arbitrary. It would be nice, however, if we can choose optimal vectors. Specifically, the desired $\lambda$ and $\mu$ would be such that maximize the lower bound $\underline{a}_{j,\lambda}$ and minimize the upper bound $\overline{a}_{j,\mu}$, respectively. The result of Carrizosa (2001) is useful for this purpose. (A related result can be found in Bravo, *et al.* (2007).) He showed that these maximization

[1] Maximization is not necessary to have $b_{j,\lambda}(\theta)$ if $\lambda$ is known to be a subgradient of $b_j(\theta)$ at some point $\theta^0 \in \Theta$. Indeed, in this case, we have $b_{j,\lambda}(\theta) = \lambda^{\mathrm{T}}(\theta - \theta^0) + b_j(\theta^0)$. The same comment applies to $c_{j,\mu}(\theta)$, too. This property will be used in Proposition 8.




(a) (b)

Figure 2. The graphs of $a(\theta)$ and its bounds. (a) The case of $a(\theta) = \sinh\theta$ and $\Theta = [-1,\ 1]$. The solid line stands for the original function $a(\theta)$; the broken lines stand for the constant bounds $\underline{a}_\lambda$ and $\overline{a}_\mu$; the dash-dotted lines stand for the parameter-dependent bounds $\underline{r}_\lambda(\theta)$ and $\overline{r}_\mu(\theta)$, to be introduced in Section 4. (b) The case of $a(\theta) = \sin\theta$ and $\Theta = [-2,\ 2]$. The usage of the lines is the same as in (a).

and minimization can be reduced to minimization of some convex functions. This minimization is however not easy in general because the minimized functions do not have analytic expressions and may not be differentiable at some points. On the other hand, in the special case that $\Theta$ is a simplex, we can obtain the optimal $\lambda$ and $\mu$ by solving linear equations. Indeed, $\lambda$ is optimal when the value of $\lambda^{\mathrm{T}}\theta - c_j(\theta)$ is identical for all $\theta \in \operatorname{ver}\Theta$. A similar property holds on the optimal $\mu$ but with $\mu^{\mathrm{T}}\theta - b_j(\theta)$ this time.

**Example 4.** We continue Example 1.

In the case that $a(\theta) = \sinh\theta$ and $\Theta = [-1,\ 1]$, the functions $b(\theta) = \mathrm{e}^{\theta}/2$ and $c(\theta) = \mathrm{e}^{-\theta}/2$ give the DC-representation. Let us compute the optimal lower bound $\underline{a}_\lambda$. Since the parameter set $\Theta$ is a simplex, the optimal $\lambda$ can be obtained by solving $\lambda \cdot (-1) - c(-1) = \lambda \cdot 1 - c(1)$, *i.e.*, $\lambda = 1/4\mathrm{e} - \mathrm{e}/4$. With this $\lambda$, we obtain the explicit expression of $b_\lambda(\theta) = \lambda\theta - \max_{\widehat{\theta}\in\Theta}[\lambda\widehat{\theta} - b(\widehat{\theta})]$. Differentiating the function in the square bracket, we see that the maximum is attained at $\widehat{\theta} = -1$, which gives $b_\lambda(\theta) = \lambda\theta + \lambda + 1/2\mathrm{e}$. To compute $\underline{a}_\lambda = \min_{\theta\in\operatorname{ver}\Theta}[b_\lambda(\theta) - c(\theta)]$, we evaluate $b_\lambda(\theta) - c(\theta)$ at $\theta = -1$ and $\theta = 1$. The two values are equal and give the desired optimal lower bound $\underline{a}_\lambda = 1/2\mathrm{e} - \mathrm{e}/2$. This bound is tight because $a(\theta)$ takes the same value at $\theta = -1$. By symmetry, the optimal upper bound is $\overline{a}_\mu = \mathrm{e}/2 - 1/2\mathrm{e}$, which is attained at $\mu = \mathrm{e}/4 - 1/4\mathrm{e}$. Figure 2 (a) shows the obtained optimal bounds $\underline{a}_\lambda$ and $\overline{a}_\mu$ together with the original function $a(\theta)$. It confirms the inequality $\underline{a}_\lambda \le a(\theta) \le \overline{a}_\mu$ ($\theta \in \Theta$) and the tightness of the bounds. The figure also shows the parameter-dependent bounds, to be discussed in the next section.

In the case that $a(\theta) = \sin\theta$ and $\Theta = [-2,\ 2]$, the functions $b(\theta) = \sin\theta + \theta^2/2$ and $c(\theta) = \theta^2/2$ give the DC-representation. The optimal lower bound $\underline{a}_\lambda$ is computed similarly to the first example. In particular, we obtain the optimal $\lambda$ as 0 by noticing that $\Theta$ is a simplex. Since $b_\lambda(\theta) = -\max_{\widehat{\theta}\in\Theta}[-\sin\widehat{\theta} - \widehat{\theta}^2/2]$, we carry out the numerical maximization to have the explicit expression $b_\lambda(\theta) \approx -0.400$. We then evaluate $b_\lambda(\theta) - c(\theta)$ at $\theta = -2$ and $\theta = 2$ and obtain the optimal lower bound $\underline{a}_\lambda \approx -2.400$. This bound is not tight. Computation of the optimal upper bound $\overline{a}_\mu$ is similar.

The result is $\overline{a}_\mu \approx 2.103$ for $\mu \approx 0.455$. This bound is not tight, either. Figure 2 (b) shows the bounds in this case. ◇

Our approach is conservative because the condition (5) is only sufficient for the original condition (1). This conservatism can be reduced by division of the parameter set $\Theta$. Namely, we divide the parameter set $\Theta$ into a finite family of closed convex polytopes and consider a condition corresponding to (5) for each subpolytope. It is also possible to make such division adaptively in order to suppress the rapid increase of the number of subpolytopes. See Oishi (2009) and Oishi and Fujioka (2010) for the details.

## 4. Approach with parameter-dependent bounds

The bounds in the previous section are constant in $\theta$ and neglect how the functions $a_j(\theta)$ depend on $\theta$. They can give a considerably conservative result when incorporation of this dependence is important for solving $P$. In this section, we introduce parameter-dependent bounds using the DC-representations of $a_j(\theta)$. These bounds give another sufficient condition for the constraint (1) of the problem $P$. Using this condition in the problem $P$, we have a solvable SDP problem, which gives an approximate solution of $P$. We hence have another approach to the problem $P$.

The considered bounds are the following.

**Proposition 5.** *For each $j = 1, 2, \ldots, q$, define the functions on $\Theta$ as*

$$\underline{r}_{j,\lambda}(\theta) := b_{j,\lambda}(\theta) - c_j(\theta), \quad \overline{r}_{j,\mu}(\theta) := b_j(\theta) - c_{j,\mu}(\theta) \tag{6}$$

*with $\lambda$ and $\mu$ being any vectors in $\mathbb{R}^p$. Then, $\underline{r}_{j,\lambda}(\theta)$ is concave and $\overline{r}_{j,\mu}(\theta)$ is convex. Moreover, they satisfy*

$$\underline{r}_{j,\lambda}(\theta) \le a_j(\theta) \le \overline{r}_{j,\mu}(\theta) \quad (\theta \in \Theta).$$

*Proof.* The concavity of $\underline{r}_{j,\lambda}(\theta)$ follows from the affinity of $b_{j,\lambda}(\theta)$ and the convexity of $c_j(\theta)$. The convexity of $\overline{r}_{j,\mu}(\theta)$ follows similarly. The desired inequality is a consequence of $b_{j,\lambda}(\theta) \le b_j(\theta)$ and $c_{j,\mu}(\theta) \le c_j(\theta)$. □

With the parameter-dependent bounds $\underline{r}_{j,\lambda}(\theta)$ and $\overline{r}_{j,\mu}(\theta)$, we have the following sufficient condition for (1). Let $\lambda(j)$ and $\mu(j)$ be some vectors in $\mathbb{R}^p$ for each $j = 1, 2, \ldots, q$. Then, for a given $x \in \mathbb{R}^n$, we consider the condition

$$F_0(x) + \sum_{i=1}^{p} \theta_i F_i(x) + \sum_{j=1}^{q} \rho_j(\theta) F_{p+j}(x) \succeq O$$
$$(\forall \rho_j \in \{\underline{r}_{j,\lambda(j)},\ \overline{r}_{j,\mu(j)}\},\ j = 1, 2, \ldots, q;\ \forall \theta \in \operatorname{ver} \Theta). \tag{7}$$

Here, $\rho_j(\theta)$ is a function chosen from the two candidate functions $\underline{r}_{j,\lambda(j)}(\theta)$ and $\overline{r}_{j,\mu(j)}(\theta)$ for each $j$. All the $2^q$ choices are considered for $\rho_j(\theta)$, $j = 1, 2, \ldots, q$, and all the vertices in $\operatorname{ver} \Theta$ are considered for $\theta$.

**Proposition 6.** *For a given $x \in \mathbb{R}^n$, the condition* (1) *is satisfied if the condition* (7) *is satisfied.*

*Proof.* Let $v$ be any nonzero vector in $\mathbb{R}^m$. The proof is complete if we can show

$$v^{\mathrm{T}} F_0(x) v + \sum_{i=1}^{p} \theta_i v^{\mathrm{T}} F_i(x) v + \sum_{j=1}^{q} a_j(\theta) v^{\mathrm{T}} F_{p+j}(x) v \geq 0 \tag{8}$$

for any $\theta \in \Theta$, which is the inequality (1) multiplied by $v^{\mathrm{T}}$ and $v$. Toward this end, multiply $v^{\mathrm{T}}$ and $v$ to the assumption (7). Defining

$$f_j(\theta) := \min\{\underline{r}_{j,\lambda(j)}(\theta) v^{\mathrm{T}} F_{p+j}(x) v,\ \overline{r}_{j,\mu(j)}(\theta) v^{\mathrm{T}} F_{p+j}(x) v\}$$

for $j = 1, 2, \ldots, q$, we have

$$v^{\mathrm{T}} F_0(x) v + \sum_{i=1}^{p} \theta_i v^{\mathrm{T}} F_i(x) v + \sum_{j=1}^{q} f_j(\theta) \geq 0 \quad (\theta \in \operatorname{ver} \Theta). \tag{9}$$

Pick up any $j$ and suppose $v^{\mathrm{T}} F_{p+j}(x) v \geq 0$. In this case, the function $f_j(\theta)$ is equal to $\underline{r}_{j,\lambda(j)}(\theta) v^{\mathrm{T}} F_{p+j}(x) v$, which is concave in $\theta$ and less than or equal to $a_j(\theta) v^{\mathrm{T}} F_{p+j}(x) v$ by Proposition 5. Repeating a similar discussion in the case of $v^{\mathrm{T}} F_{p+j}(x) v < 0$, we see that, in any case, $f_j(\theta)$ is concave and satisfies

$$a_j(\theta) v^{\mathrm{T}} F_{p+j}(x) v \geq f_j(\theta) \quad (\theta \in \Theta) \tag{10}$$

for each $j$. The concavity of $f_j(\theta)$ implies that the inequality (9) holds for all $\theta \in \Theta$ and the inequality (10) implies that the desired inequality (8) holds for all $\theta \in \Theta$. The proof is complete. □

In the present approach, the utilized sufficient condition (7) consists of $2^q |\operatorname{ver} \Theta|$ LMIs. This number is the same as the approach in the previous section. On the other hand, the present approach is less conservative than the previous one when the two approaches use the same $\lambda(j)$ and $\mu(j)$ for each $j$. This is shown by the next result.

**Proposition 7.** *Let $\lambda(j)$ and $\mu(j)$ be some vectors in $\mathbb{R}^p$ for each $j = 1, 2, \ldots, q$. Then, the condition* (7) *is satisfied if the condition* (5) *is satisfied.*

*Proof.* The claim follows if we have

$$\underline{a}_{j,\lambda(j)} \leq \underline{r}_{j,\lambda(j)}(\theta) \quad (\theta \in \Theta), \quad \overline{r}_{j,\mu(j)}(\theta) \leq \overline{a}_{j,\mu(j)} \quad (\theta \in \Theta) \tag{11}$$

for each $j = 1, 2, \ldots, q$.

Recall that

$$\underline{a}_{j,\lambda(j)} = \min_{\theta \in \operatorname{ver} \Theta} [b_{j,\lambda(j)}(\theta) - c_j(\theta)] = \min_{\theta \in \Theta} [b_{j,\lambda(j)}(\theta) - c_j(\theta)],$$
$$\underline{r}_{j,\lambda(j)}(\theta) = b_{j,\lambda(j)}(\theta) - c_j(\theta).$$

Hence, the first inequality in (11) follows immediately. The second inequality is similarly derived. □

We next consider how the vectors $\lambda(j)$ and $\mu(j)$ should be chosen for the parameter-dependent bounds. One possibility is to use the $\lambda$ and $\mu$ that optimize the constant bounds $\underline{a}_{j,\lambda}$ and $\overline{a}_{j,\mu}$. Proposition 7 then guarantees a less conservative result for the parameter-dependent bounds. A problem here is that computation of such $\lambda$ and $\mu$ is not easy except for the special case that $\Theta$ is a simplex. Even if such $\lambda$ and $\mu$ are computed, we still need concave maximization to have the explicit expressions of $b_{j,\lambda}(\theta)$ and $c_{j,\mu}(\theta)$. It is also questionable if the optimality in this criterion really means the goodness of the parameter-dependent bounds.

We here consider another criterion for the choice of $\lambda$ and $\mu$. It does not guarantee a less conservative result for the present approach in the sense of Proposition 7. It is however reasonable to use and convenient to compute the corresponding bounds. In particular, we employ the following quantity as a measure of conservatism:

$$V_j(\lambda,\mu) := \int_{\Theta} \overline{r}_{j,\mu}(\theta) - \underline{r}_{j,\lambda}(\theta)\,\mathrm{d}\theta \tag{12}$$

and choose $\lambda$ and $\mu$ so as to minimize this measure for each $j$. The measure $V_j(\lambda,\mu)$ stands for the volume of the region sandwiched by the parameter-dependent bounds $\underline{r}_{j,\lambda}(\theta)$ and $\overline{r}_{j,\mu}(\theta)$. Since we use $\underline{r}_{j,\lambda}(\theta)$ and $\overline{r}_{j,\mu}(\theta)$ to approximate $a_j(\theta)$ in the present approach, it is reasonable to choose $\lambda$ and $\mu$ so as to minimize $V_j(\lambda,\mu)$. Such $\lambda$ and $\mu$ have the following simple characterization.

**Proposition 8.** *For each $j = 1, 2, \ldots, q$, the measure of conservatism, $V_j(\lambda,\mu)$ in* (12)*, is minimized if and only if $\lambda$ and $\mu$ are subgradients of $b_j(\theta)$ and $c_j(\theta)$, respectively, at the gravity center of $\Theta$, that is,*

$$\theta^{\mathrm{c}} := \frac{\int_{\Theta} \theta\,\mathrm{d}\theta}{\int_{\Theta} \mathrm{d}\theta}.$$

*Moreover, for these $\lambda$ and $\mu$, the following relationships hold:*

$$b_{j,\lambda}(\theta) = \lambda^{\mathrm{T}}(\theta - \theta^{\mathrm{c}}) + b_j(\theta^{\mathrm{c}}), \quad c_{j,\mu}(\theta) = \mu^{\mathrm{T}}(\theta - \theta^{\mathrm{c}}) + c_j(\theta^{\mathrm{c}}).$$

*Proof.* As noticed in the previous section, the functions $b_{j,\lambda}(\theta)$ and $c_{j,\mu}(\theta)$, which are used in the definitions of $\underline{r}_{j,\lambda}(\theta)$ and $\overline{r}_{j,\mu}(\theta)$, are related to the notion of conjugate functions. Consider in general a convex function $f(\theta)$ on $\Theta$. Then, its *conjugate* $f^*(\lambda)$ is defined as

$$f^*(\lambda) := \max_{\theta\in\Theta}\bigl[\lambda^{\mathrm{T}}\theta - f(\theta)\bigr]$$

for $\lambda \in \mathbb{R}^p$ (*e.g.*, Hiriart-Urruty & Lemaréchal, 1993; Bertsekas, Nedić, & Ozdaglar, 2003). The conjugate function $f^*(\lambda)$ is convex because it is the maximum of the family of affine functions. With this notion, we can write

$$\underline{r}_{j,\lambda}(\theta) = \lambda^{\mathrm{T}}\theta - b_j^*(\lambda) - c_j(\theta), \quad \overline{r}_{j,\mu}(\theta) = b_j(\theta) - \mu^{\mathrm{T}}\theta + c_j^*(\mu)$$

for any $j$, $\lambda$, and $\mu$. The measure of conservatism can be rewritten as

$$V_j(\lambda,\mu) = \int_{\Theta}\bigl[b_j(\theta) - \mu^{\mathrm{T}}\theta + c_j^*(\mu) - \lambda^{\mathrm{T}}\theta + b_j^*(\lambda) + c_j(\theta)\bigr]\mathrm{d}\theta$$

$$= \int_{\Theta} b_j(\theta) + c_j(\theta)\,\mathrm{d}\theta - \lambda^{\mathrm{T}} \int_{\Theta} \theta\,\mathrm{d}\theta + b_j^*(\lambda) \int_{\Theta} \mathrm{d}\theta - \mu^{\mathrm{T}} \int_{\Theta} \theta\,\mathrm{d}\theta + c_j^*(\mu) \int_{\Theta} \mathrm{d}\theta$$

for each $j$. This is minimized when both $-\lambda^{\mathrm{T}}\theta^{\mathrm{c}} + b_j^*(\lambda)$ and $-\mu^{\mathrm{T}}\theta^{\mathrm{c}} + c_j^*(\mu)$ are minimized. Let us consider when the first function, $-\lambda^{\mathrm{T}}\theta^{\mathrm{c}} + b_j^*(\lambda)$, is minimized. Due to the convexity of this function, it is minimized if and only if 0 is a subgradient of $-\lambda^{\mathrm{T}}\theta^{\mathrm{c}} + b_j^*(\lambda)$ or, equivalently, $\theta^{\mathrm{c}}$ is a subgradient of $b_j^*(\lambda)$. This occurs if and only if $\lambda$ is a subgradient of $b_j(\theta^{\mathrm{c}})$ by a property of conjugate functions (Corollary X.1.4.4 of Hiriart-Urruty and Lemaréchal (1993)). Similar equivalence holds on the second function.

Suppose that $\lambda$ is a subgradient of $b_j(\theta)$ at $\theta = \theta^{\mathrm{c}}$. Then, the graph of $\lambda^{\mathrm{T}}(\theta - \theta^{\mathrm{c}}) + b_j(\theta^{\mathrm{c}})$ gives a hyperplane supporting the graph of $b_j(\theta)$ from below. This means that this function is identical with $b_{j,\lambda}(\theta)$. Similar discussion is possible on $c_{j,\mu}(\theta)$. □

**Example 9.** This is a continuation of Example 4.

We consider the first example: $a(\theta) = \sinh\theta$ and $\Theta = [-1,\ 1]$. The function $b(\theta) = \mathrm{e}^{\theta}/2$ is differentiable at the gravity center $\theta^{\mathrm{c}} = 0$ and its gradient is $1/2$ there. Hence, by Proposition 8, $\lambda = 1/2$ is optimal and the corresponding $b_\lambda(\theta)$ is $\lambda(\theta - \theta^{\mathrm{c}}) + b(\theta^{\mathrm{c}}) = \theta/2 + 1/2$. We thus obtain the optimal lower bound $\underline{r}_\lambda(\theta) = \theta/2 + 1/2 - \mathrm{e}^{-\theta}/2$. By symmetry, the optimal upper bound is $\overline{r}_\mu(\theta) = \mathrm{e}^{\theta}/2 + \theta/2 - 1/2$ for $\mu = -1/2$. The obtained bounds are shown in Figure 2 (a). In a large subset of $\Theta$, they are closer to the original function $a(\theta)$ than the constant bounds $\underline{a}_\lambda$ and $\overline{a}_\mu$ considered in Example 4. It is seen however that $\underline{a}_\lambda$ is closer to $a(\theta)$ than $\underline{r}_\lambda(\theta)$ around $\theta = -1$ and that $\overline{a}_\mu$ is closer than $\overline{r}_\mu(\theta)$ around $\theta = 1$. This is not surprising. Indeed, the uniform superiority of the parameter-dependent bounds is not guaranteed in the sense of Proposition 7 because different values are used for $\lambda$ and $\mu$ between the two types of bounds.

The procedure is the same for the second example: $a(\theta) = \sin\theta$ and $\Theta = [-2,\ 2]$. The function $b(\theta) = \sin\theta + \theta^2/2$ has the gradient 1 at the gravity center $\theta^{\mathrm{c}} = 0$. Hence, $\lambda = 1$ is optimal and the corresponding $b_\lambda(\theta)$ is $\lambda(\theta - \theta^{\mathrm{c}}) + b(\theta^{\mathrm{c}}) = \theta$. This gives the optimal lower bound $\underline{r}_\lambda(\theta) = \theta - \theta^2/2$. On the other hand, the function $c(\theta) = \theta^2/2$ has the gradient 0 at $\theta^{\mathrm{c}} = 0$. Hence, $\mu = 0$ is optimal and the corresponding $c_\mu(\theta)$ is $\mu(\theta - \theta^{\mathrm{c}}) + c(\theta^{\mathrm{c}}) = 0$. The obtained optimal upper bound is $\overline{r}_\mu(\theta) = \sin\theta + \theta^2/2$. The obtained bounds are found in Figure 2 (b). The same comments apply on the comparison with the constant bounds. ◇

When the gravity center $\theta^{\mathrm{c}}$ is computable for $\Theta$, the following approach is possible to the problem $P$. For each $j = 1, 2, \ldots, q$, we compute subgradients of $b_j(\theta)$ and $c_j(\theta)$ at $\theta = \theta^{\mathrm{c}}$ and choose them for $\lambda(j)$ and $\mu(j)$, respectively. Proposition 8 then guarantees their optimality with respect to $V_j(\lambda, \mu)$ and provides the explicit expressions of $b_{j,\lambda(j)}(\theta)$ and $c_{j,\mu(j)}(\theta)$. We obtain the optimal bounds $\underline{r}_{j,\lambda(j)}(\theta)$ and $\overline{r}_{j,\mu(j)}(\theta)$ with (6) and substitute them into the condition (7). Using this condition in place of (1), we obtain an approximate problem of $P$, which is solvable with the interior-point method. If we want to reduce the conservatism of the present approach, we can divide the parameter set $\Theta$ as in Section 3.

Computation of the gravity center $\theta^{\mathrm{c}}$ is immediate when $\Theta$ is a simplex or a parallelepiped. In many control problems, $\Theta$ is a hyperrectangle, which is a special parallelepiped. On the other hand, the exact computation of $\theta^{\mathrm{c}}$ is difficult when $\Theta$ is a general polytope (Rademacher, 2007). A solution would be approximate computation of $\theta^{\mathrm{c}}$ with a randomized technique (*e.g.*, Bertsimas & Vempala, 2004).

## 5. Examples

We applied the proposed two approaches to three example problems. The first two problems are on random matrices and the last problem is on control system design. The proposed approaches worked well for all the problems. In particular, they gave approximate solutions of the problems, whose precision became better as the division of the parameter region became finer.

**Example 10.** Set $a(\theta) = \sinh\theta$ and $\Theta = [-1,\ 1]$. Let $F$ be a $3 \times 3$ symmetric matrix whose elements are randomly generated according to the standard normal distribution. We consider the problem:

$$\begin{aligned} &\text{minimize} && z \\ &\text{subject to} && zI - \theta\,\mathrm{diag}(x_1, x_2, x_3) + a(\theta)F \succeq O \quad (\forall\theta \in \Theta), && (13) \\ &&& zI + \theta\,\mathrm{diag}(x_1, x_2, x_3) - a(\theta)F \succeq O \quad (\forall\theta \in \Theta), && (14) \end{aligned}$$

with the real scalar design variables $z$, $x_1$, $x_2$, and $x_3$. Here, $\mathrm{diag}(x_1, x_2, x_3)$ stands for the $3 \times 3$ diagonal matrix with the diagonal elements $x_1$, $x_2$, and $x_3$. In this problem, uniform approximation of $a(\theta)F$ is considered with $\theta\,\mathrm{diag}(x_1, x_2, x_3)$. Hence, it is important to take account of the parameter dependence of $a(\theta)$.

For 100 randomly generated $F$, we applied our two approaches and obtained the approximate solutions. Let $z_{\mathrm{cons}}$ stand for the minimum given by the approach with the constant bounds; let $z_{\mathrm{dep}}$ stand for the minimum given by the approach with the parameter-dependent bounds. These are upper bounds on the true minimum of the problem. For reference, we also computed its lower bound $z_{\mathrm{samp}}$ by minimizing $z$ subject to the inequalities (13) and (14) but only for the 101 values of $\theta$ equally spaced in $\Theta$. The normalized approximation error for the approach with the constant bounds, *i.e.*, $(z_{\mathrm{cons}} - z_{\mathrm{samp}})/z_{\mathrm{samp}}$, had the mean 0.96 and the standard deviation 1.81. On the other hand, the normalized approximation error for the approach with the parameter-dependent bounds, *i.e.*, $(z_{\mathrm{dep}} - z_{\mathrm{samp}})/z_{\mathrm{samp}}$, had the mean 0.22 and the standard deviation 0.30. To summarize, the approach with the parameter-dependent bounds gave better approximation on average. For some individual instances of $F$, however, this approach was inferior to the other. This is not surprising because the optimality of the parameter-dependent bounds does not imply the minimality of the approximation error. The mean computational time was 0.15 s in the approach with the constant bounds and 0.22 s in the approach with the parameter-dependent bounds. Here, the SDP problems were solved with SeDuMi (Sturm, 1999) through the modeling language YALMIP (Löfberg, 2004) on the computer equipped with Intel Core 2 Duo P8800 (2.66 GHz) and the memory of 4 GB.

We were able to improve the result by dividing the parameter set $\Theta$ to the two intervals $[-1,\ 0]$ and $[0,\ 1]$. Indeed, for the same set of 100 matrices $F$, the mean of $(z_{\mathrm{cons}} - z_{\mathrm{samp}})/z_{\mathrm{samp}}$ was improved from 0.96 to 0.65 and the mean of $(z_{\mathrm{dep}} - z_{\mathrm{samp}})/z_{\mathrm{samp}}$ was improved from 0.22 to 0.031. The mean computational time was a little longer: 0.24 s in the approach with the constant bounds and 0.34 s in the approach with the parameter-dependent bounds. ◇

**Example 11.** We consider the same problem as the previous example for the same set of $F$ but with $a(\theta) = \sin\theta$ and $\Theta = [-2,\ 2]$. The result was more or less similar to the previous one. We first applied our approaches without dividing $\Theta$. The mean of the normalized approximation error was 2.66 for the approach with the constant bounds and 3.33 for the approach with the parameter-dependent bounds. Although the result of the parameter-dependent approach was worse, the situation changed when $\Theta$ was divided into $[-2, 0]$ and $[0, 2]$. Namely, the mean of the normalized error became 0.71 for the approach with the constant bounds and 0.35 for the approach with the parameter-dependent bounds. ◇

**Example 12.** We apply our approaches to design of a sampled-data control system with an uncertain sampling interval. Consider a linear continuous-time plant $(\mathrm{d}/\mathrm{d}t)x(t) = Ax(t) + Bu(t)$ with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}.$$

We would like to design a matrix $K$ so that the sampled-data state-feedback control

$$u(t) = Kx(t_k) \quad (t_k \le t < t_{k+1};\ k = 0, 1, 2, \ldots)$$

stabilizes the plant, where $t_0, t_1, t_2, \ldots$ are sampling instants that satisfy $0 = t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. We assume here that the sampling interval $t_{k+1} - t_k$ is uncertain and can vary with $k$, motivated by the need in the networked and embedded control. In particular, we consider the (rather extreme) situation that $t_{k+1} - t_k$ can take any value in $(0, 200]$ and look for a stabilizing $K$ even in this situation.

This type of control problems has been considered by several authors (*e.g.*, Fridman, Seuret, & Richard, 2004; Hetel, Daafouz, & Iung, 2006; Mirkin, 2007; Naghshtabrizi, Hespanha, & Teel, 2008). We here follow the approach of Oishi and Fujioka (2010) and reduce the problem to a robust SDP problem:

$$\begin{array}{ll} \text{minimize} & z \\ \text{subject to} & Q - I \succeq O, \\ & \text{(the formula (15) at the bottom of this page)} \end{array}$$

with $\Theta := [0, 200]$ and

---

$$\begin{pmatrix} -E(\theta)(AQ + BL) - (AQ + BL)^{\mathrm{T}} E(\theta)^{\mathrm{T}} + zI & \sqrt{\theta} E(\theta)(AQ + BL) \\ \sqrt{\theta}(AQ + BL)^{\mathrm{T}} E(\theta)^{\mathrm{T}} & Q \end{pmatrix} \succeq O \quad (\theta \in \Theta) \tag{15}$$

$$E(\theta) := \frac{1}{\theta}\int_0^\theta \mathrm{e}^{At}\,\mathrm{d}t = \begin{pmatrix} 1 & -10 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\mathrm{e}^{-0.1\theta}-1}{-0.1\theta} \end{pmatrix} \begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix}.$$

We define $E(\theta) = I$ at $\theta = 0$. In this problem, the design variables are a scalar $z$, a $2 \times 2$ symmetric matrix $Q$, and a $1 \times 2$ matrix $L$. If a feasible solution $(z, Q, L)$ such that $z < 0$ is found for this problem, a stabilizing state-feedback gain $K$ is obtained by $LQ^{-1}$.

Let us set $a_1(\theta) = \sqrt{\theta}$ and $a_2(\theta) = (\mathrm{e}^{-0.1\theta} - 1)/(-0.1\theta)$. We define $a_2(\theta) = 1$ at $\theta = 0$. We first notice that the constraint of the problem includes the product $a_1(\theta)a_2(\theta)$. To remove this product, we apply the technique of Oishi (2009) and Oishi and Isaka (2009). Then, we can obtain a sufficient condition, which is affine in $a_1(\theta)$ and $a_2(\theta)$. Replacing the constraint by this sufficient condition, we can apply our approaches. We choose $b_1(\theta)$ and $c_2(\theta)$ to be zero noting the concavity of $a_1(\theta)$ and the convexity of $a_2(\theta)$, respectively.

The obtained minimum was $1.75 \times 10^{-5}$ in the approach with the constant bounds and $3.87 \times 10^{-4}$ in the approach with the parameter-dependent bounds. This means that both approaches failed to give a stabilizing state-feedback gain. The situation changed when $\Theta$ was divided into two subintervals $[0, 100]$ and $[100, 200]$. Indeed, with both approaches, a feasible solution with a negative $z$ was found, which gave a stabilizing state-feedback gain. The computational time in this case was 1.65 s in the approach with the constant bounds and 4.85 s in the approach with the parameter-dependent bounds.

Oishi and Fujioka (2010) gave an approach specialized for analysis and design of this type of sampled-data control systems. The performance of our present approaches were comparable to this specialized approach. Indeed, the specialized approach failed to give a state-feedback gain when $\Theta$ was not divided. It failed again with the division consisting of $[0, 100]$ and $[100, 200]$. After some trials and errors, it succeeded with the division consisting of $[0, 25]$ and $[25, 200]$. The computational time in the last case was 0.39 s, which was short due to specialization. ◇

## 6. Conclusion

Two conservative approaches are proposed for robust SDP problems with general nonlinear parameter dependence. The first approach is based on the computation of constant bounds for each nonlinear function. It is conceptually simple but has a computational issue in the optimization of the bounds. The second approach is based on the computation of parameter-dependent bounds. It has a potential to give a less conservative result by exploiting the parameter dependence of the problem. It also has a computational advantage when the gravity center is computable for the parameter set. Both approaches rely on the assumption that the nonlinear parameter dependence is expressed as the difference of two convex functions. Since this assumption is mild enough, the proposed approaches can be applied to a large class of problems.

Some extensions are possible with the present approaches. We saw in Example 12 that, even if the original constraint includes the product $a_1(\theta)a_2(\theta)$, we can apply our approaches with the help of the existing result. This technique can be generalized to the case that the constraint includes polynomial or rational functions of $a_j(\theta)$ (Peaucelle & Sato, 2009; Oishi, 2009; Oishi & Isaka, 2009). On the other hand, by mimicing the approach of Chesi and Hung (2008), it is possible to expand $a_j(\theta)$ to the finite Taylor series and compute the proposed bounds for the remainder term. This technique can reduce conservatism when the remainder term is smaller than the original function $a_j(\theta)$.

## Acknowledgments

The authors are thankful to Fabrizio Dabbene, Yoshio Ebihara, Dimitri Peaucelle, and Roberto Tempo for their valuable comments. This work is supported in part by the Grant-in-Aid for Scientific Research of the Japan Society for the Promotion of Science and the Nanzan University Pache Research Subsidy I-A-2 for the academic year 2010. This work is performed during the sabbatical stay of the first author at the MAC group, LAAS-CNRS, Toulouse, France.

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