Robust Model Predictive Controller for Tracking Changing Periodic Signals

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***Abstract*—In this paper, we propose a novel robust model predictive controller for tracking periodic signals for linear systems subject to bounded additive uncertainties based on nominal predictions and constraint tightening. The proposed controller joins optimal periodic trajectory planning and robust control for tracking in a single optimization problem and guarantees that the perturbed closed-loop system converges asymptotically to a neighborhood of an optimal reachable periodic trajectory while robustly satisfying the constraints. In addition, the closed-loop system maintains recursive feasibility even in the presence of sudden changes in the target reference.**

***Index Terms*—Predictive control for linear systems, robust control, stability of linear systems, uncertain systems.**

I. Introduction

In this paper we focus on the problem of tracking periodic references for linear systems subject to bounded additive uncertainties which appears when the optimal operation is not to remain at a single steady state, but to follow a time varying periodic or quasi-periodic trajectory such as water, electricity or energy distribution networks, chemical reactors or repetitive processes. The application of MPC to periodic systems has been studied in [1], where a model predictive control approach to repetitive control of continuous processes with periodic operations was proposed, or more recently in [2]. In addition to periodic systems and batch processes, economic operation may lead to non-steady state operation and periodic references, see for example [3], [4]. In these references stability is proved under the assumption that the provided periodic reference trajectory is never changed. However, in practice, the reference may be changed due, for instance, to variations on the desired behavior of the controlled system or derived from variations of the demands of the controlled plants. The tracking MPC must cope with the robust stability of the controlled system and convergence to the (possibly changing) periodic reference.

The tracking problem is considerably more difficult when the reference varies in a way not known a-priori because MPC is naturally suited to deterministic control problems. Therefore, the study of the closed-loop stability properties of predictive controllers for tracking periodic references is an open issue. In [5] a predictive controller for the offset-free tracking of reference signals generated by arbitrary dynamics is proposed. Another approach is to adopt a hierarchical architecture in which a trajectory planner decides the optimal reachable trajectory that is provided to a MPC controller [4]. In [6] is shown that this control scheme may lead to a loss of feasibility when a sudden change in the reference takes place. This issue is inherent to tracking problems and the design of MPC strategies robust to target changes is a relevant problem as most MPC schemes in the literature can exhibit feasibility issues. The latest MPC state of the art reviews mention this problem as one of the open issues of the field, see [7]–[9]. A scheme for tracking arbitrary periodic references was proposed in [10] based on a single layer that unites dynamic trajectory planning and control. This control scheme was extended to the economic case in [11], [12]. The above mentioned methods do not take into account possible disturbances or modeling errors although recently, in [13], a robust periodic controller for linear systems was proposed in the economic MPC framework; that is, instead of assuming that a target reference is given, the controller tries to optimize directly an economic target function.

Following these ideas, the main contribution of this work is a novel robust model predictive controller for tracking periodic signals for linear systems based on extending the control scheme proposed in [10] to take into account bounded additive uncertainties following the constraint tightening method proposed in [14]. A set of design conditions that guarantees that the closed-loop system converges to neighborhood of a robust optimal trajectory that minimizes the cost function while robustly satisfying the constraints even in the presence of sudden reference changes is also provided. These design condition can be fulfilled using a robust positive invariant terminal region or a equality terminal constraint, avoiding the calculation of invariants. The properties of the proposed controller are demonstrated with a simulation of a ball and plate system.

Notation: Bold letters are used to denote a sequence of T values of a trajectory, i.e. $\mathbf{z} = \{z(0), \ldots, z(T-1)\}$, while $\mathbf{z}_N(\theta)$ denotes the sequence which cardinality, N, is not T. $\mathbb{I}_a$ denotes the set of positive integer numbers including the origin, that is $\{0, 1, \ldots, a\}$. $\|\cdot\|$ denotes quadratic norm.

II. Problem Formulation

In this work, we consider the following class of discrete time linear systems subject to bounded additive uncertainties

$$x(k+1) = Ax(k) + Bu(k) + w(k) \tag{1a}$$

$$y(k) = Cx(k) + Du(k) \tag{1b}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $w(k) \in \mathbb{R}^n$ are the state, the input and the uncertainty at sampling time k respectively. The uncertainty is known to be confined in the convex compact polytope containing the origin in its interior $\mathcal{W}$. We denote as nominal model system (1) with $w(k)$ null.

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The state and input trajectories must satisfy the $x(k) \in \mathcal{X}$ and $u(k) \in \mathcal{U}$ for all sampling times k where the sets $\mathcal{X}$ and $\mathcal{U}$ are convex, compact polytopes containing the origin in its interior.

The control objective is to steer the output of the system as close as possible to an exogenous periodic target reference with period T defined as $\mathbf{r}$. We assume that the estate of the system is fully measured. It is important to remark that no assumption is considered in the reference beyond its periodicity; that is, there may not exist a control law capable of steering the system to this reference signal satisfying the constraints. In any case the closed-loop system must converge to the best reachable periodic trajectory, i.e. that optimizes a certain criterion while satisfying the constraints for all possible uncertainties. Besides, the reference can be subject to sudden changes. For this reason, the reference is included as one of the input parameters of the controller in addition to the state measurement.

Standard tracking schemes are usually based on a hierarchical architecture in which a trajectory planner computes the optimal reachable trajectory which is then used by a MPC as a reference trajectory. This implies that the MPC controller depends on this optimal trajectory and that two different optimization problems have to be solved. In addition, the feasibility region of the optimization problem of standard MPC methods generally depends on the reference signal. This implies that feasibility can be lost if a sudden change in the reference takes place. As it will be demonstrated later on, the proposed control law will deal with the case that the periodic reference signal is suddenly changed without loosing feasibility.

III. Controller Formulation

The proposed robust controller is based on the MPC for tracking periodic references presented in [10] but extended to cope with the uncertainty using the ideas of [14]. The controller is based on augmenting the decision variables with a set of auxiliary variables that describe a future, periodic and admissible artificial reference. The objective cost takes into account both the deviation of the predicted trajectory from the artificial reference, and the deviation of the artificial reference from the target periodic reference. The prediction horizon is N.

To this end, the predictive control law is derived from the solution of an optimization problem that minimizes a cost function $V_N(\cdot)$ that includes two terms. The first term $V_t(\cdot)$ penalizes the deviation of a N step predicted trajectory starting from the current state, from the artificial reference. The second term $V_p(\cdot)$ penalizes the deviation of the artificial reference from the target reference over a period. The cost function is defined as follows:

$$V_N(x, \mathbf{r}; \mathbf{x}_{N+1}, \mathbf{u}_N, \mathbf{x}^r_{T+1}, \mathbf{u}^r) = V_t(x; \mathbf{x}_{N+1}, \mathbf{u}_N, \mathbf{x}^r_{T+1}, \mathbf{u}^r) + V_p(\mathbf{r}; \mathbf{x}^r_{T+1}, \mathbf{u}^r)$$

with

$$\begin{aligned} V_t(x; \mathbf{x}_{N+1}, \mathbf{u}_N, \mathbf{x}^r_{T+1}, \mathbf{u}^r) = & \sum_{i=0}^{N-1} \|x(i) - x^r(i)\|^2_Q \\ & + \sum_{i=0}^{N-1} \|u(i) - u^r(i)\|^2_R \\ & + \|x(N) - x^r(N)\|^2_P \\ V_p(\mathbf{r}; \mathbf{x}^r_{T+1}, \mathbf{u}^r) = & \sum_{i=0}^{T-1} \|y^r(i) - r(i)\|^2_S \end{aligned}$$

where Q, R, P and S are suitable positive definite matrices and

$$y^r(i) = Cx^r(i) + Du^r(i)$$

The optimization variables $(\mathbf{x}_{N+1}, \mathbf{u}_N, \mathbf{x}^r_{T+1}, \mathbf{u}^r)$ are the predicted sequences of states and inputs of the system and the artificial state and input trajectories respectively.

Following [14], the proposed robust controller requires the design of a local robustly stabilizing control gain K. Based on this gain, the following sets are defined:

$$\mathcal{X}(i) = \mathcal{X} \ominus \left\{ \bigoplus_{j=0}^{i-1} (A + BK)^j \mathcal{W} \right\} \quad (2)$$

$$\mathcal{U}(i) = \mathcal{U} \ominus \left\{ K \bigoplus_{j=0}^{i-1} (A + BK)^j \mathcal{W} \right\} \quad (3)$$

where $\oplus$ stands for the Minkowski addition of sets and $\ominus$ stands for the Pontryagin difference of sets with $\mathcal{X}(0) = \mathcal{X}$ and $\mathcal{U}(0) = \mathcal{U}$. The sets $\mathcal{X}(i)$ and $\mathcal{U}(i)$ are tightened versions of the set of constraints $\mathcal{X}$ and $\mathcal{U}$. These sets depend both on the size of the uncertainty set $\mathcal{W}$ and the local control law K and they are easily obtained off-line. These conservative sets will be considered as set of constraints of the predicted trajectories in order to ensure robust constraint satisfaction, as it will be proved in the next section.

In order to derive a stabilizing constraint, a suitable terminal control gain K_f and a suitable (robust) invariant set Ω is computed. Thus, the terminal region is given by

$$\mathcal{X}_f = \Omega \ominus (A + BK)^{N-1} \mathcal{W} \quad (4)$$

Similarly to the set of constraints, the terminal region $\mathcal{X}_f$ is a (robust invariant) set Ω tightened by a measure of the effect of the uncertainty in the predicted terminal state. On the other hand, the artificial trajectory is also subject to tighter constraints given by the sets

$$\mathcal{X}^r = \mathcal{X}(N) \ominus \mathcal{X}_f \quad (5)$$

$$\mathcal{U}^r = \mathcal{U}(N-1) \ominus K_f \Omega \quad (6)$$

In the following section the design assumptions that these ingredients must satisfy to guarantee some closed-loop properties are defined.

The proposed robust model predictive for tracking periodic references is derived from the solution of the following optimization problem

$$\min_{\mathbf{x}_{N+1}, \mathbf{u}_N, \mathbf{x}^r_{T+1}, \mathbf{u}^r} V_N(x, \mathbf{r}; \mathbf{x}_{N+1}, \mathbf{u}_N, \mathbf{x}^r_{T+1}, \mathbf{u}^r) \quad (7a)$$

$$s.t.\ x(0) = x \quad (7b)$$

$$x(i+1) = Ax(i) + Bu(i) \quad i \in \mathbb{I}_{N-1} \quad (7c)$$

$$x(i) \in \mathcal{X}(i) \quad i \in \mathbb{I}_N \quad (7d)$$

$$u(i) \in \mathcal{U}(i) \quad i \in \mathbb{I}_{N-1} \quad (7e)$$

$$x(N) - x^r(N) \in \mathcal{X}_f \quad (7f)$$

$$x^r(i+1) = Ax^r(i) + Bu^r(i) \quad i \in \mathbb{I}_{T-1} \quad (7g)$$

$$x^r(i) \in \mathcal{X}^r \quad i \in \mathbb{I}_T \quad (7h)$$

$$u^r(i) \in \mathcal{U}^r \quad i \in \mathbb{I}_{T-1} \quad (7i)$$

$$x^r(T) = x^r(0) \quad (7j)$$

The optimal solution of this optimization problem is denoted as $(\mathbf{x}^*_{N+1}, \mathbf{u}^*_N, \mathbf{x}^{r*}_{T+1}, \mathbf{u}^{r*})$ and it's a function of x and $\mathbf{r}$. Analogously, the optimal cost function is denoted as $V^*_N(x, \mathbf{r})$.

Constraints (7b)–(7c) define the predicted trajectories of the system starting from the current state. Constraints (7g) and (7j) define the planned periodic reachable reference starting from the free initial state $x^r(0)$. Constraints (7d) and (7e) include the state and input constraints for the predicted states and inputs. These constraints depend on the tightened sets defined above and are different for each prediction step i. Constraints (7h) and (7i) include the state and input constraints for the artificial reference. These constraints depend on the tightened sets defined above but are constant for all prediction steps i. In addition, a terminal constraint is included to guarantee closed-loop convergence to the optimal reachable trajectory. Constraint (7f) guarantees that the terminal state of the predicted trajectory of the plant reaches a neighborhood of the planned reachable trajectory at the end of the prediction horizon.

At each time step k, the periodic reference signal $\mathbf{r}(k)$ used to define the controller is different because the initial time of the sequence changes. With a slight abuse of notation, we define $\mathbf{r}$ as the target periodic reference, and $\mathbf{r}(k)$ the reference fed to the controller which takes into account the time shift. The control law is given by the first input of the optimal reachable predicted trajectory,

$$\kappa_N(x(k), \mathbf{r}(k)) = u_N^*(0; x(k), \mathbf{r}(k)) \quad (8)$$

IV. Robust Stability of the Controlled System

In this section we study the closed-loop properties of the proposed controller. We prove that the closed-loop system converges asymptotically to a neighborhood of an optimal reachable trajectory while constraints are robustly satisfied. The optimal reachable trajectory can be obtained solving an optimization problem. To this end we prove that the deviation of the system from the optimal reachable trajectory is input-to-state stable with respect to the uncertainty. In order to guarantee these properties, the controller must be designed appropriately. In particular, the following design assumptions must hold:

Assumption 1: The weighting matrices Q, R, P and S, the controller gains K, K_f and the set Ω satisfy the following conditions:

1) $(A+BK_f)^T P(A+BK_f) - P \leq -(Q + K_f^T R K_f)$
2) The set Ω is compact polytope such that

$$(A+BK_f)\Omega \subseteq \Omega \ominus (A+BK)^{N-1}\mathcal{W}$$

3) The optimization problem (7) is strictly convex and the sets $\mathcal{X}_r$ and $\mathcal{U}_r$ are non-empty.

It is important to remark that the local control gain K must be designed in a way such that the tightened sets are not empty. There is a trade-off between perturbation rejection (which affects the size of $\mathcal{X}(i)$) and the amount of control effort used (which affects the size of $\mathcal{U}(i)$). This design challenge is inherent to semi-feedback prediction schemes, see for example [15].

The optimal reachable trajectory of the plant $(\mathbf{x}^\circ, \mathbf{u}^\circ)$ is the nominal trajectory that minimizes the following optimization problem

$$\arg\min_{\mathbf{x}^r_{T+1}, \mathbf{u}^r} \quad V_p(\mathbf{r}; \mathbf{x}^r_{T+1}, \mathbf{u}^r) \quad (9a)$$

$$\text{s.t.} \quad x^r(i+1) = Ax^r(i) + Bu^r(i) \quad (9b)$$

$$i \in \mathbb{I}_{T-1}$$

$$x^r(i) \in \mathcal{X}^r \quad i \in \mathbb{I}_T \quad (9c)$$

$$u^r(i) \in \mathcal{U}^r \quad i \in \mathbb{I}_{T-1} \quad (9d)$$

$$x^r(T) = x^r(0) \quad (9e)$$

This is the admissible trajectory (according to the tighter set of constraints) that minimizes the cost function $V_p(\cdot)$ which measures the distance to the reference $\mathbf{r}$. It is important to remark that this trajectory is uniquely defined since from the Assumption 1 we derive that the optimization problem (9) is feasible and strictly convex.

We will prove that the optimal admissible trajectory is a robustly stable trajectory of the system in the input-to-state stability sense, which is defined as follows.

Definition 1: The periodic trajectory $\mathbf{x}^\circ$ is an input-to-state stable trajectory for the controlled system with a domain of attraction $\mathcal{X}_N$ if for all $x(0) \in \mathcal{X}_N$, then $x(k) \in \mathcal{X}_N$ and there exists a $\mathcal{KL}$ function $\beta(\cdot)$ and a $\mathcal{K}$ function $\sigma(\cdot)$ such that

$$\|x(k) - x^\circ(k)\| \leq \beta(\|x(0) - x^\circ(0)\|, k) + \sigma(\|\mathbf{w}_k\|_\infty)$$

for all $k \geq 0$. $\|\mathbf{w}_k\|_\infty$ denotes the maximum value of $\|w(i)\|$ for all $i \in \mathbb{I}_{k-1}$.

In the following theorem it is stated the input-to-state stability of the trajectory $\mathbf{x}^\circ$ which is the main result of the paper.

Theorem 1: Assume that conditions given in Assumption 1 hold. Then system (1) controlled by the proposed control law $u(k) = \kappa_N(x(k), \mathbf{r}(k))$ is recursively feasible and the optimal periodic reachable trajectory $\mathbf{x}^\circ$ is input-to-state stable with a region of attraction $\mathcal{X}_N$, i.e. the closed loop system is stable and $x(k)$ converges asymptotically to a neighborhood of $\mathbf{x}^\circ(k)$ for all $x(0) \in \mathcal{X}_N$.

Proof: In order to proof the theorem, we first prove that if the initial state is inside the feasibility region of the optimization problem, the closed-loop system will remain inside this region; i.e. the closed-loop system is recursively feasible. Then, asymptotic stability will be proved by demonstrating that for the system that models the error between the state of the reachable optimal trajectory and the closed loop trajectory of the system the function

$$W(x(k) - x^\circ(k)) = V_N^*(x(k), \mathbf{r}(k)) - V_p^\circ(\mathbf{r}) \quad (10)$$

is an input-to-state Lyapunov function [16]. This function is defined as the difference between the optimal cost of the MPC problem at time k and the cost value of the optimal reachable trajectory.

As the sets $\mathcal{X}_r$ and $\mathcal{U}_r$ are non-empty, then the sets $\mathcal{X}(i)$ and $\mathcal{U}(j)$ are non-empty for $i \in \mathbb{I}_N$ and $j \in \mathbb{I}_{N-1}$. This ensures that there exists an initial state where the optimization problem is feasible.

The proof is divided into two parts: first recursive feasibility of the optimization problem is demonstrated and then, input-to-state stability of the optimal reachable trajectory is proved.

Recursive feasibility: We define next the shifted solution at time $k+1$ obtained from the optimal solution at time k and some corrections provided by the feedback policy K as proposed in [14]. We use the notation $(i|k)$ to denote the time step to which a given variable is referred.

$$u^{rs}(i|k+1) = u^{r*}(i+1|k) \quad i \in \mathbb{I}_{T-2} \quad (11a)$$

$$u^{rs}(T-1|k+1) = u^{r*}(0|k) \quad (11b)$$

$$x^{rs}(i|k+1) = x^{r*}(i+1|k) \quad i \in \mathbb{I}_{T-1} \quad (11c)$$

$$x^{rs}(T|k+1) = x^{r*}(1|k) \quad (11d)$$

$$u^s(i|k+1) = u^*(i+1|k)$$

$$+ \quad K(x^s(i|k+1) - x^*(i+1|k)) \quad i \in \mathbb{I}_{N-2} \quad (11e)$$

$$u^s(N-1|k+1) = u^{rs}(N-1|k+1) \quad (11f)$$

$$+ \quad K_f(x^s(N-1|k+1) - x^{rs}(N-1|k+1)) \tag{11a}$$

$$x^s(0|k+1) = x(k+1) = Ax(k) + Bu^*(0|k)$$

$$+ \quad w(k) \tag{11b}$$

$$x^s(i+1|k+1) = Ax^s(i|k+1) + Bu^s(i|k+1) \tag{11c}$$

$$i \in \mathbb{I}_{N-1}$$

Since $x^*(0|k) = x(k)$, we have that

$$x^s(0|k+1) - x^*(1|k) = w(k)$$

As it has been proved in [14] the error between the optimal trajectory at k and the shifted one is given by

$$x^s(i+1|k+1) - x^*(i+2|k) = A_K(x^s(i|k+1) - x^*(i+1|k))$$

for $i \in \mathbb{I}_{N-1}$ and $A_K = A + BK$. Since $w(k) \in \mathcal{W}$, the following condition holds

$$x^s(i|k+1) - x^*(i+1|k) \in (A+BK)^i\mathcal{W} \tag{12}$$

Proceeding similarly for the shifted input, the following inequalities hold for $i \in \mathbb{I}_{N-2}$

$$u^s(i|k+1) - u^*(i+1|k) \in K(A+BK)^i\mathcal{W}$$

Based on these results, it will be proved that the shifted solution is feasible for the optimization problem at $k+1$.

It is immediate to see that the constraints (7b), (7c), and (7g)–(7j) are satisfied. The remaining constraints are proved next.

- Constraint (7f): The shifted solution is such that

$$x^s(N-1|k+1) - x^*(N|k) \in A_K^{N-1}\mathcal{W}$$

Since the optimal solution at k satisfies the constraint (7f), we can infer that

$$\begin{aligned}\Delta_x^s &= x^s(N-1|k+1) - x^{rs}(N-1|k+1)\\ &= x^s(N-1|k+1) - x^{rs}(N|k)\\ &= (x^s(N-1|k+1) - x^*(N|k)) +\\ &\quad +(x^*(N|k) - x^{rs}(N|k))\\ &\in (A+BK)^{N-1}\mathcal{W} \oplus \mathcal{X}_f \subseteq \Omega\end{aligned}$$

Applying the shifted input $u^s(N-1|k+1)$, then

$$x^s(N|k+1) - x^{rs}(N|k+1)$$

is posed as

$$(A+BK_f)(x^s(N-1|k+1) - x^{rs}(N-1|k+1))$$

Therefore $x^s(N|k+1) - x^{rs}(N|k+1) \in (A+BK_f)\Omega \subseteq \Omega \ominus (A+BK)^{N-1}\mathcal{W} = \mathcal{X}_f$.

- Constraint (7d): The optimal solution satisfies that

$$x^*(i+1|k) \in \mathcal{X}(i+1)$$

On the other hand

$$x^s(i|k+1) - x^*(i+1|k) \in (A+BK)^i\mathcal{W}$$

and then

$$x^s(i|k+1) = x^*(i+1|k) + (x^s(i|k+1) - x^*(i+1|k))$$

Thus

$$x^s(i|k+1) \in \mathcal{X}(i+1) \oplus (A+BK)^i\mathcal{W} \subseteq \mathcal{X}(i)$$

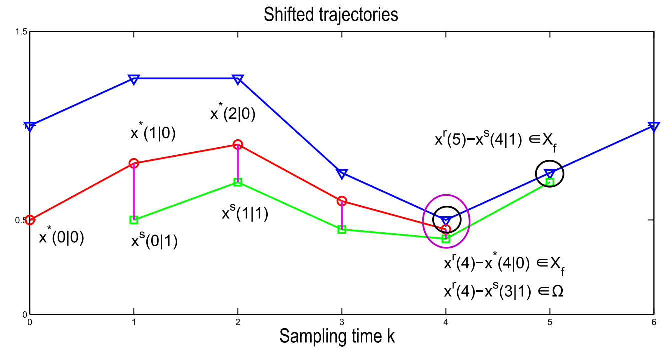

Fig. 1. Example of shifted state trajectories considering a terminal region and a terminal controller.

for all $i \in \mathbb{I}_{N-1}$. For $i = N$, we have proved that

$$x^s(N|k+1) - x^{rs}(N|k+1) \in X_f$$

As $x^{rs}(N|k+1) = x^{r*}(N+1|k) \in \mathcal{X}^r = \mathcal{X}(N) \ominus X_f$, then $x^s(N|k+1) \in \mathcal{X}(N) \ominus X_f \oplus X_f \subseteq \mathcal{X}(N)$.

- Constraint (7e): The optimal solution satisfies that

$$u^*(i+1|k) \in \mathcal{U}(i+1)$$

On the other hand

$$x^s(i|k+1) - x^*(i+1|k) \in (A+BK)^i\mathcal{W}$$

and then

$$u^s(i|k+1) = u^*(i+1|k) + K(x^s(i|k+1) - x^*(i+1|k))$$

Thus

$$u^s(i|k+1) \in \mathcal{U}(i+1) \oplus K(A+BK)^i\mathcal{W} \subseteq \mathcal{U}(i)$$

for all $i \in \mathbb{I}_{N-2}$. For $i = N-1$, since

$$x^s(N-1|k+1) - x^{rs}(N-1|k+1) \in \Omega$$

and

$$\begin{aligned}u^{rs}(N-1|k+1) &= u^{rs}(N|k)\\ &\in \mathcal{U}^r = \mathcal{U}(N-1) \ominus K_f\Omega\end{aligned}$$

We have that $u^s(N-1|k+1)$ is posed as

$$u^{rs}(N-1|k+1) + K_f(x^s(N-1|k+1) - x^{rs}(N-1|k+1))$$

thus

$$u^s(N-1|k+1) \in \mathcal{U}(N-1) \ominus K_f\Omega \oplus K_f\Omega$$

$$u^s(N-1|k+1) \subseteq \mathcal{U}(N-1)$$

Fig. 1 shows an example of one dimensional shifted state trajectories. In blue triangles the optimal artificial trajectory is shown for both time step $k = 0$ and $k = 1$. In red circles the optimal trajectory at time step $k = 0$ is shown. After four time steps (the prediction horizon), the difference between the artificial reference and the optimal trajectory is inside the terminal region denoted with a magenta ellipsoid[1]. In green

[1]The terminal regions are shown as ellipsoids for aesthetic reasons, although they should be a segment because the state has dimension one.

squares the shifted trajectory at time step $k = 1$ is shown. It can be seen that it deviates from the previous optimal trajectory because of the uncertainty. However, the deviation is corrected in the predictions by the local controller K, and this deviation converges asymptotically to zero as $(A + BK)^i w(0)$. The shifted trajectory last input is defined by the artificial reference, so it follows the blue trajectory. Although there is a difference between the shifted trajectory and the artificial reference, is inside the robust positive invariant of the local controller K_f for an uncertainty bounded in $(A + BK)^N \mathcal{W}$, so it is guaranteed that the nominal prediction lies inside X_f.

Stability: Stability is proved by demonstrating that the function

$$W(x - x^\circ) = V_N^*(x, \mathbf{r}) - V_p(\mathbf{r}; \mathbf{x}^\circ, \mathbf{u}^\circ)$$

is an ISS Lyapunov function [16]. Using similar arguments to the stability proof of the nominal case in [10], we have that there exists positive constants α_1, α_2 and α_3 such that

$$\begin{aligned} \alpha_1 \|x(k) - x^\circ(k)\|^2 &\leq W(x(k) - x^\circ(k)) \\ \alpha_2 \|x(k) - x^\circ(k)\|^2 &\geq W(x(k) - x^\circ(k)) \end{aligned}$$

On the other hand, the increment of $W(\cdot)$ for the nominal system, $\Delta W = W(x(1|k) - x^\circ(k+1)) - W(x(k) - x^\circ(k))$ satisfies that

$$\Delta W \leq -\alpha_3 \|x(k) - x^\circ(k)\|^2.$$

This is derived from lemma 1 of [10] that can be applied for the nominal prediction.

Since the optimal cost function $V_N^*(x, \mathbf{r})$ is a convex function of x defined in a compact set, $V_N^*(\cdot)$, and hence $W(\cdot)$, is Lipschitz continuous. Given that $x(k+1) = x(1|k) + w(k)$, from the Lipschitz continuity of $W(\cdot)$ there exists a positive constant γ such that

$$\begin{aligned} W(x(k+1) - x^\circ(k+1)) - W(x(1|k) \\ - x^\circ(k+1))\| \leq \gamma \|w(k)\| \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \Delta W = \; & W(x(k+1) - x^\circ(k+1)) \\ & - W(x(1|k) - x^\circ(k+1)) \\ & + W(x(1|k) - x^\circ(k+1)) \\ & - W(x(k) - x^\circ(k)) \\ \leq \; & \gamma \|w(k)\| - \alpha_3 \|x(k) - x^\circ(k)\|^2 \end{aligned}$$

and then $W(\cdot)$ is an ISS Lyapunov function which completes the proof.

A. Equality Terminal Constraint

The proposed controller requires the calculation of a stabilizing gain K_f and a robust positively invariant set Ω. There exists efficient algorithms to compute these ingredients, but their complexity grows exponentially with the dimension of the system to be controlled. Then, from a practical point of view, it is very interesting to take K such that the eigenvalues $\lambda_i(A + BK) = 0$, since in this case $(A + BK)^{N-1}\mathcal{W}$ is $\{0\}$ and then the stabilizing terminal ingredients can be chosen as $K_f = 0$, $P = 0$ and $X_f = \{0\}$. This is equivalent to consider a terminal equality constraint $x(N) = x^r(N)$. In Fig. 2, it is illustrated the different trajectories in the case of using a terminal equality constraint.

The robust model predictive control for tracking periodic references using the proposed equality terminal constraint is derived from the solution of the following optimization problem

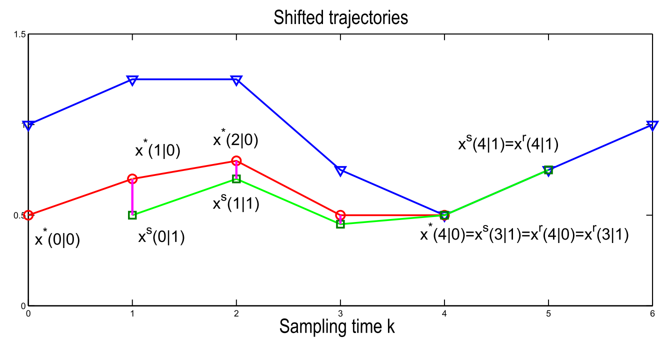

Fig. 2. Example of shifted state trajectories considering an equality terminal constraint.

$$\min_{\mathbf{x}_{N+1}, \mathbf{u}_N, \mathbf{x}^r_{T+1}, \mathbf{u}^r} V_N(x, \mathbf{r}; \mathbf{x}_{N+1}, \mathbf{u}_N, \mathbf{x}^r_{T+1}, \mathbf{u}^r) \quad (13a)$$

$$\text{s.t. } x(0) = x \quad (13b)$$

$$x(i+1) = Ax(i) + Bu(i) \quad i \in \mathbb{I}_{N-1} \quad (13c)$$

$$x(i) \in \mathcal{X}(i) \quad i \in \mathbb{I}_N \quad (13d)$$

$$u(i) \in \mathcal{U}(i) \quad i \in \mathbb{I}_{N-1} \quad (13e)$$

$$x(N) = x^r(N) \quad (13f)$$

$$x^r(i+1) = Ax^r(i) + Bu^r(i) \quad i \in \mathbb{I}_{T-1} \quad (13g)$$

$$x^r(i) \in \mathcal{X}^r \quad i \in \mathbb{I}_T \quad (13h)$$

$$u^r(i) \in \mathcal{U}^r \quad i \in \mathbb{I}_{T-1} \quad (13i)$$

$$x^r(T) = x^r(0) \quad (13j)$$

In practice it suffices to design a K such that $\max_{w \in \mathcal{W}} \|(A + BK)^{N-1} w\|$ is sufficiently small w.r.t. to the precision of the optimization solver. For instance if the set $\mathcal{W}$ is the unitary ball and $N = 30$, then taking a control gain K such that $\|A + BK\| \leq 0.28$, then $\max_{w \in \mathcal{W}} \|(A + BK)^{N-1} w\|$ is lower than 10^{-16} and if $\|A + BK\| \leq 0.57$ then $\max_{w \in \mathcal{W}} \|(A + BK)^{N-1} w\|$ is lower than $\leq 10^{-7}$. These values are smaller than the Matlab spacing for double and single precision respectively. Therefore, introducing this practical relaxation makes the design more flexible since it is not necessary to guarantee that $\lambda_i(A + BK) = 0$.

V. Application to a Ball and Plate System

In this section we apply the proposed controller to a linear approximation of a ball and plate system. The system consists of a plate pivoted at its center such that the slope of the plate can be manipulated in two perpendicular directions. A servo system consisting of two motors is used for tilting the plate and control the two angles of rotation θ_1, θ_2. An appropriate sensor for measurement of the ball position z_1, z_2 is assumed to be available, for example an intelligent vision system. The basic control task is to control the position of a ball freely rolling on a plate. This system is a dynamic system with two inputs and two outputs.

To carry out the simulations a nonlinear continuous time model is obtained taking into account the rigid body dynamics of the ball on the plate. In particular, applying the Lagrange-Euler formulation to each coordinate $(z_1, z_2, \theta_1, \theta_2)$ and assuming that the ball holds always contact with the plate and does not slip when moving, the following model is obtained:

$$\ddot{z}_1 = \frac{5}{7}(z_1\dot{\theta}_1^2 + \dot{\theta}_1 z_2 \dot{\theta}_2 + g\sin\theta_1) + w_{e1} \quad (14)$$

$$\ddot{z}_2 = \frac{5}{7}(z_2\dot{\theta}_2^2 + z_1\dot{\theta}_1 \dot{\theta}_2 + g\sin\theta_2) + w_{e2} \quad (15)$$

The inputs of the ball and plate system are the accelerations applied in each rotation axis and they are denoted as $\mathcal{U} = [u_1, u_2]^t = [\ddot{\theta}_1, \ddot{\theta}_2]^t$. In this model, we have included an unknown perturbation in the acceleration of the ball denoted as $w_e(t) = [w_{e1}, w_{e2}]$. This perturbation is supposed to satisfy $\|w_e\|_\infty \leq 0.2$ m/s^2. This class of uncertainty can describe unknown external forces acting on the ball. The state $x \in \mathbb{R}^8$ is defined as follows

$$x^T = [z_1, \dot{z}_1, \theta_1, \dot{\theta}_1, z_2, \dot{z}_2, \theta_2, \dot{\theta}_2]^T$$

We consider the following constraints in the position, angles and inputs:

$$|z_i| \leq 0.06 \quad \text{m}, i = 1, 2$$

$$|\theta_i| \leq \frac{\pi}{2} \quad \text{rad}, i = 1, 2$$

$$|u_i| \leq 110 \quad \text{rad/s}^2, i = 1, 2$$

To apply the proposed MPC control scheme, a discrete time linear system with a sampling time of 0.05 seconds is obtained from the linearization of the model at the origin using the Euler approximation. The details of this model can be found in [17] and in [18].

The dynamics of variables z_1 and z_2 are decoupled. This model is used both to design the controller and to carry out the simulations. The weighting matrices of the controller are $R = 10 \cdot I^2$, $Q = 100 \cdot I^8$, $S = 7000 \cdot I^2$ where I^n is the identity matrix of dimension n. The simulations were done in Matlab 2013a using the solver `quadprog`.

In the proposed scenario the prediction horizon is $N = 28$. The number of decision variables of the optimization problem posed in sequential formulation are $n_u \cdot (N + T) + n_x = 120$. In addition, in order to prove that recursive feasibility is not lost even in the presence of a sudden change in the target reference, in this scenario the reference switches between two geometric figures. First the ball must draw a rectangle of size 6×4 cm and is centered in $\{4, 5\}$cm with a speed of $11.43\frac{\text{cm}}{\text{s}}$. At time 3.5 seconds the reference changes in order to draw a circumference with center on $\{-4, -4\}$cm and a radius of 1cm. The target speed of the second trajectory is $2.3\frac{\text{cm}}{\text{s}}$. The period length of both references is the same, that is $T = 28$. The initial state of this scenario is the ball in equilibrium at $\{z_1, z_2\} = \{-5, 5\}$cm.

The proposed robust controller has been designed using a terminal equality constraint. This avoids the calculation of the robust invariant set Ω, which in this case is a cumbersome procedure. In order to guarantee recursive feasibility and convergence, the local controller must guarantee that any possible perturbation is rejected in N time steps. The local control feedback K used to obtain the reduced set of constraints that guarantee both robust constraint satisfaction and recursive feasibility of the closed-loop controller is the following

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_1 \end{pmatrix}$$

where $K_1 = 10^3[-7.2087, -1.3139, -0.7509, -0.0349]$. This local control gain satisfies that $\max_{w \in \mathcal{W}} \|(A + BK)^{N-1} B_d w\| \leq 2.8475 \cdot 10^{-11}$. This value is lower than the tolerance of the optimization solver used.

The simulations have been executed for the uncertainty trajectory $w_{e1}(k) = w_{e2}(k)$ shown in Fig. 3. This uncertainty realization is divided into two stages. During the first stage (the first 3.5 seconds while the ball follows the rectangle), the uncertainty switches between the extreme values every 0.5 seconds. This extreme uncertainty realization aims to demonstrate that the controller is robust to any possible uncertainty included in the set $\mathcal{W}$. During the second stage (the last 3.5 seconds while the ball follows the circle), the uncertainty vanishes exponentially, demonstrating that in this case, the closed-loop system will converge to the optimal reachable trajectory with zero error.

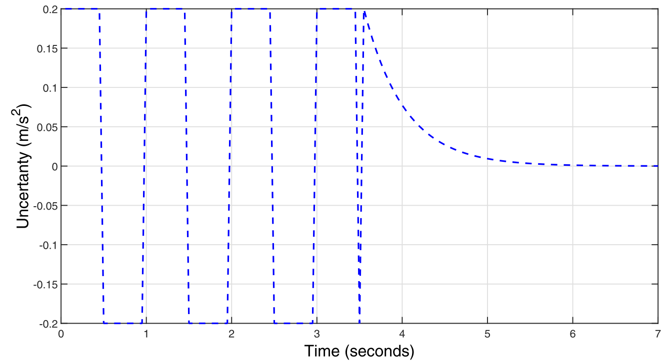

Fig. 3. Uncertainty trajectory.

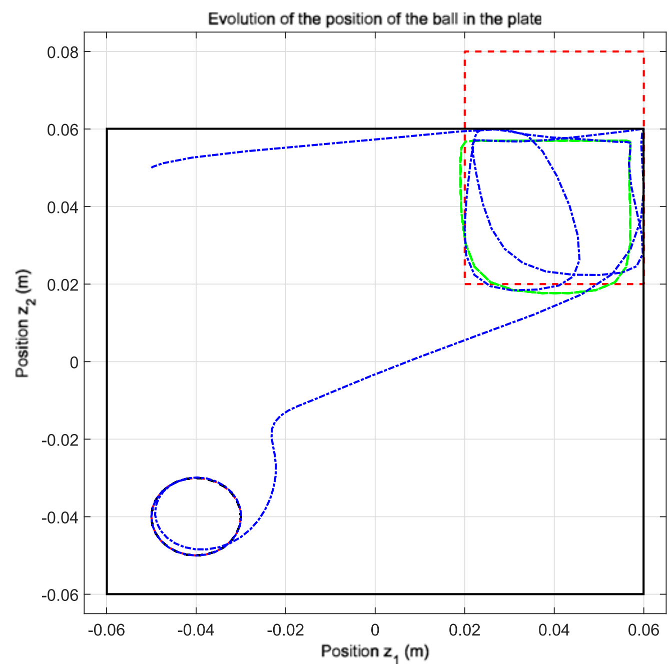

Fig. 4. Trajectories of z_1, z_2 for the closed loop system (dash-dot blue), the trajectory planners (discontinuous green and black), the target reference (discontinuous red) and the limit of the plate (dot black).

The trajectory of the ball converges to a neighborhood new optimal reachable trajectory satisfying the constraints and without losing feasibility even when the prediction horizon is much lower than the period. Fig. 4 shows the trajectories of (z_1, z_2) for the closed loop system (dash-dot blue), the trajectory planner (discontinuous green and black) and the target reference (discontinuous red) in the z_1, z_2 plane. Figs. 5 and 6 show the trajectory of the ball on each axis. It can be seen that there exists a deviation between the trajectory of the planner and the target reference for the rectangle, because one of the sides of the

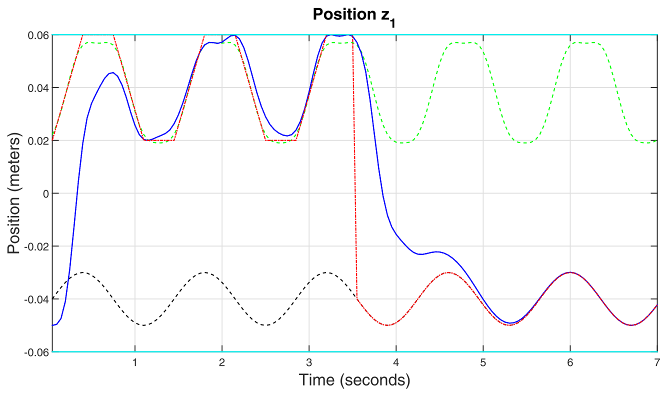

Fig. 5. Trajectories of z_1 for the closed loop system (continuous blue), the trajectory planners (discontinuous green and black) and the target reference (discontinuous red).

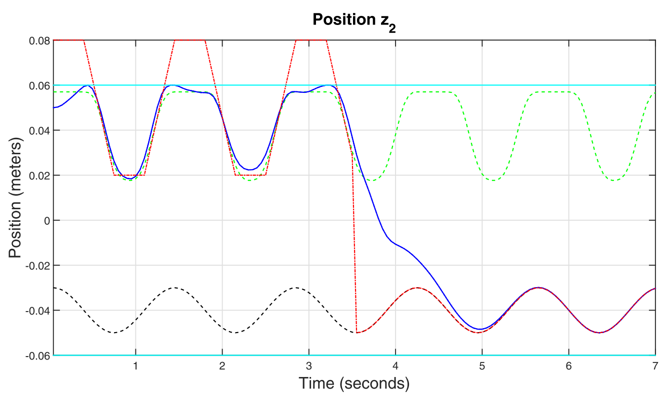

Fig. 6. Trajectories of z_2 for the closed loop system (continuous blue), the trajectory planners (discontinuous green and black) and the target reference (discontinuous red).

rectangles lies outside the position limits. The robust planner trajectory tries to get as close as possible to the target reference while satisfying the tightened constraints.

The constraints are robustly satisfied for all times including during in stage 1 in which the uncertainty takes extreme values and the target trajectory is close to the physical limits. In stage 2, it can be seen that the trajectory of the closed-loop system converges to the optimal reachable tightened reference trajectory with zero error as the perturbation vanishes with time and that the planner reachable trajectory converges to the target when the target is a robust reachable reference.

Fig. 7 shows the trajectories of the optimal cost V_N^* (discontinuous blue) and trajectory planner cost V_p^o (discontinuous green). The simulations include a sudden change in the reference when it switches from the rectangle to the circle which as a clear effect on the optimal costs, in particular the difference between both values increases suddenly when the reference changes, but then it converges again to the new optimal trajectory planner cost as the uncertainty vanishes. The cost evolution is non-strictly decreasing during stage 1 because of the effect of the uncertainties of the closed-loop system. The cost of the proposed controller converges to the cost of the trajectory planners in a non-increasing manner in stage 2, demonstrating that the optimal trajectory is input-to-state stable, so as the uncertainties vanish, the closed-loop system converges asymptotically with zero error.

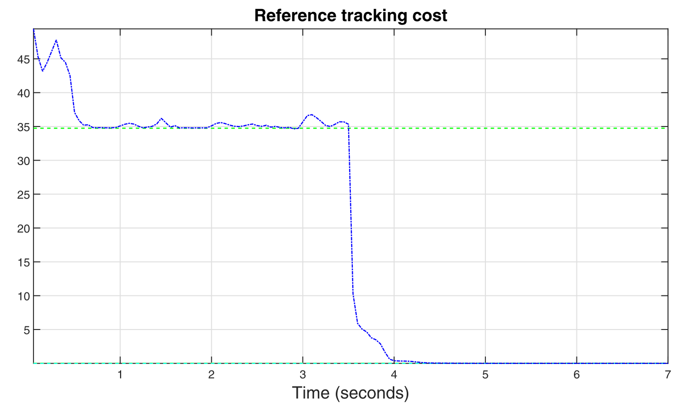

Fig. 7. Trajectories of the optimal cost V_N^* (discontinuous blue) and trajectory planner cost V_p^o (discontinuous green).

VI. Conclusions

In this paper, we propose a novel robust MPC formulation based on a constraint tightening method. This controller joins a dynamic and robust trajectory planning and a robust MPC for tracking in a single layer taking into account periodic references. It is proved that under certain assumptions the perturbed closed loop system is input-to-state stable, converges asymptotically to the optimal reachable periodic trajectory, robustly satisfies all the constraints and maintains feasibility even in the presence of a sudden change in the target reference. In addition, it is not necessary the computation of the minimal robust positive invariant set.

References

[1] J. Lee, S. Natarajan, and K. Lee, "A model-based predictive control approach to repetitive continuous processes with periodic operations," *Journal of Process Control*, vol. 11, pp. 195–207, 2001.

[2] R. Gondhalekar and C. Jones, "MPC of constrained discrete-time linear periodic systems: A framework for asynchronous control: Strong feasibility, stability and optimality via periodic invariance," *Automatica*, vol. 2, pp. 326–333, 2011.

[3] R. Huang, E. Harinath, and L. T. Biegler, "Lyapunov stability of economically oriented NMPC for cyclic processes," *Journal of Process Control*, vol. 21, pp. 501–509, 2011.

[4] D. Angeli, R. Amrit, and J. Rawlings, "On average performance and stability of economic model predictive control," *IEEE Transactions on Automatic Control*, vol. 57, no. 7, pp. 1615–1626, Jul. 2012.

[5] U. Mäder and M. Morari, "Offset-free reference tracking with model predictive control," *Automatica*, vol. 46, no. 9, pp. 1469–1476, 2010.

[6] D. Limon and T. Alamo, "Tracking model predictive control," in *Encyclopedia of Systems and Control*, J. Baillieul and T. Samad, Eds. Springer, 2014.

[7] D. Q. Mayne, "Model predictive control: Recent developments and future promise," *Automatica*, vol. 50, no. 12, pp. 2967–2986, 2014.

[8] D. Angeli, A. Casavola, and F. Tedesco, "Theoretical advances on economic model predictive control with time-varying costs," *IFAC-PapersOnLine*, vol. 48, no. 23, pp. 272–277, 2015.

[9] J. Baillieul and T. Samad, *Encyclopedia of Systems and Control*. Springer Publishing Company, Incorporated, 2015.

[10] D. Limon, M. Pereira, D. Muñoz de la Peña, T. Alamo, C. Jones, and M. Zeilinger, "MPC for tracking periodic references," *IEEE Transactions on Automatic Control*, vol. PP, 2015.

[11] D. Limon, M. Pereira, D. Muñoz de la Peña, T. Alamo, and J. Grosso, "Single-layer economic model predictive control for periodic operation," *Journal of Process Control*, vol. 24, no. 8, pp. 1207–1224, 2014.

[12] M. Pereira, D. Limon, D. Muñoz de la Peña, L. Valverde, and T. Alamo, "Periodic economic control of a nonisolated microgrid," *IEEE Transactions on Industrial Electronics*, vol. 62, no. 8, pp. 5247–5255, Aug. 2015.

[13] T. J. Broomhead, C. Manzie, R. C. Shekhar, and P. Hield, "Robust periodic economic mpc for linear systems," *Automatica*, vol. 60, pp. 30–37, 2015.

[14] I. Alvarado, D. Limon, D. Muñoz de la Peña, T. Alamo, and E. Camacho, “Enhanced ISS nominal MPC based on constraint tightening for constrained linear systems,” in *UKACC International Conference on Control 2010*, Sept. 92010, pp. 1–6.

[15] D. Q. Mayne, M. M. Seron, and S. V. Rakovic, “Robust model predictive control of constrained linear systems with bounded disturbances,” *Automatica*, vol. 41, pp. 219–224, 2005.

[16] D. Limon, T. Alamo, D. Raimondo, D. Muñoz de la Peña, J. Bravo, A. Ferramosca, and E. Camacho, “Input-to-state stability: A unifying framework for robust model predictive control,” in *International Workshop on Assessment and Future Direction of Nonlinear Model Predictive Control*, L. Magni, D. M. Raimondo, and F. Allgöwer, Eds. Springer, 2009, pp. 1–26.

[17] M. A. Moreno-Armendáriz, C. A. Pérez-Olvera, F. O. Rodríguez, and E. Rubio, “Indirect hierarchical FCMAC control for the ball and plate system,” *Neurocomputing*, vol. 73, pp. 2454–2463, 2010.

[18] Y. Wang, M. Sun, Z. Wang, Z. Liu, and Z. Chen, “A novel disturbance-observer based friction compensation scheme for ball and plate system,” *ISA Transactions*, vol. 53, pp. 671–678, 2014.