Constrained min-max predictive control: Modifications of the objective function leading to polynomial complexity

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and using (26), we obtain

$$2j\Delta_1(j)s'_+(0) - k_0 j\Delta_1(j) - jg_0^2\Delta_2(j) = 0$$

so

$$s'_+(0) = \frac{1}{2}\left(k_0 + g_0^2\frac{\Delta_2(j)}{\Delta_1(j)}\right).$$

Thus, again from Lemma 3, we see that $\sigma'(0) = \Re[s'_+(0)]$ is positive and the transversality condition for the occurrence of a Hopf bifurcation is satisfied.

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Constrained Min–Max Predictive Control: Modifications of the Objective Function Leading to Polynomial Complexity

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***Abstract*—In this note, an efficient way of implementing a constrained min–max predictive controller is presented. The new approach modifies the objective function in such a way that the resulting min–max problem can be solved in polynomial time. Different modifications are proposed. The main contribution of the note is to provide a robust constrained min–max predictive controller that can be implemented in real time. The new controller stabilizes the uncertain system.**

***Index Terms*—Min–max, optimization algorithms, predictive control of linear systems, robust control.**

I. Introduction

Min–max robust receding horizon control was first proposed by Witsenhausen [14]. In the context of robust model predictive control (MPC), the problem was tackled by Campo and Morari [5]. In general, solving a min–max problem subject to constraints and bounded additive disturbances is computationally too demanding for practical implementation. Some approaches to overcome this complexity can be found in literature. Lee and Kouvaritakis proposed a linear programming approach in [9]. In [10], the worst case value of the objective function is bounded by means of a linear matrix inequality (LMI). Langson *et al.* presented a feedback model predictive control that maintains the trajectories in a tube [8]. Min–max MPC can also be addressed by the use of multiparametric programming [3], [6], [13].

In this note, an efficient way of implementing a constrained quadratic min–max predictive controller is presented. The new approach relies on a slight modification of the objective function. This modification allows us to solve the min–max problem in polynomial time. The proposed controller inherits the stability and robustness of the standard min–max controller.

The note is organized as follows. In Section II, the min–max problem is stated. It is shown in Section III that there are instances in which solving the maximization problem can be made in polynomial time. Different modifications of the objective function are proposed in Section IV. The stability of the proposed controller is addressed in Section V. An illustrative example is presented in Section VI. The note draws to a close with a section of conclusions.

II. Min–Max MPC With Global Uncertainties

Consider the discrete-time linear system with bounded uncertainties

$$x_{k+1} = Ax_k + Bu_k + Dw_k \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input, and $w_k \in W$ is the uncertainty that is assumed to be bounded by the hypercube $W = \{w \in \mathbb{R}^{n_w} : \|w\|_\infty \leqslant \epsilon\}$. The set of possible disturbance sequences of length N will be denoted W_N

$$W_N = \{\{w_0, w_1, \ldots, w_{N-1}\} : w_i \in W, \, i = 0, \ldots, N-1\}.$$

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It will be assumed that the control input is given by $u_k = Kx_k + v_k$, where K is chosen in order to achieve some desired property for the unconstrained problem. In this way, some amount of feedback is provided to the predictions [7], [9]. The linearity of the system and the prestabilization policy guarantee that the disturbance response of the controlled system is bounded. Note that this prestabilization policy $u_k = Kx_k + v_k$ is sub-optimal (when compared with theoretical closed-loop min–max approaches).

The MPC controller will compute the sequence of correction control signals along the control horizon $\{v_0, \ldots, v_{N_c-1}\}$. Defining $A_K = (A + BK)$, the dynamics of the system can be rewritten as: $x_{k+1} = A_K x_k + Bv_k + Dw_k$.

Given an initial condition x, a sequence of N_c correction control inputs $\mathbf{v} = \{v_0, v_1, \ldots, v_{N_c-1}\}$, and a sequence of disturbances $\mathbf{w} = \{w_0, w_1, \ldots, w_{N-1}\} \in W_N$, the predicted state $x_j(x, \mathbf{v}, \mathbf{w})$ is given by

$$\begin{cases} x, & \text{if } j = 0 \\ A_k x_{j-1}(x, \mathbf{v}, \mathbf{w}) + Bv_{j-1} + Dw_{j-1}, & \text{if } 1 \leqslant j \leqslant N_c \\ A_k x_{j-1}(x, \mathbf{v}, \mathbf{w}) + Dw_{j-1}, & \text{if } N_c < j \leqslant N \end{cases} \tag{2}$$

and the predicted control input $u_j(x, \mathbf{v}, \mathbf{w})$ by

$$\begin{cases} Kx_j(x, \mathbf{v}, \mathbf{w}) + v_j, & \text{if } 0 \leqslant j \leqslant N_c - 1 \\ Kx_j(x, \mathbf{v}, \mathbf{w}), & \text{if } N_c \leqslant j \leqslant N - 1 \end{cases}. \tag{3}$$

The objective function $V_N(x, \mathbf{v}, \mathbf{w})$ is

$$\sum_{j=0}^{N-1} \left(\|x_j(x, \mathbf{v}, \mathbf{w})\|_Q^2 + \|u_j(x, \mathbf{v}, \mathbf{w})\|_R^2 \right) + \|x_N(x, \mathbf{v}, \mathbf{w})\|_P^2$$

where $\|x\|_A = \sqrt{x^T Ax}$. Matrices Q and P are positive definite and R is positive semidefinite. Linear constraints in state and input, $x_k \in X$ and $u_k \in U$ are considered. In order to achieve stability, a polytopic terminal region constraint $(x_N(x, \mathbf{v}, \mathbf{w}) \in \Omega)$ is also considered [11]. In this way the min–max constrained predictive controller results in the solution of the following min–max optimization problem [denoted $P_N(x)$]:

$$J_N^*(x) = \min_{\mathbf{v}} \max_{\mathbf{w} \in W_N} V_N(x, \mathbf{v}, \mathbf{w})$$
$$s.t. \begin{cases} x_j(x, \mathbf{v}, \mathbf{w}) \in X, & \forall \mathbf{w} \in W_N, & j = 0, \ldots, N-1 \\ x_N(x, \mathbf{v}, \mathbf{w}) \in \Omega & \forall \mathbf{w} \in W_N & \\ u_j(x, \mathbf{v}, \mathbf{w}) \in U & \forall \mathbf{w} \in W_N, & j = 0, \ldots, N-1. \end{cases}$$

This optimization problem is solved at each sample instant. An optimal vector of control correction signals $\mathbf{v}^*$ is obtained and the control input $u_0 = Kx + v_0^* = K_{\text{MPC}}(x)$ is applied.

Terminal region $\Omega \subseteq X$ is assumed to satisfy the following conditions:

- **C1)** If $x \in \Omega$, then $A_K x + Dw \in \Omega$, for every $w \in W$.
- **C2)** If $x \in \Omega$, then $Kx \in U$.

Matrix P, that characterizes the terminal cost $\|x_N(x, \mathbf{v}, \mathbf{w})\|_P^2$, is assumed to satisfy

- **C3)** $P - A_K^T P A_K > Q + K^T RK$.

The stability of $A_K = A + BK$ guarantees the existence of a positive–definite matrix P satisfying **C3)**.

Observe that the predictions $x_j(x, \mathbf{v}, \mathbf{w})$ and $u_j(x, \mathbf{v}, \mathbf{w})$ depend linearly on x, $\mathbf{v}$ and $\mathbf{w}$. This means that it is possible to find a vector $d \in \mathbb{R}^p$ and matrices G_x, G_v and G_w such that the linear constraints of problem $P_N(x)$ can be rewritten as

$$G_x^i x + G_v^i \mathbf{v} + G_w^i \mathbf{w} \leqslant d_i, \qquad i = 1 \ldots, p \qquad \forall \mathbf{w} \in W_N$$

where G_x^i, G_v^i, G_w^i denote the ith rows of G_x, G_v and G_w, respectively, and d_i is the i-esime component of $d \in \mathbb{R}^p$. Denote now $\|G_w^i\|_1$ the sum of the absolute values of row G_w^i. Taking into account that $\max_{\mathbf{w} \in W_N} G_w^i \mathbf{w} = \max_{\|\mathbf{w}\|_\infty \leqslant \epsilon} G_w^i \mathbf{w} = \epsilon \|G_w^i\|_1$, the robust fulfillment of the constraints is satisfied if and only if: $G_x^i x + G_v^i \mathbf{v} + \epsilon \|G_w^i\|_1 \leqslant d_i$, $i = 1 \ldots, p$.

III. Tractable Cases of the Max Function

The objective function $V_N(x, \mathbf{v}, \mathbf{w})$ is a quadratic function of x, $\mathbf{v}$ and $\mathbf{w}$. That is, matrices H_x, H_v and H_w can be found in such a way that

$$\begin{aligned} V_N(x, \mathbf{v}, \mathbf{w}) &= \|H_x x + H_v \mathbf{v} + H_w \mathbf{w}\|_2^2 \\ &= \mathbf{w}^T M \mathbf{w} + q(x, \mathbf{v})^T \mathbf{w} + V_N(x, \mathbf{v}, 0) \end{aligned}$$

where $M = H_w^T H_w$ and $q(x, \mathbf{v}) = 2H_w^T (H_x x + H_v \mathbf{v})$. Therefore, the computation of the max function

$$V_N^*(x, \mathbf{v}) = \max_{\mathbf{w} \in W_N} V_N(x, \mathbf{v}, \mathbf{w})$$

belongs to the following class of maximization problems:

$$\mu^* = \max_{\|\mathbf{w}\|_\infty \leqslant \epsilon} \mathbf{w}^T \tilde{M} \mathbf{w} + q^T \mathbf{w}.$$

In principle, computing μ^* is an NP-hard problem. However, there are some instances in which μ^* can be obtained in polynomial time. The complexity of the computation can be dramatically reduced if matrix $\tilde{M} \in \mathbb{R}^{n \times n}$, where $n = n_w N$, belongs to one of the following categories.

1) **$\tilde{M}$ is a positive semidefinite band matrix:** Matrix $\tilde{M}$ is an L-band matrix if $|i - j| \geqslant L$, implies $\tilde{M}_{ij} = 0$. It has been recently shown (see [1]) that under the assumption of band structure, the maximization problem can be solved in polynomial time $\text{O}(n^2 2^L)$.
2) **All the elements of $\tilde{M}$ are non-negative:** In this case, it is well known (see [2] and [12]) that the maximization problem can be posed as a min cut graph problem. This graph problem can be solved in polynomial time $\text{O}(n^3)$.
3) **$\tilde{M}$ is a negative–semidefinite matrix:** In this case, $\mu^* = -\min_{\|\mathbf{w}\|_\infty \leqslant \epsilon} (\mathbf{w}^T (-\tilde{M}) \mathbf{w} - q^T \mathbf{w})$. It results that $-\tilde{M}$ is positive semidefinite. Therefore, μ^* can be computed solving a quadratic convex problem. This can be accomplished in polynomial time.
4) **$\tilde{M}$ is a positive–semidefinite diagonal matrix:** In this case $\mu^* = \epsilon^2 \text{tr } \tilde{M} + \epsilon \|q\|_1$, where $\text{tr } \tilde{M}$ denotes the trace of matrix $\tilde{M}$. The maximum is attained at $\mathbf{w}^* = \epsilon \cdot \text{sign}(q)$.

Note that for the considered tractable cases, not only the value of the max function is obtained in polynomial time, but also the vertex at which this maximum is attained. This means that a subgradient (with respect to $\mathbf{v}$) can also be obtained in polynomial time.

IV. Modification of the Objective Function

In this section, a modification of the objective function that allows us to solve the min–max problem in polynomial time while preserving the stability and robustness properties of the standard approach is presented. The new objective function is an upper bound of the original one and it will be denoted as $\tilde{V}_N(x, \mathbf{v}, \mathbf{w})$. The proposed objective function differs only in a quadratic term on $\mathbf{w}$ and a constant

$$\begin{aligned} \tilde{V}_N(x, \mathbf{v}, \mathbf{w}) &= V_N(x, \mathbf{v}, \mathbf{w}) + \mathbf{w}^T F \mathbf{w} + c\epsilon^2 \\ &= \mathbf{w}^T (M + F) \mathbf{w} + q(x, \mathbf{v})^T \mathbf{w} + V_N(x, \mathbf{v}, 0) + c\epsilon^2. \end{aligned}$$

Note that with an appropriate choice of matrix F, it is always possible to make $\tilde{M} = M + F$ belong to any of the tractable cases of the max function. The modified max function will be denoted as

$$\tilde{V}_N^*(x, \mathbf{v}) = \max_{\mathbf{w} \in W_N} \tilde{V}_N(x, \mathbf{v}, \mathbf{w}).$$

It is clear that $\tilde{V}_N(x, \mathbf{v}, \mathbf{w}) = V_N(x, \mathbf{v}, \mathbf{w}) + \mathbf{w}^T F \mathbf{w} + c\epsilon^2 = \|H_x x + H_v \mathbf{v} + H_w \mathbf{w}\|_2^2 + \mathbf{w}^T F \mathbf{w} + c\epsilon^2$ is a convex function in x and $\mathbf{v}$. Since the pointwise supremum of an arbitrary infinite set of convex functions is convex [4], it follows that $\tilde{V}_N^*(x, \mathbf{v}) = \max_{\|\mathbf{w}\|_\infty \leqslant \epsilon} \tilde{V}_N(x, \mathbf{v}, \mathbf{w})$ is also convex in x and $\mathbf{v}$. Choosing conveniently F and c, the following assumption will be satisfied:

- **C4)** $V_N^*(x, \mathbf{v}) \leqslant \tilde{V}_N^*(x, \mathbf{v}) \leqslant V_N^*(x, \mathbf{v}) + \sigma\epsilon^2$.

It will be shown how to compute F and c in such a way that **C4** is satisfied and σ minimized. For this purpose, it is important to introduce the following auxiliary lemmas.

Lemma 1: Let us suppose that $T \geqslant 0$ is a diagonal matrix and that $0 \leqslant F \leqslant T$ and $c = 0$. Then, making $\sigma = \operatorname{tr} T$, **C4)** is satisfied.

Proof: From $F \geqslant 0$ it is inferred that $\mathbf{w}^T F \mathbf{w} \geqslant 0$ and, therefore, $V_N^*(x, \mathbf{v}) \leqslant \tilde{V}_N^*(x, \mathbf{v})$. From $F \leqslant T$ and the diagonal nature of T, it is inferred that $\mathbf{w}^T F \mathbf{w} \leqslant \mathbf{w}^T T \mathbf{w} \leqslant (\operatorname{tr} T)\epsilon^2$. Thus, $\tilde{V}_N^*(x, \mathbf{v}) \leqslant V_N^*(x, \mathbf{v}) + (\operatorname{tr} T)\epsilon^2$. ∎

Lemma 2: Let us suppose that $T \geqslant 0$ is a diagonal matrix; $0 \geqslant F \geqslant -T$ and $c = \operatorname{tr} T$. Then, making $\sigma = \operatorname{tr} T$, **C4)** is satisfied.

Proof: From the assumptions of the lemma, $\mathbf{w}^T F \mathbf{w} \geqslant -\mathbf{w}^T T \mathbf{w} \geqslant -(\operatorname{tr} T)\epsilon^2$. From this inequality and the fact that $c = \operatorname{tr} T$ it is inferred that $\mathbf{w}^T F \mathbf{w} + c\epsilon^2 \geqslant 0$. Thus $V_N^*(x, \mathbf{v}) \leqslant \tilde{V}_N^*(x, \mathbf{v})$. On the other hand, $0 \geqslant F$ implies that $\mathbf{w}^T F \mathbf{w} \leq 0$ and therefore $\tilde{V}_N^*(x, \mathbf{v}) \leqslant V_N^*(x, \mathbf{v}) + (\operatorname{tr} T)\epsilon^2$. ∎

Using previous lemmas, F and c will be chosen in such a way that $\tilde{M} = M + F$ belongs to one of the categories for which the maximization problem can be solved in polynomial time.

A. $\tilde{M} = M + F$ *is a Positive–Semidefinite Band Matrix*

Obtain matrix F and the diagonal matrix T (of minimum trace) such that $0 \leqslant F \leqslant T$, $M_{ij} + F_{ij} = 0, \forall |i - j| \geqslant L$. Then making $c = 0$ and $\sigma = \operatorname{tr} T$, $\tilde{M} = M + F$ is a positive–definite L-band matrix. Moreover, Lemma 1 guarantees that **C4)** holds.

The control input to the system $u_k = Kx_k + v_k$ is chosen in such a way that $A_K = A + BK$ is stable. This implies that the elements M_{ij} of matrix M vanish with the absolute value of $|i - j|$. Thus, the original matrix M can be approximated by an L-band and, therefore, the value of σ, that measures the difference between $V_N^*(x, \mathbf{v})$ and $\tilde{V}_N^*(x, \mathbf{v})$, decreases in an exponential way with the width of the band matrix. Moreover, if K is chosen in such a way that all the eigenvalues of A_K are at the origin (dead-beat control), then $A_K^{n_x} = 0$. Under this control, it is not difficult to show that matrix M has a band structure and no approximation is required.

B. $\tilde{M} = M + F$ *has Non-Negative Elements*

Note that there are systems for which M has non negative elements (see [2]). In this case, no approximation is required and $F = 0$, $c = \sigma = 0$. If M does have negative elements, obtain matrix F and the diagonal matrix T (of minimum trace) such that $0 \leqslant F \leqslant T$, $M_{ij} + F_{ij} \geqslant 0, \forall i, \forall j$. Then making $c = 0$ and $\sigma = \operatorname{tr} T$ it results that $\tilde{M} = M + F$ has nonnegative elements. Moreover, Lemma 1 guarantees that **C4)** holds again.

C. $\tilde{M} = M + F$ *is a Negative–Semidefinite Matrix*

Obtain matrix F and the diagonal matrix T (of minimum trace) such that $0 \geqslant F \geqslant -T$ and $M + F \leq 0$. Then, making $c = \sigma = \operatorname{tr} T$, $\tilde{M} = M + F$ is a negative–semidefinite matrix. It follows that Lemma 2 guarantees that **C4)** holds.

This modification of the objective function is closely related to H_∞ control when F is chosen to be a diagonal matrix. Let us suppose that $F = -\rho \mathrm{I}$ and that $M - \rho \mathrm{I} < 0$, then $\tilde{V}_N(x, \mathbf{v}, \mathbf{w}) - c\epsilon^2$ is equal to

$$\|x_N(x, \mathbf{v}, \mathbf{w})\|_P^2 + \sum_{j=0}^{N-1} \Big(\|x_j(x, \mathbf{v}, \mathbf{w})\|_Q^2 + \|u_j(x, \mathbf{v}, \mathbf{w})\|_R^2 - \rho w_j^T w_j \Big).$$

Thus, $\tilde{V}_N(x, \mathbf{v}, \mathbf{w}) - c\epsilon^2$ is a finite quadratic cost, strictly concave in $\mathbf{w}$, with the same structure as the objective functions encountered in the literature of H_∞ control (see [11, Sec. 4.7]).

D. $\tilde{M} = M + F$ *is Positive Semidefinite and Diagonal*

Obtain diagonal matrix S and the diagonal matrix T (of minimum trace) such that $0 \leqslant S - M \leqslant T$. Then, making $F = S - M$, $c = 0$ and $\sigma = \operatorname{tr} T$, $\tilde{M} = M + F = S$ is a positive–semidefinite diagonal matrix and Lemma 1 guarantees that **C4)** is satisfied.

V. Proposed Min–Max Predicted Control: Stability

The new min–max problem (that will be denoted $\tilde{P}_N(x)$) is stated as

$$\tilde{J}_N^*(x) = \min_{\mathbf{v}} \tilde{V}_N^*(x, \mathbf{v})$$
$$s.t. \begin{cases} x_j(x, \mathbf{v}, \mathbf{w}) \in X, & \forall \mathbf{w} \in W_N, \quad j = 0, \ldots, N-1 \\ x_N(x, \mathbf{v}, \mathbf{w}) \in \Omega, & \forall \mathbf{w} \in W_N \\ u_j(x, \mathbf{v}, \mathbf{w}) \in U, & \forall \mathbf{w} \in W_N, \quad j = 0, \ldots, N-1 \end{cases}$$

where $\tilde{V}_N^*(x, \mathbf{v})$ denotes one of the approximations of the max function proposed in the last section. Note that the feasibility region of $\tilde{P}_N(x)$ equals the feasibility region of $P_N(x)$.

It is clear that the optimal solution $\tilde{\mathbf{v}}^*$ of problem $\tilde{P}_N(x)$ is a suboptimal feasible solution for problem $P_N(x)$. As it is claimed in the following property, the difference between the optimal value of the original objective function and the value obtained with $\tilde{\mathbf{v}}^*$ is bounded by $\sigma\epsilon^2$.

Property 1: Denote $\tilde{\mathbf{v}}^*$ the optimal solution to problem $\tilde{P}_N(x)$. If assumption ***C*₄**
is satisfied, then

- $V_N^*(x, \tilde{v}^*) \geqslant J_N^*(x) \geqslant V_N^*(x, \tilde{v}^*) - \sigma\epsilon^2$;
- $\tilde{J}_N^*(x) \geqslant J_N^*(x) \geqslant \tilde{J}_N^*(x) - \sigma\epsilon^2$.

Proof: First claim: The first inequality stems directly from the suboptimality of $\tilde{\mathbf{v}}^*$. Denote now $\mathbf{v}^*$ the optimal solution to problem $P_N(x)$, the second inequality is obtained from **C4)**

$$J_N^*(x) = V_N^*(x, \mathbf{v}^*) \geqslant \tilde{V}_N^*(x, \mathbf{v}^*) - \sigma\epsilon^2 \geqslant \tilde{V}_N^*(x, \tilde{\mathbf{v}}^*) - \sigma\epsilon^2 \geqslant V_N^*(x, \tilde{\mathbf{v}}^*) - \sigma\epsilon^2.$$

Second claim: Again, from **C4)**

$$\tilde{J}_N^*(x) = \tilde{V}_N^*(x, \tilde{\mathbf{v}}^*) \geqslant V_N^*(x, \tilde{\mathbf{v}}^*) \geqslant V_N^*(x, \mathbf{v}^*) = J_N^*(x).$$
$$J_N^*(x) = V_N^*(x, \mathbf{v}^*) \geqslant \tilde{V}_N^*(x, \mathbf{v}^*) - \sigma\epsilon^2 \geqslant \tilde{V}_N^*(x, \tilde{\mathbf{v}}^*) - \sigma\epsilon^2 = \tilde{J}_N^*(x) - \sigma\epsilon^2.$$

∎

From a computational point of view, the main properties of the proposed objective function $\tilde{V}_N^*(x, \mathbf{v})$ are that it is convex in $\mathbf{v}$ and that its evaluation for a given pair $(x, \mathbf{v})$ can be made in polynomial time. Moreover, given $(x, \mathbf{v})$, a subgradient (with respect to $\mathbf{v}$) can also be obtained in polynomial time. This class of optimization problem can be solved in polynomial time using standard convex algorithms [4]. Note also that all the LMI problems proposed in the previous section

TABLE I
PERFORMANCE COMPARISON

	σ	J
Original min-max	-	108.47
Band matrix ($L = 5$)	0.82	108.49
Nonnegative matrix	0.53	108.72
Diagonal matrix	129.6	113.49
Semidefinite negative	151,55	112.06
Nominal MPC	-	119.81

are always feasible and solved offline. The following property, which is proved in Appendix I, plays an important role when analyzing the stability of the proposed controllers.

Property 2: Denote $\Gamma(x,w) = \|x\|_P^2 - \|A_K x + Dw\|_P^2 - x^T(Q + K^T RK)x + \gamma\epsilon^2$. If the system $x_{k+1} = A_K x_k$ is asymptotically stable then there exists a positive–definite matrix P and a positive scalar γ such that **C3)** is satisfied and $\Gamma(x,w) \geqslant 0, \forall x \in \mathbb{R}^{n_x}, \forall w \in W$.

As it is stated in the following theorem (proved in Appendix II), the new min–max controller guarantees that the uncertain system evolves to a bounded set that contains the origin.

Theorem 1: Let us suppose the following.

1) Assumptions **C1)–C4)** are satisfied.
2) $\|x\|_P^2 - \|A_K x + Dw\|_P^2 - x^T(Q + K^T RK)x + \gamma\epsilon^2 \geq 0$, $\forall x \in \mathbb{R}^{n_x}, \forall w \in W$.
3) $\tilde{\mathbf{v}}^* = \{\tilde{v}_0^*, \tilde{v}_1^*, \ldots, \tilde{v}_{N_c-1}^*\}$ is the optimal solution to problem $\tilde{P}_N(x_k)$.

Then, the min–max controller ($\tilde{K}_{MPC}(x_k) = Kx_k + \tilde{v}_0^*$) guarantees that the state is ultimately bounded and for every $w_k \in W$, $x_{k+1} = Ax_k + B\tilde{K}_{MPC}(x_k) + Dw_k$ is such that

1) $\mathbf{v}_s = \{\tilde{v}_1^*, \tilde{v}_2^*, \ldots, \tilde{v}_{N_c-1}^*, 0\}$ is a feasible solution to problems $P_N(x_{k+1})$ and $\tilde{P}_N(x_{k+1})$;
2) $J_N^*(x_k) - J_N^*(x_{k+1}) \geqslant x_k^T Q x_k - (\gamma + \sigma)\epsilon^2$.

It is important to observe that γ and σ do not depend on the size of the uncertainty.

VI. ILLUSTRATIVE EXAMPLE

Let us consider the linear uncertain system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_k$$

where both the state and the control action are constrained, namely $\|x_k\|_\infty \leqslant 5$ and $|u_k| \leqslant 5$. The uncertainty is bounded, $\|w_k\| \leqslant \epsilon$, $\epsilon = 1$. The objective function is defined by matrices $Q = I$ and $R = 1$. The control gain matrix $K = [-0.4221 \ \ -1.2439]$ corresponds to an LQR control law. The terminal region Ω is chosen as the maximal robust invariant set of the system for K. The terminal cost function is defined by $P = \begin{bmatrix} 4.0696 & 3.8641 \\ 3.8641 & 6.6199 \end{bmatrix}$.

As it is stated in Property 1, the difference between the optimum value $J_N^*(x_k)$ and the one obtained with the optimal solution of the modified problem is bounded by $\sigma\epsilon^2$. Table I shows the value of σ for the proposed controllers for a prediction horizon $\mathrm{N} = 15$.

The values of $J = \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k$ obtained when the different controllers are applied to the system are shown in Table I. For the computation of these values, a vanishing disturbance was applied to the system; the initial condition was $x_0 = [1 \ 2.5]^T$ and the prediction horizon N was equal to 15. As it can be observed in Table I, the value of J corresponding to the nominal MPC is equal to 119.81, 10.45% greater than the one corresponding to the original min–max problem. Given x_k, the nominal MPC is obtained minimizing, with respect $\mathbf{v}$, $V_N(x_k, \mathbf{v}, 0)$. The constraints of the nominal MPC problem are the same as the ones of the original min–max problem.

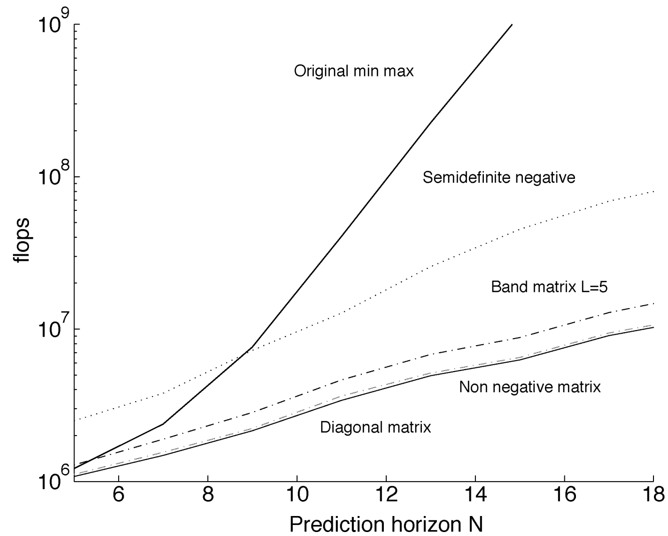

Fig. 1. Computational burden comparison.

Fig. 1 shows the computational burden comparison between the original min–max problem and the modified approaches. These results are given for the system and controller presented in this section, for different prediction horizons and over a hundred random samples. A logarithmic scale is used in the figure.

VII. CONCLUSION

A new formulation of the min–max predictive control is presented in this note. The new controller is based on a modification of the objective function that allows us to compute the max function in polynomial time. This makes possible the implementation of min–max predictive control in real-time applications. It has been shown that the proposed controller guarantees the robust satisfaction of the constraints and the convergence to a bounded set that contains the origin.

APPENDIX I
PROOF OF PROPERTY 2

As $x_{k+1} = A_K x_k$ is asymptotically stable, there exists a symmetric matrix $P > 0$ such that ***C*3** is satisfied. This implies that matrix $S = P - A_K^T P A_K - Q - K^T RK$ is definite positive. Note that $\Gamma(x,w)$ can be rewritten as $\Gamma(x,w) = x^T Sx - 2x^T A_K^T PDw - w^T D^T PDw + \gamma\epsilon^2$. Since S is a positive–definite matrix, $\Gamma(x,w)$ is a convex function on x. It can be easily shown that the minimum of $\Gamma(x,w)$ is attained at $x^* = S^{-1}A_K^T PDw$. Thus, $\Gamma(x,w) \geq -w^T D^T(PA_K S^{-1} A_K^T P + P)Dw + \gamma\epsilon^2$. Taking into account that the term on the right is a concave function on w, it is concluded that $\Gamma(x,w) \geq 0$ if

$$\gamma\epsilon^2 \geqslant \max_{w \in \mathrm{vert}\{W\}} w^T D^T \left(PA_k S^{-1} A_k^T P + P\right) Dw$$

where $\mathrm{vert}\{W\}$ denotes the vertices of W. Note that S is a positive–definite matrix. This implies that there is a finite value of γ that satisfies previous inequality. Dividing the last inequality by ϵ^2, the following equivalent inequality is obtained:

$$\gamma \geqslant \max_{\vartheta \in \mathrm{vert}\left\{B_1^{nw}\right\}} \vartheta^T D^T \left(PA_k S^{-1} A_k^T P + P\right) D\vartheta$$

where $\mathrm{vert}\{B_1^{n_w}\}$ denotes the vertices of the unit hypercube in $\mathbb{R}^{n_w}$. It can be shown by means of Schur's complement that this inequality is satisfied if and only if for every $\vartheta \in \mathrm{vert}\{B_1^{n_w}\}$

$$\begin{bmatrix} \gamma - \vartheta^T D^T P D \vartheta & \vartheta^T D^T P A_k \\ A_k^T P D \vartheta & P - A_k^T P A_k - Q - K^T R K \end{bmatrix} > 0. \quad (4)$$

APPENDIX II
PROOF OF THEOREM 1

First Claim: Given $\hat{\mathbf{w}} = \{\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_{N-1}\} \in W_N$ and $w_k \in W$, denote

$$d(w_k, \hat{\mathbf{w}}) = \{w_k, \hat{w}_0, \hat{w}_1, \ldots, \hat{w}_{N-2}\} \in W_N.$$

With this notation, and taking into account (2) and (3), it results that for all $\hat{\mathbf{w}} \in W_N$ and for all $w_k \in W$

$$x_j(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) = x_{j+1}(x_k, \tilde{\mathbf{v}}^*, d(w_k, \hat{\mathbf{w}})), 0 \leqslant j \leqslant N-1 \quad (5)$$
$$u_j(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) = u_{j+1}(x_k, \tilde{\mathbf{v}}^*, d(w_k, \hat{\mathbf{w}})), 0 \leqslant j \leqslant N-2. \quad (6)$$

As $\tilde{\boldsymbol{v}}^*$ is a feasible solution for problem $\tilde{P}_N(x_k)$ it results (from the previous equalities) that for all $\hat{\mathbf{w}} \in W_N$ and for all $w_k \in W$: $x_j(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) \in X, j = 0, \ldots, N-1$ and $u_j(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) \in U$, $j = 0, \ldots, N-2$.

Thus, it only remains to show that $u_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) \in U$ and $x_N(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) \in \Omega$, $\forall \hat{\mathbf{w}} \in W_N$. From the previous equalities and the feasibility of $\tilde{\mathbf{v}}^*$: $x_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) = x_N(x_k, \tilde{\mathbf{v}}^*, d(w_k, \hat{\mathbf{w}})) \in \Omega$, $\forall \hat{\mathbf{w}} \in W_N$. Taking into account the assumptions on Ω, it is inferred that $u_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) = K x_{N-1}(x_{k+1}, \tilde{\mathbf{v}}_s^*, \hat{\mathbf{w}}) \in U$, $\forall \hat{\mathbf{w}} \in W_N$ and $x_N(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) = A_K x_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) + D\hat{w}_{N-1} \in \Omega$, $\forall \hat{\mathbf{w}} \in W_N$.

Second Claim: From equalities (5) and (6), it is inferred that, for every $w_k \in W$ and for every $\hat{\mathbf{w}} = \{\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_{N-1}\} \in W_N$, $V_N(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}})$ is equal to

$$\begin{aligned}
&\sum_{j=1}^{N-1} \|x_j\,(x_k, \tilde{\mathbf{v}}^*, d(w_k, \hat{\mathbf{w}}))\|_Q^2 + \|u_j\,(x_k, \tilde{\mathbf{v}}^*, d(w_k, \hat{\mathbf{w}}))\|_R^2 \\
&\quad + \|x_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}})\|_Q^2 + \|u_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}})\|_R^2 \\
&\quad + \|x_N(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}})\|_P^2 \\
&\quad\; = V_N\,(x_k, \tilde{\mathbf{v}}^*, d(w_k, \hat{\mathbf{w}})) - x_k^T Q x_k - u_k^T R u_k \\
&\quad\quad - \|x_N\,(x_k, \tilde{\mathbf{v}}^*, d(w_k, \hat{\mathbf{w}}))\|_P^2 \\
&\quad\quad + \|x_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}})\|_Q^2 + \|u_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}})\|_R^2 \\
&\quad\quad + \|x_N(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}})\|_P^2.
\end{aligned}$$

Taking into account that $u_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) = K x_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}})$ and that $x_N(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) = A_K x_{N-1}(x_{k+1}, \mathbf{v_s}, \hat{\mathbf{w}}) + D\hat{w}_{N-1}$ it results that for every $w_k \in W$ and every $\hat{\mathbf{w}} \in W_N$

$$V_N(x_{k+1}, \mathbf{v}_s, \hat{\mathbf{w}}) - V_N\,(x_k, \tilde{\mathbf{v}}^*, d(w_k, \hat{\mathbf{w}}))$$

is equal to

$$\begin{aligned}
&\|x_{N-1}(x_{k+1}, \mathbf{v}_s, \hat{\mathbf{w}})\|_{Q+K^TRK}^2 \\
&\quad + \|A_K x_{N-1}(x_{k+1}, \mathbf{v}_s, \hat{\mathbf{w}}) + D\hat{w}_{N-1}\|_P^2 \\
&\quad - x_k^T Q x_k - u_k^T R u_k - \|x_{N-1}(x_{k+1}, \mathbf{v}_s, \hat{\mathbf{w}})\|_P^2.
\end{aligned}$$

Thus, from the second assumption of the theorem, $\forall w_k \in W$, $\forall \hat{\mathbf{w}} \in W_N$

$$V_N(x_{k+1}, \mathbf{v}_s, \hat{\mathbf{w}}) - V_N\,(x_k, \tilde{\mathbf{v}}^*, d(w_k, \hat{\mathbf{w}})) \leqslant -x_k^T Q x_k + \gamma\epsilon^2.$$

From this inequality, it is inferred that $V_N^*(x_{k+1}, \mathbf{v}_s) - V_N^*(x_k, \tilde{\mathbf{v}}^*) \leqslant -x_k^T Q x_k + \gamma\epsilon^2, \forall w_k \in W$. From the last inequality and the first claim of property 1), it results that for all $w_k \in W$

$$\begin{aligned}
J_N^*(x_{k+1}) &\leqslant V_N^*(x_{k+1}, \mathbf{v}_s) \leqslant V_N^*(x_k, \tilde{\mathbf{v}}^*) - x_k^T Q x_k + \gamma\epsilon^2 \\
&\leqslant J_N^*(x_k) - x_k^T Q x_k + (\gamma + \sigma)\epsilon^2.
\end{aligned}$$

Define $\Phi_\epsilon = \{x \in \mathbb{R}^n : P_N(x)$ is feasible and $x^T Q x \leqslant (\gamma + \sigma)\epsilon^2\}$. Then, the system evolves into set $\Omega_\alpha = \{x \in \mathbb{R}^n : J_N^*(x) \leqslant \alpha(\epsilon)\}$ where $\alpha(\epsilon) = \max_{x \in \Phi_\epsilon} J_N^*(x) + (\gamma + \sigma)\epsilon^2$.

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