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Estimation of the domain of attraction for saturated discrete-time systems

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The domain of attraction of a given non-linear system constitutes a zone of safe operation that can avoid unnecessary operational restrictions. In this paper, an alternative approach to the estimation of the domain of attraction of a saturated linear system is presented. Given a system with m saturated control inputs, we show how to choose a linear difference inclusion (LDI) in such a way that the conservativeness in the estimation is reduced. For that purpose, an LMI problem with 2^m+m constraints must be solved. In this paper, an algorithm that estimates the domain of attraction of the non-linear system is provided. Moreover, sufficient conditions to guarantee that the proposed algorithm obtains the greatest domain of attraction for the linear difference inclusion are given. Some illustrative examples are presented.

Keywords: Constrained non-linear systems; Domain of attraction; Invariant sets; Saturation; Linear difference inclusion

1. Introduction

The estimation of stability regions of non-linear systems is important for many fields in engineering. Regions of asymptotic stability are zones of safe operation that can avoid unnecessary operational restrictions if they are non-conservative (Genesio *et al.* 1985, Chiang and Thorp 1989, Gilbert and Tan 1991, Blanchini 1999).

Saturation is probably the most commonly encountered non-linearity in control engineering. For this reason, the estimation of the domain of attraction of linear systems subject to control saturation has received the attention of many authors in the last years (see, for example, Gomes da Silva Jr. and Tarbouriech 1999, Aracil *et al.* 2000, Hu and Lin 2001, Hu *et al.* 2002 and references therein).

The circle and Popov criteria have been used to obtain an estimation of the domain of attraction of a saturated control system (Pittet *et al.* 1997, Hindi and Boyd 1998). In Johansson (2002b), a piecewise quadratic estimate of the domain of attraction of a continuous-time saturated system is obtained. The estimation is based on an initial ellipsoidal estimation obtained by means of the circle criterion (see also Johansson 2002a). Also, in Alamo *et al.* (2005), a new concept of invariance for saturated systems is introduced. This new notion allows the computation of invariant sets for this class of non-linear systems.

One of the most relevant approaches to the analysis of saturated systems is based on a linear difference inclusion (LDI) of the saturation non-linearity. For example, in Gomes da Silva Jr. and Tarbouriech (2001) and Hu *et al.* (2002), an invariant ellipsoid for the saturated system is obtained by means of a linear difference inclusion. This approach has also been used in Limon *et al.* (2003) to obtain a polyhedral invariant set for a saturated system.

In this paper we present an approach to the estimation of the domain of attraction of a saturated linear system using linear difference inclusions. Given a system with m saturated control inputs, we show how to choose a linear difference inclusion (LDI) in such a way that the conservativeness in the estimation is reduced.

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For that purpose, an LMI problem with $2^m + m$ constraints must be solved. Moreover, given the obtained LDI, we characterize the maximum domain of attraction provided by the LDI (denoted H-domain of attraction in the paper). We provide an algorithm that estimates the domain of attraction of the non-linear system. Sufficient conditions are given to guarantee that the proposed algorithm obtains the H-domain of attraction of the system.

The paper is organized as follows: section 2 presents the problem statement. In section 3 the LDI under consideration is presented. The concept of H-domain of attraction is introduced in section 4. How to choose the LDI in order to reduce the conservativeness of the estimation is described in section 5. The one-step operator for the LDI is presented in section 6. An algorithm that provides an estimation of the domain of attraction of the non-linear system is provided in section 8. Some illustrative examples are given in sections 8 and 9. The paper draws to a close with a section of conclusions.

2. Problem statement

Let us consider the following system:

$$x^+ = Ax + B\sigma(Kx), \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state vector and x^+ the successor state vector. The function $\sigma\colon \mathbb{R}^m \to \mathbb{R}^m$ is the vector-valued standard saturation function defined as follows:

$$\sigma(u) = [\sigma(u_1)\ \sigma(u_2)\ \ldots\ \sigma(u_m)]^\top,$$

where $\sigma(u_i) = \text{sign}(u_i)\min\{1, |u_i|\}$.

Denoting B_i, $i = 1, \ldots, m$, the columns of matrix B and K_i, $i = 1, \ldots, m$, the rows of matrix K, system (1) can be rewritten as

$$x^+ = Ax + \sum_{i=1}^{m} B_i\sigma(K_i x) \tag{2}$$

The purpose of this paper is to provide an estimation of the domain of attraction for this class of non-linear systems. Such an estimation is obtained using a suitable linear difference inclusion of the non-linearity.

3. Linear difference inclusion

In this section we present the linear difference inclusion (LDI) that is going to be used throughout the paper. This linear difference inclusion is the one adopted in recent works, such as Hu *et al.* (2002), Cao and Lin (2003), and it is a generalization that improves the one presented in Gomes da Silva Jr. and Tarbouriech (1998) (see also Gomes da Silva Jr. and Tarbouriech 2001).

In the following, and in order to introduce the LDI, some auxiliary definitions are given. Denote $\mathcal{M} = \{1, 2, \ldots, m\}$. With this notation, system (2) can be rewritten as

$$x^+ = Ax + \sum_{i \in \mathcal{M}} B_i\sigma(K_i x)$$

Definition 1: Given $\mathcal{M} = \{1, 2, \ldots, m\}$, set $2^{\mathcal{M}}$ is the set of all subsets of $\mathcal{M}$. That is,

$$2^{\mathcal{M}} = \{S\colon S \subseteq \mathcal{M}\}$$

Example: If $m = 2$, then $\mathcal{M} = \{1, 2\}$ and $2^{\mathcal{M}} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Note that the empty set $\emptyset$ belongs to $2^{\mathcal{M}}$.

Throughout this paper, S^c denotes the complementary of S in $\mathcal{M}$. That is, $S^c = \{i \in \mathcal{M} : i \notin S\}$.

Definition 2: Given matrix $H \in \mathbb{R}^{m\times n}$, and set $S \in 2^{\mathcal{M}}$, $G_H(x, S)$ is defined as follows:

$$G_H(x, S) = \left(A + \sum_{i\in S^c} B_i K_i + \sum_{i \in S} B_i H_i\right)x.$$

Note that with these definitions, we have that $G_H(x, \mathcal{M}) = (A + BH)x$ and $G_H(x, \emptyset) = (A + BK)x$.

Definition 3: Given matrix $H \in \mathbb{R}^{m\times n}$, $\mathcal{L}(H)$ denotes the following symmetric polyhedron:

$$\mathcal{L}(H) = \{x \in \mathbb{R}^n\colon ||Hx||_\infty \leq 1\}$$

The following lemma (Hu and Lin 2001, Hu *et al.* 2002) provides, given matrix H, a linear difference inclusion that is valid in $\mathcal{L}(H)$:

Lemma 1: *Let $H \in \mathbb{R}^{m\times n}$ be given. If $x \in \mathcal{L}(H)$ then*

$$Ax + \sum_{i=1}^{m} B_i\sigma(K_i x) \in co\{G_H(x, S)\colon S \in 2^{\mathcal{M}}\},$$

where $co\ \{\cdot\}$ denotes the convex hull of a set.

4. H-domain of attraction

In this section the concept of H-domain of attraction is introduced.

Definition 4: We say that the initial condition x_0 belongs to the domain of attraction of system $x^+ = Ax + B\sigma(Kx)$ if the recursion

$$x_{k+1} = Ax_k + B\sigma(Kx_k)$$

converges to the origin.

Definition 5: We say that a sequence $\{S_0, S_1, S_2, \ldots\}$ is admissible if all the elements of the sequence belong to $2^{\mathcal{M}}$.

Definition 6: Given matrix H, we say that the initial condition $x_0 \in \mathcal{L}(H)$ belongs to the H-domain of attraction of system $x^+ = Ax + B\sigma(Kx)$ if the recursion

$$x_{k+1} = G_H(x_k, S_k)$$

satisfies the following two conditions:

1. $x_k \in \mathcal{L}(H)$, for all $k \geq 0$ and for every admissible sequence $\{S_0, S_1, \ldots, S_{k-1}\}$.
2. The recursion converges to the origin for every admissible infinite sequence $\{S_0, S_1, S_2, \ldots\}$.

It is clear from the linear difference inclusion provided by means of lemma 1 that any H-domain of attraction constitutes an estimation of the domain of attraction of the non-linear system. (See, for example, Limon *et al.* 2003.)

5. Obtaining matrix H

In this section we show how to obtain matrix H in such a way that a lower bound of the size of the H-domain of attraction is maximized.

Notation 1: Given a positive definite matrix P, and a positive scalar ρ, $\mathcal{E}(P, \rho)$ represents the following ellipsoid:

$$\mathcal{E}(P, \rho) = \{x : x^\top Px \leq \rho\}.$$

Definition 7: Given matrix $H \in \mathbb{R}^{m\times n}$, we say that the ellipsoid $\mathcal{E}(P, 1)$ is H-contractive if $\mathcal{E}(P, 1) \subset \mathcal{L}(H)$ and there is $\alpha \in (0, 1)$ such that for every $S \in 2^{\mathcal{M}}$:

$$G_H^\top(x, S)PG_H(x, S) \leq \alpha x^\top Px.$$

Property 1: If the linear system $x^+ = (A + BK)x$ is asymptotically stable then there exists a matrix $H \in \mathbb{R}^{m\times n}$ and a matrix P such that $\mathcal{E}(P, 1)$ is an H-contractive ellipsoid.

Proof: Note that the asymptotical stability of $x^+ = (A + BK)x$ implies that there is $\hat{P} > 0$ and $\alpha \in (0, 1)$ such that

$$(A + BK)^\top \hat{P}(A + BK) < \alpha\hat{P}$$

Note that if we set H equal to K then $G_H(x, S) = (A + BK)x$ for every set $S \in 2^{\mathcal{M}}$. Therefore:

$$\begin{aligned} &G_H^\top(x, S)\hat{P}G_H(x, S) \\ &\quad = x^\top(A + BK)^\top \hat{P}(A + BK)x \leq \alpha x^\top \hat{P}x \end{aligned} \tag{3}$$

It is always possible to find a scalar ζ such that $\mathcal{E}(\zeta\hat{P}, 1) \subset \mathcal{L}(H) = \mathcal{L}(K)$. Therefore, if we make P equal to $\zeta\hat{P}$ then $\mathcal{E}(P, 1) \subset \mathcal{L}(H)$. Moreover, the inequality (3) is also satisfied for the scaled matrix $P = \zeta\hat{P}$. □

The following result (Hu and Lin 2001, Hu *et al.* 2002) provides a characterization of the H-contractive ellipsoids of a given system.

Theorem 1: *Let us suppose that given $\alpha \in (0, 1)$, matrices $W \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{m\times n}$ satisfy the following linear matrix inequalities (LMIs):*

$$\begin{bmatrix} \alpha W & \left(A + \sum_{i\in S^c} B_iK_iW + \sum_{i\in S} B_iY_i\right)^\top \\ A + \sum_{i\in S^c} B_iK_iW + \sum_{i\in S} B_iY_i & W \end{bmatrix} > 0, \quad \forall\, S \in 2^{\mathcal{M}} \tag{4}$$

$$\begin{bmatrix} 1 & Y_i \\ Y_i^\top & W \end{bmatrix} < 0, \quad i = 1, \ldots, m \tag{5}$$

where Y_i denotes the i-th row of Y. Then, denoting $H = YW^{-1}$ and $P = W^{-1}$ it results that $\mathcal{E}(P, 1)$ is an H-contractive ellipsoid.

In order to obtain the greatest H-contractive ellipsoid, different approaches can be considered. In Hu and Lin (2001) and Hu *et al.* (2002) the concept of reference set is applied to give a measure of a given set.

In this paper, we propose the maximization of the trace of matrix W (this corresponds to the maximization of the sum of the axes of the H-contractive ellipsoid). That is, the following LMI problem can be solved in order to obtain W and $H = YW^{-1}$:

$$\begin{gathered} \max_{W,\,Y} \quad \operatorname{tr} W \\ \text{subject to LMIs (4)} \quad \text{and} \quad (5) \end{gathered}$$

Note that the above LMI problem has $2^m + m$ constraints. Therefore, the computational burden associated to the LMI problem becomes unmanageable when m grows beyond a certain limit.

Solving the previous optimization problem, a measure of the size of the H-contractive ellipsoid is maximized. Note that the ellipsoid is included in the H-domain of attraction. Therefore, the size of the obtained ellipsoid is a lower bound of the size of the H-domain of attraction.

Denote H^* the matrix obtained from the solution of the proposed maximization problem. We shall use the linear difference inclusion corresponding to matrix H^* (see lemma 1). Note that with this choice a lower bound of the size of the H-domain of attraction is maximized.

6. One-step operator for the linear difference inclusion

In the context of set invariance theory, the one-step set plays an important role (Blanchini 1999, Raković *et al.* 2004). In order to obtain an estimation of the domain of attraction of the saturated non-linear system, the one-step operator for the linear difference inclusion is presented in the following definition.

Definition 8:

- *Given a set* Ω *and* $S \in 2^{\mathcal{M}}: Q_H(\Omega, S) = \{x: G_H(x, S) \in \Omega\}$.
- *Given a set* Ω: $\hat{Q}_H(\Omega) = \bigcap_{S\in 2^{\mathcal{M}}} Q_H(\Omega, S)$.

One of the most important properties of $\hat{Q}(\cdot)$ is that given a convex polyhedral set Ω, $\hat{Q}_H(\Omega)$ is a convex polyhedron.

7. Proposed algorithm

In this section we propose an algorithm that provides an estimation of the domain of attraction of a saturated system. As claims property 1, if system $x^+ = (A + BK)x$ is asymptotically stable then it is possible to find an H-contractive ellipsoid for the saturated system. Based on the existence of such an ellipsoid, we propose an algorithm that converges to the H-domain of attraction. Moreover, we show that if the H-domain of attraction is bounded then the algorithm is finitely determined.

The main advantage of this algorithm is that the sequence of sets obtained in the execution of the algorithm belong to the H-domain of attraction of the system.

In order to present the algorithm, we need the following auxiliary result (see Appendix A for a proof):

Lemma 2: *Let us consider the ellipsoid* $\mathcal{E}(P, 1) \subset \mathbb{R}^n$. *Suppose that* v_i, $i = 1, \ldots, n$, *are the orthonormal eigenvectors of matrix* P *and* λ_i, $i = 1, \ldots, n$ *their corresponding eigenvalues. Denote*

$$\Gamma(P) = \mathcal{L}\left(\begin{bmatrix} \sqrt{n\lambda_1}v_1^{\mathrm{T}} \\ \sqrt{n\lambda_2}v_2^{\mathrm{T}} \\ \vdots \\ \sqrt{n\lambda_n}v_n^{\mathrm{T}} \end{bmatrix}\right)$$

Then

$$\mathcal{E}\left(P, \frac{1}{n}\right) \subseteq \Gamma(P) \subseteq \mathcal{E}(P, 1)$$

The following theorem establishes the theoretical support of the algorithm proposed to obtain an estimation of the domain of attraction of the saturated system.

Theorem 2: *Let us suppose that* $\mathcal{E}(P, 1) \subset \mathbb{R}^n$ *is an H-contractive ellipsoid for a given matrix* H *and a given scalar* $\alpha \in (0, 1)$. *Set* $\hat{C}_0 = \Gamma(P)$ *and consider the following recursion*:

$$\hat{C}_{k+1} = \hat{Q}_H(\hat{C}_k) \cap \mathcal{L}(H).$$

Each obtained set $\hat{C}_k$ *has the following properties*:

(i) $\hat{C}_k$ *is a convex polyhedron that can be obtained by means of definition* (8).
(ii) $\hat{C}_k$ *belongs to the H-domain of attraction of system* $x^+ = Ax + B\sigma(Kx)$.
(iii) *If* $\hat{x}$ *belongs to the H-domain of attraction, then* $\hat{x} \in \hat{C}_j$, *where* j *is the smallest integer that satisfies*:

$$j \geq \frac{\ln(n\hat{x}^{\top} P\hat{x})}{\ln(1/\alpha)}$$

(iv) *The sequence* $\hat{C}_0, \hat{C}_1, \hat{C}_2, \ldots,$ *converges to the H-domain of attraction.*
(v) *If the H-domain of attraction is bounded then the H-domain of attraction is finitely determined. That is, there is a finite integer* j^* *such that* $\hat{C}_{j^*}$ *equals the H-domain of attraction.*

Proof:

(i) The first point stems directly from the fact that $\hat{C}_0 = \Gamma(P)$ is a polyhedron and the definition of $\hat{Q}_H(\cdot)$.
(ii) From the properties of the one-step operator, it is inferred that every x_0 in $\hat{C}_k$ satisfies that the recursion

$$x_{i+1} = G_H(x_i, S_i)$$

is such that $x_i \in \mathcal{L}(H)$, $i = 0, \ldots, k$ and $x_k \in \Gamma(P)$ for every admissible sequence $S_0, S_1, \ldots, S_{k-1}$. Due to the fact that $\Gamma(P)$ belongs to the H-domain of attraction of the system, we conclude that $\hat{C}_k$ belongs to the H-domain of attraction.

(iii) Suppose that $x_0 = \hat{x}$ belongs to the H-domain of attraction of the system. Then, from the definition of H-domain of attraction, the recursion $x_{i+1} = G_H(x_i, S_i)$ remains in $\mathcal{L}(H)$ for every admissible sequence $\{S_0, S_1, \ldots\}$. Moreover, the fact that $\mathcal{E}(P,1)$ is an H-contractive ellipsoid guarantees that $x_{i+1}^\top P x_{i+1} \le \alpha x_i^\top P x_i$, for all $i \ge 0$ and all admissible sequence $\{S_0, S_1, \ldots\}$. From this it is inferred that

$$x_i^\top P x_i \le \alpha^i x_0^\top P x_0.$$

It can easily be seen that if

$$j \ge \frac{\ln(n \hat{x}^\top P \hat{x})}{\ln(1/\alpha)}$$

then $x_j^\top P x_j \le 1/n$. This implies, by means of lemma (2) that

$$x_j \in \mathcal{E}\left(P, \frac{1}{n}\right) \in \Gamma(P).$$

Note that $x_j \in \Gamma(P)$ implies that $\hat{x} = x_0 \in \hat{C}_j$.

(iv) It has been proved that if x_0 belongs to the H-domain of attraction then there is j such that $x_0 \in \hat{C}_j$. This proves the claim.

(v) If the H-domain of attraction is bounded then the maximum value of $x^\top P x$ in the H-domain can be bounded by a finite constant. Suppose that ρ is such a constant. Then, using similar arguments such as the ones used in the proof of claim (*iii*), it is obtained that the H-domain of attraction is equal to $\hat{C}_{j*}$, where $j*$ is the smallest integer that satisfies:

$$j^* \ge \frac{\ln(n\rho)}{\ln(1/\alpha).} \qquad \square$$

The previous theorem justifies the use of the following algorithm to obtain an estimation of the domain of attraction of a saturated linear system.

Algorithm:

1. Obtain matrix H and the corresponding H-contractive ellipsoid $\mathcal{E}(P,1)$ solving the LMI problem proposed in section 5.
2. Set the initial region $\hat{C}_0$ equal to $\Gamma(P)$ (see lemma 2). Set k equal to 0.
3. $\hat{C}_{k+1} = \hat{Q}_H(\hat{C}_k) \bigcap \mathcal{L}(H)$.
4. Obtain a polyhedral representation of $\hat{C}_{k+1}$ without redundant inequalities.
5. If $\hat{C}_{k+1} = \hat{C}_k$ then $\hat{C}_k$ is the H-domain of attraction. Stop. Else, set $k = k+1$ and return to step (3).

Note that one of the main advantages of this algorithm with respect to other existing ones (see, for example, Limon *et al.* 2003) is that the finite determinedness of the H-domain of attraction is not necessary. That is, every obtained set $\hat{C}_k$ constitutes an estimation of the domain of attraction of the non-linear system. This allows us to obtain estimations of the domain of attraction with a given limit of computational burden and complexity of the polyhedral set representation.

8. Example 1: A family of single input systems

Let us consider the following family of systems:

$$G(s) = \frac{1}{s^n}, \quad n = 2, \ldots, 5$$

For $n = 2$, the discrete-time state space representation of the system (sample time equal to one) is given by $x^+ = Ax + Bu$, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

The closed-loop system is given by $x^+ = Ax + B\sigma(Kx)$ where gain matrix $K = [-0.6167 \; -1.2703]$ corresponds to the discrete LQR controller with $Q = \mathrm{I}$ and $R = 0.1$. Note that the closed-loop system corresponds to a double integrator controlled by a saturated linear controller.

Figures 1 and 2 show the application of the algorithm to the proposed system ($n = 2$). In figure 1, the optimal H-invariant ellipsoid is displayed. In the same figure set $\mathcal{L}(H)$ (dotted lines) and $\Gamma(P)$ (solid lines) are represented. The obtained H-domain of attraction is depicted in figure 2, where the intermediate iterations of the algorithm are also shown. In this case, the algorithm converged in 7 steps and hence, the H-domain of attraction is equal to $\hat{C}_7$.

Table 1 shows a comparison between the volume of the H-contractive ellipsoid regions and the polyhedral H-domains of attraction for the family $G(s) = 1/s^n$, $n = 2, \ldots, 5$. The corresponding discrete-time closed loop systems $x^+ = A_n x + B_n \sigma(K_n x)$, $n = 2, \ldots, 5$ are obtained with the sample time equal to one and $u = \sigma(K_n x)$, where K corresponds to the discrete-time LQR controller ($Q = \mathrm{I}$, $R = 0.1$). As can be observed, the volume of the H-domain of attraction is considerably greater than the

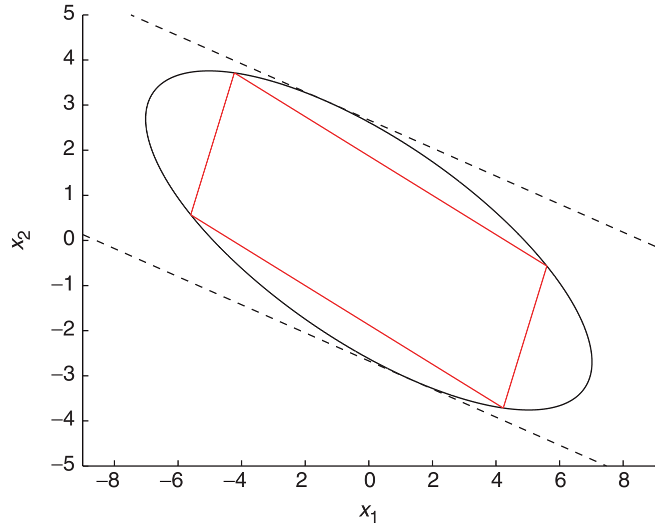

Figure 1. H-contractive ellipsoid.

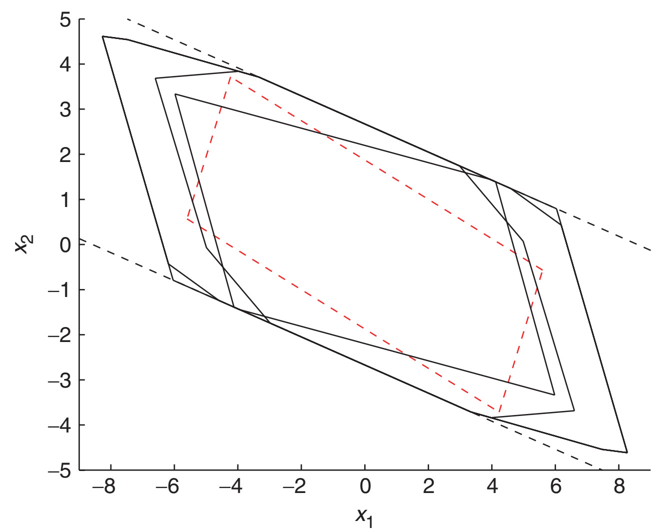

Figure 2. H-domain of attraction.

Table 1. Comparison between the volume of the obtained H-contractive ellipsoid and the polyhedral H-domain of attraction (single input systems).

	Volume ellipsoid	Volume H-domain	Volume increment (%)	Number of constraints
$n=2$	60.15	77.10	28.18	8
$n=3$	150.87	238.98	58.39	26
$n=4$	791.28	1680.9	112.42	66
$n=5$	6868.21	19884.17	189.51	188

one corresponding to the H-contractive ellipsoid. The last column of the table shows the number of non-redundant linear constraints required to represent each of the H-domains of attraction.

Let us now consider a family of two-input systems ($m=2$):

$$Y(s) = \frac{U_1(s) + sU_2(s)}{s^n}, \quad n = 2, \ldots, 5$$

Table 2. Comparison between the volume of the obtained H-contractive ellipsoid and the polyhedral H-domain of attraction (two-input systems).

	Volume ellipsoid	Volume H-domain	Volume increment (%)	Number of constraints
$n=2$	154.29	193.67	25.52	10
$n=3$	222.63	384.56	72.74	24
$n=4$	372.69	1055.6	183.24	82
$n=5$	1208.6	6094.7	404.28	288

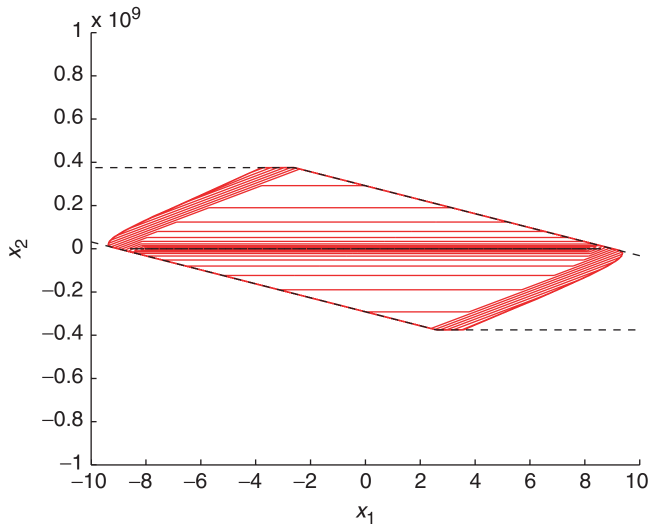

Figure 3. H-domain or attraction for example 2.

The inputs u_1 and u_2 are supposed to be saturated. The corresponding closed loop systems $x^+ = A_n x + B_n \sigma(K_n x)$, $n = 2, \ldots, 5$ are obtained with sample time equal to one and $[u_1, u_2]^\top = \sigma(K_n x)$, where K_n corresponds to the discrete-time LQR controller ($Q = \mathrm{I}$, $R = 0.11$). Table 2 shows a comparison between the volume of the H-contractive ellipsoid regions and the polyhedral H-domains of attraction for this family of two-input systems.

9. Example 2: A multiple input system

Consider system $x^+ = Ax + B\sigma(Kx)$, where

$$A = \begin{bmatrix} 1.2 & 0 \\ 0.4 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

$$K = \begin{bmatrix} -0.475 & 0 \\ 0.55 & 0.075 \end{bmatrix}$$

This example has been used in Gomes da Silva Jr. and Tarbouriech (1999). In that paper, the authors use an initial contractive polyhedral set $\Upsilon \subseteq \mathcal{L}(K)$ and show how to enlarge it in such a way that the contractiveness of the enlarged polyhedral set is not lost. That is, they obtain the maximum value of the scalar α such that $\alpha\Upsilon$ is a contractive polyhedron for the saturated system. In that paper, the authors obtained the box $\|x\|_\infty < 10$ as an estimation of the domain of attraction of the saturated system.

Figure 3 shows the application of the algorithm to the proposed system. It can be observed that the obtained region is a better estimation of the domain of attraction.

10. Conclusions

In this paper, an alternative approach to the estimation of the domain of attraction of a saturated linear system is presented. We show how to choose a linear difference inclusion (LDI) in such a way that the conservativeness in the estimation is reduced. An algorithm has been provided that estimates the domain of attraction of the non-linear system. Under mild assumptions, the proposed algorithm obtains the greatest domain of attraction for the linear difference inclusion.

A. Appendix

Proof of Lemma 2: Suppose that v_i, $i=1,\dots,n$, are the orthonormal eigenvectors of matrix P and λ_i, $i=1,\dots,n$ their corresponding eigenvalues. Then,

$$P=[\,v_1 \quad v_2 \quad \dots \quad v_n\,]\begin{bmatrix}\lambda_1 & 0 & \dots & 0\\ 0 & \lambda_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \lambda_n\end{bmatrix}\begin{bmatrix}v_1^T\\ v_2^T\\ \vdots\\ v_n^T\end{bmatrix}$$

From this equality it is inferred that

$$x^\top Px=\sum_{i=1}^{n}\lambda_i(v_i^\top x_i)^2.$$

The lemma is proved if it is shown that $\mathcal{E}(P,1/n)\subseteq\Gamma(P)$ and $\Gamma(P)\subseteq\mathcal{E}(P,1)$.

- $\mathcal{E}(P,1/n)\subseteq\Gamma(P)$:

  Let us suppose that $x\in\mathcal{E}(P,1/n)$. That is,

  $$x^{\mathrm{T}}Px=\sum_{i=1}^{n}\lambda_i(v_i^\top x_i)^2\le\frac{1}{n}$$

  This can be rewritten as

  $$\sum_{i=1}^{n}(\sqrt{n\lambda_i}v_i^Tx)^2\le 1.$$

  This implies that

  $$|(\sqrt{n\lambda_i}v_i^Tx)|\le 1,\quad i=1,\dots,n.$$

  Therefore it is inferred that $x\in\Gamma(P)$.

- $\Gamma(P)\subseteq\mathcal{E}(P,1)$:

  Let us suppose that $x\in\Gamma(P)$. That is,

  $$|(\sqrt{n\lambda_i}v_i^Tx)|\le 1,\quad i=1,\dots,n.$$

  Thus,

  $$n\lambda_i(v_i^Tx)^2\le 1,\quad i=1,\dots,n.$$

  From this,

  $$\sum_{i=1}^{n}n\lambda_i(v_i^Tx)^2\le n,$$

  and finally,

  $$\sum_{i=1}^{n}\lambda_i(v_i^Tx)^2=x^TPx\le 1,$$

  that is, $x\in\mathcal{E}(P,1)$.

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