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Brief paper

An algorithm for bounded-error identification of nonlinear systems based on DC functions☆

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Abstract

This paper presents a guaranteed method for the parameter estimation of nonlinear models in a bounded-error context. This method is based on functions which consists of the difference of two convex functions, called DC functions. The method considers DC representations of the functional form of the dynamic system to obtain an outer bound of the set of parameters that are consistent with the measurements, the system and the considered bounded error. At each iteration, the proposed algorithm solves several convex optimization problems to discard from the initial search region subregions that are proved not consistent. This operation is repeated while the obtained solution is improved. Four examples are provided to clarify the proposed identification algorithm.

Keywords: Nonlinear identification; Set-membership identification

1. Introduction

System identification is an active research area and several identification techniques have been developed in recent years. These techniques consider different ways to describe the uncertainty. The stochastic approach (Ljung, 1999; (Walter & Pronzato, 1997) assumes a probabilistic description of the uncertainty. An alternative possibility is to consider an unknown but bounded uncertainty, that is assumed by the set-membership identification approach. This paper considers an additive bounded error in the measurements and a parametric model. The set of parameters that is compatible with the model structure, obtained measurements and the considered uncertainty is named feasible solution set (*FSS*). The aim of set-membership identification methods is to obtain these feasible solution sets.

A polytope that can be computed exactly is obtained when the set-membership approach is applied to a model affine in the parameters. In this case, an algorithm that provides the facets of the solution set is presented in Broman and Shensa (1988). Alternatively, the *FSS* is represented by its vertices in Mo and Norton (1988). In Walter and Piet-Lahanier (1989), a polyhedric cone is used to represent the *FSS*. This representation can be used when the initial set of parameters is not bounded.

The complexity of the representation of the exact feasible solution set and the associated computational burden are the main drawbacks of the exact methods. Approximate feasible solution sets (*AFSS*) that bound the corresponding *FSS* are used to reduce these limitations. Boxes (Goffin & Vial, 1990; Milanese & Belforte, 1982), ellipsoids (Belforte, Bona, & Cerone, n.d.; Fogel & Huang, 1982), parallelotopes (Vicino & Zappa, 1996) and limited-complexity polyhedrons (Belforte & Tay, 1990; Piet-Lahanier & Walter, 1993) are used to represent the *AFSS* when a time-invariant parametric linear system is considered. The time-varying parameters case is treated in

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Bravo, Alamo, and Camacho (2006) where zonotopes are considered to capture the variable nature of the parameters.

When the model of the system is not expressed as an affine function of the set of parameters, the complexity of the identification problem can be very high. In Hanebeck (2001), an equivalent representation in a higher dimensional space is used to represent the *AFSS* with a pseudo ellipsoid. If the model is polynomial in the parameters of the system, signomial programming can be used to find a box that bounds the *FSS* (Milanese & Vicino, 1991). An interval branch and bound algorithm is used in Jaulin and Walter (1993) and Jaulin (2000) to describe the *AFSS* as a union of boxes. The set-membership approach is adopted in Milanese and Novara (2004) to develop a nonlinear system identification method. It does not require information on the functional form of the regression function describing the relations between measured input and output. Only bounds on the regression function gradient are assumed.

This paper presents a set-membership identification method for systems that exhibit a nonlinear dependence with respect to the parameters. An equivalent DC representation of the functional form of the system is used to obtain convex relaxations. Simplexes are used to represent the approximate feasible solution set that bounds the exact solution set. The method is an iterative algorithm that solves several convex optimization problems at each iteration to improve the *AFSS*.

The paper is organized as follows. In Section 2, the problem is formulated. Preliminary concepts are presented in Section 3. Section 4 provides a brief review of DC functions and DC programming. In Section 5, a detailed version of the proposed algorithm is presented. Section 6 reports some examples before the conclusions.

2. Problem formulation

In the context of set membership identification, it will be assumed that each measurement is a function of a regression vector $r_k \in \mathbb{R}^{n_r}$, a parameter vector $\theta \in \mathbb{R}^n$ and a given bounded error:

Assumption 1. A set of measurements $y_1, y_2, \ldots, y_N$ is provided. Each measurement $y_k \in \mathbb{R}$ is related with the parameter vector $\theta \in \mathbb{R}^n$ and the regression vector $r_k \in \mathbb{R}^{n_r}$ by means of the following expression:

$$y_k = f(r_k, \theta) + e_k, \tag{1}$$

where $f(\cdot)$ is continuous with respect to θ and e_k represents the considered error. This error belongs to a bounded set: $e_k \in E = \{e \in \mathbb{R} : |e| \leqslant \sigma\}$, where σ is a known constant.

In this paper it will be assumed that $f(r_k, \theta)$ is nonlinear with respect to the parameter θ. The term e_k bounds the effect of nonmodelled dynamics, perturbations to the system, noise in the measurements, etc.

The results presented in this paper can be easily generalized to the case in which the measurement y_k is a vector. It suffices to consider each component of y_k as an individual measurement.

Definition 1 (*Feasible solution set*). Suppose that the pairs (y_k, r_k), $k = 1, \ldots, N$ are given. The feasible solution set (*FSS*) is defined by

$$FSS = \{\theta : |y_k - f(r_k, \theta)| \leqslant \sigma,\ k = 1, \ldots, N\}.$$

The aim of set-membership identification methods is to find the set of parameters that are consistent with the assumed parametric model and bounded error in the measurements. When $f(r_k, \theta)$ in (1) is affine in θ, the *FSS* is the intersection of N strips in the parameter domain and can be computed exactly. However, the complexity of the *FSS* is greatly increased when the dependence of the model with respect to the parameters is nonlinear. In this case the exact *FSS* can be a complex, not convex and not-connected shape. An alternative solution is to consider outer bounds of the *FSS*.

Definition 2 (*Approximate feasible solution set*). An approximate feasible solution set, denoted *AFSS*, is a set that satisfies: $FSS \subseteq AFSS$.

The next section presents some preliminary concepts required to introduce the proposed iterative set-membership identification method.

3. Preliminary concepts

The proposed new identification method is an iterative algorithm that considers the measurements and regression vectors (y_k, r_k) with $k = 1, \ldots, N$, the error bound σ and the regression function of (1) to obtain a sequence of approximate feasible solution sets $AFSS_j$ with $j \geqslant 1$. A set of convex optimization problems is used at each iteration of the algorithm to improve the approximate feasible set obtained in the previous iteration. These convex optimization problems are defined using convex and concave relaxations of $f(r_k, \theta)$.

Fig. 1 can be used to illustrate the proposed algorithm (in this case a unique pair (y_k, r_k) is taken into account). The set

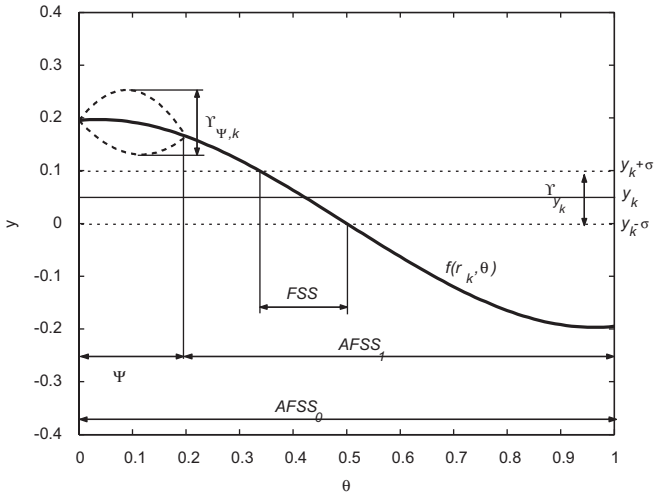

Fig. 1. Proposed algorithm illustration.

$FSS=\{\theta : y_k-\sigma \leqslant f(r_k,\theta) \leqslant y_k+\sigma\}$ is consistent with the measurement y_k, the regressor r_k, the considered error bound σ and the functional form of the system. An initial set $AFSS_0$ such that $FSS \subseteq AFSS_0$ and a candidate set $\Psi \subseteq AFSS_0$ are considered. Note that $\Psi \cap FSS=\emptyset$ and a new candidate $AFSS_1=AFSS_0\setminus\Psi$, where $\setminus$ is the set subtraction, can be considered to bound the FSS. To prove the equality $\Psi\cap FSS=\emptyset$ the proposed identification method uses convex $\check{f}(r_k,\theta)$ and concave $\hat{f}(r_k,\theta)$ relaxations (represented by dashed lines in Fig. 1) of the regression function $f(r_k,\theta)$ in Ψ. Next, both definitions are given.

Definition 3 (*Valid convex relaxation*). Consider the set Ψ and the pairs (y_k, r_k) with $k=1,\ldots,N$. The functions $\check{f}(r_k,\theta)$ with $k=1,\ldots,N$ are valid convex relaxations of $f(r_k,\theta)$ in Ψ if they are convex and $\check{f}(r_k,\theta)\leqslant f(r_k,\theta)$ for all $\theta\in\Psi$ with $k=1,\ldots,N$.

Definition 4 (*Valid concave relaxation*). Consider the set Ψ and the pairs (y_k, r_k) with $k=1,\ldots,N$. The functions $\hat{f}(r_k,\theta)$ with $k=1,\ldots,N$ are valid concave relaxations of $f(r_k,\theta)$ in Ψ if they are concave and $\hat{f}(r_k,\theta)\geqslant f(r_k,\theta)$ for all $\theta\in\Psi$ with $k=1,\ldots,N$.

Valid convex and concave relaxations of $f(r_k,\theta)$ are used by the algorithm to obtain the set $\Upsilon_{\Psi,k}=[\inf_{\theta\in\Psi}\check{f}(r_k,\theta), \sup_{\theta\in\Psi}\hat{f}(r_k,\theta)]$. Considering $\Upsilon_{y_k}=\{y : |y_k-y|\leqslant\sigma\}$, if $\Upsilon_{\Psi,k}\cap\Upsilon_{y_k}=\emptyset$ then it is inferred that $\Psi\cap FSS=\emptyset$ and consequently, Ψ can be discarded from $AFSS_0$ and a new outer bound $AFSS_1$ can be computed by set subtraction.

Once $AFSS_1$ is obtained, a new candidate set $\Psi\subseteq AFSS_1$ can be considered and this process is iterated while the obtained approximate feasible set is improved.

As will be shown in the following section, DC programming can be used to obtain valid convex and concave relaxations.

4. DC functions

A DC function is a function composed by the difference of two convex functions. The structure of these kind of functions supplies an efficient way to obtain convex and concave relaxations (Carrizosa, 2001; Tuy, 1995). In this section, it is assumed that $f(r_k,\theta)$ is a DC function or that it is possible to build an equivalent DC representation. This assumption provides an efficient method to obtain convex and concave relaxations of the functions $f(r_k,\theta)$ with $k=1,\ldots,N$. Next, some notations about DC functions are introduced.

Definition 5. Let S be a convex polytope (bounded polyhedral set) of $\mathbb{R}^n$. A real-valued function $f : S\to\mathbb{R}$ is called DC on S, if there exists two convex functions $g, h : S\to\mathbb{R}$ such that f can be expressed in the form: $f(x)=g(x)-h(x)$.

It is known that the set of DC functions defined on a compact convex set of $\mathbb{R}^n$ is dense in the set of continuous functions of this set (Horst & Thoai, 1999; Tuy, 1995). Therefore, every continuous function on a compact convex set can be approximated by a DC function with any desired precision and every $\mathscr{C}^2$-function is a DC function. Given a twice differentiable function it is possible to obtain a DC function adding and subtracting convex terms.

Consider for example the function $f(x)=x^3+x^2+1$ in the domain $x\in[-1,1]$. Note that $(\partial^2/\partial x^2)f(x)=6x+2$ and if $x\in[-1,1]$ then $(\partial^2/\partial x^2)f(x)\in[-4,8]$. Defining $g(x)=f(x)+2x^2$ and $h(x)=2x^2$, the equivalent function $f(x)=g(x)-h(x)$ is a DC function in $x\in[-1,1]$. Indeed, $h(x)$ and $g(x)$ are convex functions as they satisfy $(\partial^2/\partial x^2)h(x)\geqslant 0$ and $(\partial^2/\partial x^2)g(x)\geqslant 0$ for all $x\in[-1,1]$. A general method to compute DC representations can be obtained using the results presented in Adjiman, Dallwig, Floudas, and Neumaier (1998) and Floudas (2000).

In this paper, it will be assumed that polytope S is a simplex:

$$S=Co\{v_1,v_2,\ldots,v_{n+1}\},$$

where $v_i\in\mathbb{R}^n$, $i=1,\ldots,n+1$, are the $n+1$ vertices of S and $Co\{\cdot\}$ denotes the convex hull. The following property will be used to obtain a bound of the range of a DC function over a simplex.

Property 1. *Given a simplex S of nonzero volume and a convex function $h : S\to\mathbb{R}$, define the linear function $h_M(x)$ as $h_M(x)=h_0+h_L^T x$, where $h_0\in\mathbb{R}$ and $h_L\in\mathbb{R}^n$ are obtained from*

$$\begin{bmatrix} h_0 \\ h_L \end{bmatrix} = \begin{bmatrix} 1 & v_1^T \\ 1 & v_2^T \\ \vdots & \\ 1 & v_{n+1}^T \end{bmatrix}^{-1} \begin{bmatrix} h(v_1) \\ h(v_2) \\ \vdots \\ h(v_{n+1}) \end{bmatrix},$$

where v_i, $i=1,\ldots,n+1$ are the vertices of S. Then $h_M(v_i)=h(v_i)$, $i=1,\ldots,n+1$ and $h(x)\leqslant h_M(x)$, $\forall x\in S$.

Proof. As the volume of the simplex is different from zero, the expression that provides h_0 and h_L is well defined. Moreover, by construction, the equality $h_M(v_i)=h(v_i)$, $i=1,\ldots,n+1$ is trivially satisfied. In order to prove the second claim of the property it suffices to show that

$$\min_{x\in S} h_M(x)-h(x)\geqslant 0.$$

As $h_M(x)$ is an affine function and $h(x)$ is convex in S it results that $h_M(x)-h(x)$ is concave in S. This implies that the minimum of $h_M(x)-h(x)$ is attained at the vertices of the simplex. Thus,

$$\min_{x\in S} h_M(x)-h(x)=\min_{i=1,\ldots,n+1} h_M(v_i)-h(v_i)=0. \qquad \square$$

Property 2. *Given a DC function $f(x)=g(x)-h(x)$, the function $\check{f}(x)=g(x)-h_M(x)$ is convex and $f(x)\geqslant\check{f}(x)$, $\forall x\in S$.*

Proof. It is clear that $\check{f}(x)=g(x)-h_M(x)$ is convex as it is the difference of a convex function and an affine one. By property 1,

$h_M(x) - h(x) \geqslant 0,\ \forall x \in S$ then $f(x) = g(x) - h(x) \geqslant g(x) - h_M(x) = \check{f}(x), \forall x \in S$. □

Note that the obtained convex bound is a second order approximation (in a Taylor sense) and it is exact in the vertices $v_1, \ldots, v_{n+1}$ of S. A concave overestimation $\hat{f}(x)$ of $f(x) = g(x) - h(x)$ can be obtained by the convex relaxation of the function $-f(x) = h(x) - g(x)$.

Remark 1. In principle, the application of DC-representations is not limited to simplexes. It is possible to obtain a convex relaxation of a DC representation of a nonlinear function in any polytopic domain S. Next, we summarize the advantages of using simplexes. The proposed method is based on the use of an affine function $h_M(x)$ that bounds the convex term $h(x)$ of the DC function. This affine function must fulfill $h_M(x) \geqslant h(x)$ with $x \in S \subseteq \mathbb{R}^n$. Because of the concavity of $h_M(x) - h(x)$, this condition can be verified checking only that the condition is satisfied at the vertices of S. If S is a simplex, $n+1$ vertices must be visited. If S is a box, 2^n vertices must be visited, which implies an exponential number of vertices with respect to the dimension of the problem. If $S \subset \mathbb{R}^n$ is a simplex with nonzero volume, then $h_M(x)$ is obtained from the equalities, $h_M(v_i) = h(v_i)$, $i = 1, \ldots, n+1$, where v_i, $i = 1, \ldots, n+1$ represent the vertices of the simplex (in this way, the approximation is exact at the vertices of the simplex). Furthermore, it was proved in Carrizosa (2001) that this choice of $h_M(x)$ is optimal in the sense that it provides the tightest convex relaxation. If S is not a simplex, then it is not easy to obtain the optimal value for $h_M(x)$.

Remark 2. It is clear that the DC representation of a given nonlinear function is not unique. The performance of the method can be improved choosing an optimal DC representation of the function. There exist some results in the literature addressing this point. In particular, the systematic and efficient result presented in Adjiman et al. (1998) can be used to obtain an appropriate DC approximation. Note that given $f : S \to \mathbb{R}$, and a definite positive matrix A, the representation $f(x) + x^T Ax - x^T Ax$ is DC in S if the Hessian of $f(x) + x^T Ax$ is definite positive in S. Thus, an optimal way to select a DC representation is to find an optimal matrix A providing the least overestimation. In Adjiman et al. (1998) a systematic method to obtain a suboptimal diagonal matrix A is provided. The convexity of $f(x) + x^T Ax$ is determined in an efficient way by an interval arithmetic evaluation of the Hessian of $f(x) + x^T Ax$ in S.

As will be shown in the following section, the potential advantages of the use of DC programming methods in bounded-error identification are that DC programming provides convex second order relaxations of the original identification problem, and all the measurements can be considered at the same time.

5. Proposed identification algorithm

This section presents the new identification algorithm considering DC functions and simplexes. Given a set $\Psi \subseteq \mathbb{R}^n$ in the parameter space, the next theorem constitutes a method to study its consistence with the considered measurements, system and bounded error. To check if the intersection of Ψ and *FSS* is an empty set a convex optimization problem can be solved. The set Ψ can be discarded from the final outer bound solution if this intersection is empty.

Theorem 1. *Given* (1), *the pairs* (y_k, r_k) *with* $k = 1, \ldots, N$, *a convex set* $\Psi \subseteq \mathbb{R}^n$, *valid convex and concave relaxations* $\check{f}(r_k, \theta)$ *and* $\hat{f}(r_k, \theta)$ *in* Ψ, $k = 1, \ldots, N$ *and the convex optimization problem* $\mathscr{P}(\Psi)$

$$\gamma_{\min} = \min_{\theta, \gamma} \quad \gamma$$

$$\text{s.t.}$$

$$\theta \in \Psi,$$

$$\check{f}(r_k, \theta) - (y_k + \sigma) \leqslant \gamma, \quad k = 1, \ldots, N,$$

$$y_k - \sigma - \hat{f}(r_k, \theta) \leqslant \gamma, \quad k = 1, \ldots, N$$

then, if $\gamma_{\min} > 0$ *it is inferred that* $\Psi \cap FSS = \emptyset$.

Proof. Suppose that $\Psi \cap FSS \neq \emptyset$, then there is $\theta^* \in \Psi \cap FSS$. As $\theta^* \in \Psi$, note that by definition: $\check{f}(r_k, \theta^*) \leqslant f(r_k, \theta^*) \leqslant \hat{f}(r_k, \theta^*)$ with $k = 1, \ldots, N$. As $\theta^* \in FSS$ then $f(r_k, \theta^*) - (y_k + \sigma) \leqslant 0$ and $(y_k - \sigma) - f(r_k, \theta^*) \leqslant 0$ with $k = 1, \ldots, N$. It is deduced that $\check{f}(r_k, \theta^*) - (y_k + \sigma) \leqslant 0$ and $(y_k - \sigma) - \hat{f}(r_k, \theta^*) \leqslant 0$ with $k = 1, \ldots, N$ which yields to $\gamma_{\min} \leqslant 0$. From this it is inferred that $\gamma_{\min} > 0$ implies $\Psi \cap FSS = \emptyset$. □

Now, it is possible to introduce the proposed identification algorithm. At iteration j a simplex that bounds the *FSS* is available. This simplex is composed by $n+1$ vertices and it is denoted by $AFSS_{j-1} = Co\{v_1, v_2, \ldots, v_{n+1}\}$. Then a new $AFSS_j$ can be computed by the following algorithm:

Algorithm.

for $p = 1, \ldots, n+1$
 for $q = 1, \ldots, n+1$ with $q \neq p$
  $v_B = \frac{v_q + v_p}{2}$
  while $\|v_B - v_p\| > \varepsilon_1$
   $\Psi_{p,q}(v_B) = Co\{v_1, v_2, \ldots, v_{q-1}, v_B, v_{q+1}, \ldots, v_{n+1}\}$
   Build valid convex $\check{f}_{p,q}(r_k, \theta)$ and concave
   $\hat{f}_{p,q}(r_k, \theta)$ relaxations in $\Psi_{p,q}$ of system (1)
   with $k = 1, \ldots, N$, following the method of Section 4.
   $\gamma_{\min} = \mathscr{P}(\Psi_{p,q}(v_B))$
   if $\gamma_{\min} > 0$ then $v_p = v_B$
   else $v_B = \frac{v_B + v_p}{2}$
   endif
  endwhile
 endfor
endfor
$AFSS_j = Co\{v_1, v_2, \ldots, v_{n+1}\}$

End of algorithm

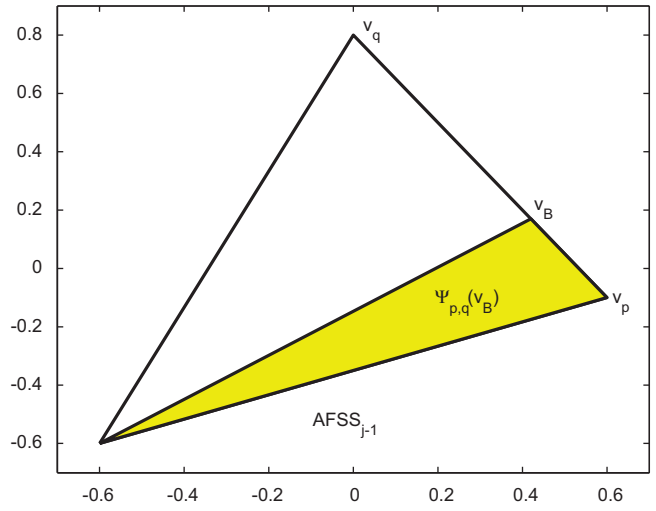

Fig. 2. Illustration of sets $AFSS_{j-1}$ and $\Psi_{p,q}(v_B)$.

The algorithm considers vertices $v_p, v_q \in AFSS_{j-1}$ to visit all edges of the simplex $AFSS_{j-1}$. A new vertex $v_B = (v_q + v_p)/2 \in AFSS_{j-1}$ is obtained. Using this vertex, the set $\Psi_{p,q}(v_B) = Co\{v_1, v_2, \ldots, v_{q-1}, v_B, v_{q+1}, \ldots, v_{n+1}\}$ is defined, see Fig. 2. Next, valid convex $\check{f}_{p,q}(r_k, \theta)$ and concave $\hat{f}_{p,q}(r_k, \theta)$ relaxations of $f(r_k, \theta)$ in $\Psi_{p,q}$ with $k = 1, \ldots, N$ are built following the method explained in Section 4. Then, the convex optimization problem $\mathscr{P}(\Psi_{p,q}(v_B))$ is solved, obtaining the minimum value of γ, denoted $\gamma_{\min}$. By Theorem 1, if $\gamma_{\min} > 0$ then $\Psi_{p,q}(v_B) \cap FSS = \emptyset$ and $\Psi_{p,q}(v_B)$ can be discarded. The set $AFSS_{j-1}$ is updated to $AFSS_{j-1} = AFSS_{j-1} \setminus \Psi_{p,q}(v_B) = Co\{v_1, v_2, \ldots, v_{p-1}, v_B, v_{p+1}, \ldots, v_{n+1}\}$. If $\gamma_{\min} \leqslant 0$ and $\|v_B - v_p\| > \varepsilon_1$ (where $\varepsilon_1 \in \mathbb{R}^+$ is a design parameter), the vertex v_B is updated to $v_B = (v_B + v_p)/2$. These operations are iterated with all edges of $AFSS_{j-1}$. Finally, the algorithm returns a set $AFSS_j$ that is composed by the updated vertices.

The algorithm can be used while the volume of the obtained approximate feasible set decreases, that is: $Volume(AFSS_{j-1}) - Volume(AFSS_j) > \varepsilon_2$, where $\varepsilon_2 \in \mathbb{R}^+$ is a design parameter.

Remark 3. The number of optimization problems to solve at each iteration j is bounded by $O(\log_2(d_v/\varepsilon_1) * (n + 1) * n)$ where d_v is the maximum distance between two vertices of $AFSS_{j-1}$. Note that standard convex optimizations algorithms can be used to solve the aforementioned convex problems. See, for instance Boyd and Vandenberghe (2004).

Remark 4. Theorem 1 provides only a sufficient condition to check if the intersection of a given convex set and the *FSS* is empty. The conservativeness of the sufficient condition can be arbitrarily reduced using a branch and bound scheme. The idea is simple, as is shown in the following. The original simplex is divided into two simplexes. If Theorem 1 determines that both simplexes have an empty intersection with the FSS we are done. If not, those simplexes satisfying the sufficient condition provided by Theorem 1 are discarded. The division process is done again with the simplex (simplexes) not discarded in the previous step. Using Theorem 1, the discarding process is done again. All this is repeated till the size of the simplexes reaches a pre-specified threshold or till the discarding process determines that the intersection of the original simplex with the FSS is empty. Of course, this means a worst case exponential computational time, but the convergence to the exact solution set is assured because the approximation error incurred when using DC approximations decreases quadratically with the size of the simplex. Note that the exponential complexity corresponding to the exact solution to the problem is not a surprise due to the NP-hard nature of the addressed problem. The solution returned by the Branch and Bound algorithm is a set of simplexes that bounds the FSS. A convex envelope of this AFSS (a box or an ellipsoid) can be computed easily using the vertices of the simplexes.

The following lemma proves that if $FSS \subseteq AFSS_0$ then the sequence $AFSS_j$ bounds the exact feasible solution set for all $j > 0$.

Lemma 1. *Given system* (1), *the pairs* (y_k, r_k) *with* $k = 1, \ldots, N$ *and an initial* $AFSS_0$ *such that* $FSS \subseteq AFSS_0$ *then, the sets computed by the proposed algorithm are approximate solution sets*:

$$FSS \subseteq AFSS_j, \quad \forall j \geqslant 1.$$

Proof. The proposed algorithm updates set $AFSS_{j-1}$ to $AFSS_{j-1} \setminus \Psi_{p,q}(v_B)$ where $\Psi_{p,q}(v_B) \subseteq AFSS_{j-1}$ and $\gamma_{\min} > 0$. By Theorem 1, $FSS \cap \Psi_{p,q}(v_B) = \emptyset$, so $FSS \subseteq AFSS_{j-1} \setminus \Psi_{p,q}(v_B)$. Then it follows that $FSS \subseteq AFSS_j$, $\forall j \geqslant 1$. □

6. Examples

The design parameters used in all examples are $\varepsilon_1 = \varepsilon_2 = 0.01$.

6.1. Example 1

The proposed identification method is used to identify the nonlinear system (Esposito & Floudas, 1998)

$$y_k = \theta_1 + \frac{1}{r_k - \theta_2} + e_k,$$

where y_k is the output, r_k is the regressor, θ_1 and θ_2 are the parameters to identify and e_k is an error term. It is considered that $r_k - \theta_2 \neq 0$. The initial search space is the simplex $AFSS_0 = Co\{(-20, 5), (0, 20), (20, 5)\}$. The functional form of the regression function of the system is a DC function where θ_1 and $-1/(r_k - \theta_2)$ are convex functions in $AFSS_0$. Five regressors $r_k = k$ with $k = 0, \ldots, 4$ are considered to obtain the measurements corresponding to $\theta_1 = 2$ and $\theta_2 = 6$. The parameters of system (1) are defined by

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad \sigma = 0.1.$$

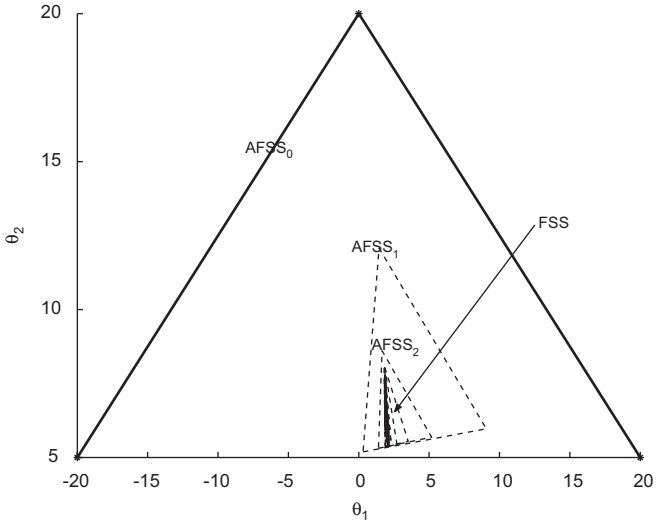

Fig. 3. The cloud of points is the *FSS*. The initial simplex is represented with solid lines. The evolution of the $AFSS_j$ is represented by dotted lines.

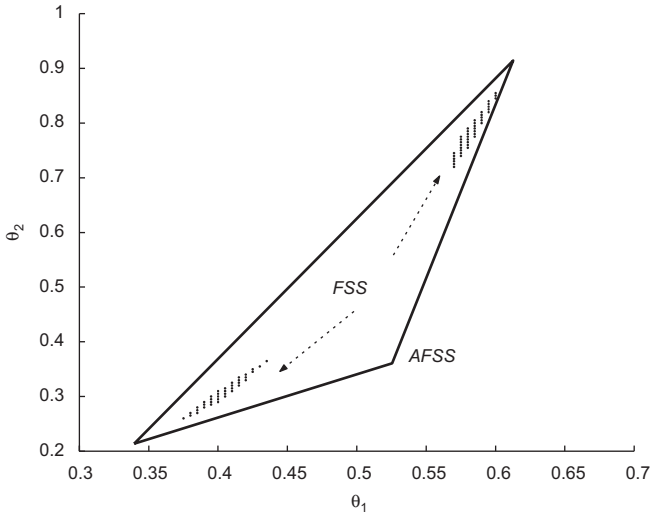

Fig. 4. The clouds of points represents the *FSS*. Solid lines represent the final simplex obtained.

Fig. 3 shows the evolution of the $AFSS_j$ with $j \geqslant 0$. The cloud of points represents the *FSS* consistent with the measurements and the bounded error. It has been obtained by a sufficiently dense random search in $AFSS_0$. The algorithm provides a sequence of $AFSS_j$ whose volume is reduced at each iteration. The final *AFSS* obtained by the proposed algorithm is defined by $Co\{(1.8, 8),\ (1.9, 5.4), (2.7, 5.5)\}$.

6.2. Example 2

The proposed identification method is used to identify the nonlinear system (Jaulin & Walter, 1999)

$$y_k = 20\mathrm{e}^{-\theta_1 r_k} - 8\mathrm{e}^{-\theta_2 r_k} + e_k,$$

where y_k is the output, $r_k = k$ is the time, θ_1 and θ_2 are the parameters to identify and e_k is an error term where $\sigma = 0.1$. The initial search space is $AFSS_0 = Co\{(-2, 0), (1, 3)(4, 0)\}$. Note that the functional form of the regression function of the system is a DC function in $AFSS_0$.

Ten sample times, $k = 1, \ldots, 10$, are considered to obtain the measurements corresponding to $\theta_1 = 0.4$ and $\theta_2 = 0.3$. Fig. 4 shows the final simplex obtained by the algorithm. The cloud of points represents the *FSS* consistent with the measurements and the bounded error. It can be seen that this set is formed by two nonconnected regions. To improve the solution a single bisection of the final simplex can be considered. The couple of obtained simplexes are reduced applying the algorithm again. The new sequence is represented in Fig. 5.

6.3. Example 3

The proposed identification method is used to identify the nonlinear system:

$$y_k = \theta_1 \mathrm{e}^{-\theta_2 r_k} - \theta_3 \mathrm{e}^{-\theta_4 r_k} + e_k,$$

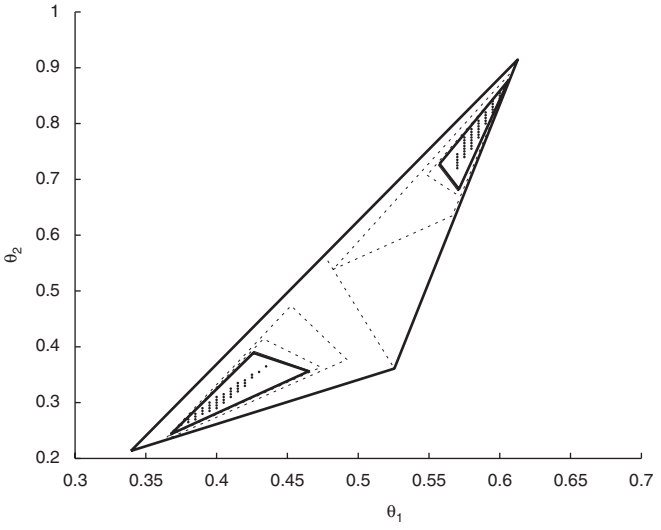

Fig. 5. Bisection of the final simplex. New $AFSS_j$ evolution of the couple of obtained simplexes is shown.

where $|e_k| \leqslant \sigma = 0.1$. It will be assumed that $\theta_1, \theta_2, \theta_3, \theta_4 > 0$. In order to obtain an appropriate DC representation, a change of variables is considered: $\theta'_1 = \log(\theta_1)$ and $\theta'_3 = \log(\theta_3)$. Then, the modified nonlinear system is

$$y_k = \mathrm{e}^{-\theta_2 r_k + \theta'_1} - \mathrm{e}^{-\theta_4 r_k + \theta'_3} + e_k.$$

Note that the functional form of the regression function of the system is a DC function. The corresponding values of the system parameters used to obtain the input-output data are $\theta'_1 = \log(20) = 2.9957$, $\theta_2 = 1$, $\theta'_3 = \log(8) = 2.0794$ and $\theta_4 = 0.1$. The initial search space is $AFSS_0 = Co\{(0, 0, 0, 0), (7, 0, 0, 0), (0, 5, 0, 0,), (0, 0, 7, 0), (0, 0, 0, 5)\}$ with volume $= 51.0417$. The final *AFSS* obtained simplex is $Co\{(1.94, 0.05, 1.95, 0.05), (5.01, 0, 1.98, 0), (3.13, 1.61, 1.60, 0), (3.02, 0, 3.97, 0), (2.65, 0, 3.54, 0.57)\}$ with volume $= 0.2282$. The volume has been drastically reduced.

Next, in order to improve the obtained solution a Branch and Bound scheme has been applied (see Remark 4). In this case the final obtained simplex constitutes a sharp bound with volume equal to 6.4343×10^{-5}. The proposed algorithm uses 2124 branches.

6.4. Example 4

The proposed method is used to identify the parameters θ_1, θ_2, θ_3 satisfying the following constraints:

$$0 = 2\theta_1^2 - 2\theta_1\theta_2 + 2\theta_2^2 - 2\theta_1\theta_3 + 2\theta_3^2 - 2\theta_2\theta_3,$$

$$0 = 2\theta_1^2 + 2\theta_1\theta_2 + 2\theta_2^2 + 2\theta_1\theta_3 + 2\theta_3^2 + 2\theta_2\theta_3,$$

$$0 \geqslant \theta_1^2 + \theta_2^2 + \theta_3^2 - 1.$$

Note that there is a unique solution to these constraints ($\theta_1 = \theta_2 = \theta_3 = 0$). An implementation of the proposed algorithm is executed in a Pentium IV (3.2 GHz) computer. The initial simplex considered is $AFSS_0 = Co\{(0, 0, 3), (2\sqrt{2}, 0, -1), (-\sqrt{2}, \sqrt{6}, -1), (-\sqrt{2}, -\sqrt{6}, -1)$. A simplex with volume 1.2986×10^{-5} that bounds the exact solution is obtained in 8.9 s solving 48 convex optimization problems. An implementation of SIVIA algorithm (Jaulin & Walter, 1993) has been developed with the interval arithmetic library INTLAB (Rump, 1999). Using the same computer and considering the unitary initial box $AFSS_0 = \{[-1, 1], [-1, 1], [-1, 1]\}$, SIVIA algorithm provides a list of boxes that bounds the feasible solution. In order to obtain a volume approximately equal to the one obtained with the method proposed in this paper, SIVIA algorithm requires 784 branchings and a computational time of 16.7 s.

7. Conclusions

This paper has presented a guaranteed method for the parameter estimation of nonlinear models in a bounded-error context. The proposed iterative algorithm is based on a DC representation of the functional form of the considered system. Simplexes are used to bound the set of parameters that are consistent with the measurements, the system and bounded error. At each iteration of the proposed algorithm, a simplex that bounds the exact solution set is considered. To improve this outer bound, the algorithm discards the subsets of the initial simplex that are not consistent with the bounded error. This operation can be made solving several convex optimization problems. The number of optimization problems to solve is polynomial in the number of considered parameters to identify. Finally, three examples have been included to clarify the proposed algorithm.

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