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Min–Max MPC based on a network problem

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Abstract

In general, min–max model predictive controllers have a high computational burden. In this work, an efficient implementation of this class of controllers that can be applied to linear plants with additive uncertainties and quadratic cost functions is presented. The new approach relies on the equivalence of the maximization problem with a network problem. If a given condition is satisfied, the computational burden of the proposed implementation grows polynomially with the prediction horizon. In particular, the resulting optimization problem can be posed as a quadratic programming problem with a number of constraints and variables that grows in a quadratic manner with the prediction horizon. An alternative controller has been proposed for those systems that do not satisfy this condition. This alternative controller approximates the original one with a given bound on the error.

Keywords: Predictive control; Robust control; Uncertain linear systems; Networks; Quadratic programming

1. Introduction

Most control strategies are based on a mathematical model of the system to be controlled. Using this model, the controller evaluates the control input. This means that the efficiency of a controller depends on how precisely the mathematical model represents the real behavior of the system. This is even more critical in the case of model predictive control (MPC) where the control decision is taken on the base of the future predicted evolution of the system which is obtained using the model. Nominal MPC algorithms, do not take directly into account possible model uncertainties and disturbances. Although the feedback mechanism itself is able to partially compensate for them, robust MPC controllers that cope with uncertainties in an explicit way are of interest.

One approach used in MPC when uncertainties are present, is to minimize the objective function for the worst possible case. This strategy is known as min–max and was originally proposed in [32] in the context of robust receding control. In robust MPC the problem was first tackled in [9]. In this paper we consider bounded additive uncertainties, for polytopic and linear fractional uncertainty see [16,31,18,6]. Min–max MPC schemes can be classified into open-loop and feedback min–max controllers (see [22]). Feedback min–max MPC obtains a sequence of feedback control laws that minimize the worst case cost while assuring robust constraint handling. It requires the solution of a very high dimensional problem that makes its practical implementation very hard (see [18,30,25]). For cost functions based on $\|.\|_\infty$ and $\|.\|_1$ norm, the explicit solution has been obtained (see [6,14]). This result has not been extended to quadratic cost functions.

We consider open-loop min–max MPC control. In this strategy, a single control input sequence that minimizes the worst case cost is obtained (see [9,8,3]). In order to introduce some feedback in the predictions, a linear feedback stabilizing control law for the nominal plant is considered [5,2,20]. Based on these ideas, several reduced complexity robust MPC control strategies have been proposed in the literature [24,17,23]. For quadratic cost functions, the open-loop min–max problem results in an optimization problem with a very high computational burden. This is due to the NP-Hard nature of the maximization problem that arises when the worst case is evaluated for a given future input trajectory. On-line and

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off-line algorithms for reducing the complexity of this problem have been studied in [20,26,27,29].

In this work, the preliminary results of [1] are extended following the ideas presented in [2], and an efficient way of implementing a constrained min–max predictive controller is proposed. The new approach relies on the equivalence of the maximization problem with a min-cut network problem. Using this equivalence, the min–max problem is solved using a single quadratic programming (QP) problem. The size (i.e., number of variables and constraints) of this problem is a quadratic function of the prediction horizon. Note that quadratic programming solvers are nowadays a wide-spread technology with plenty of commercial and non-commercial efficient software available. A modified controller is proposed for those systems that do not satisfy the required condition. This controller approximates the original one with a given bound on the error.

This paper is organized as follows: Section 2 introduces min–max model predictive control and defines the corresponding optimization problem. Section 3 shows the computational burden required to implement this controller. In Section 4 we introduce network problems. In Section 5 the equivalence between the maximization problem and a network problem is proved and the quadratic programming formulation based on this equivalence is presented. Section 6 introduces the modified controller for those systems which do not satisfy the necessary condition. The paper draws to a close with a section of conclusions.

2. Problem formulation

Consider the following discrete-time linear system with bounded additive uncertainties:

$$x_{k+1} = Ax_k + Bu_k + Dw_k, \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the control input, and $w_k \in \mathbb{R}^{n_w}$ is the uncertainty that is supposed to be bounded by the unit hypercube; that is

$$w_k \in W = \{w \in \mathbb{R}^{n_w} : \|w\|_\infty \leqslant 1\}.$$

Note that if W is an hyper-rectangle, it is always possible to scale matrix D in order to bound w by means of the unit hypercube.

The control input is given by $u_k = Kx_k + v_k$, where K is chosen in order to achieve some desired property for the unconstrained nominal problem [5,2,20]. The MPC controller computes the optimal sequence of control correction efforts v_k. Defining $A_K = (A + BK)$, the dynamics of the system can be rewritten as:

$$x_{k+1} = A_K x_k + Bv_k + Dw_k.$$

The objective function is defined as

$$V(x, \mathbf{v}, \mathbf{w}) = \sum_{j=0}^{N-1} [x_j^{\mathrm{T}} Q x_j + u_j^{\mathrm{T}} R u_j] + x_N^{\mathrm{T}} P x_N, \tag{2}$$

with $Q \geqslant 0$, $P \geqslant 0$ and $R > 0$. Matrix P defines the terminal cost function. The initial state is $x_0 = x$, vector $\mathbf{v} = [v_0^{\mathrm{T}}, v_1^{\mathrm{T}}, \ldots, v_{N-1}^{\mathrm{T}}]^{\mathrm{T}}$ denotes the sequence of control correction efforts and $\mathbf{w} = [w_0^{\mathrm{T}}, w_1^{\mathrm{T}}, \ldots, w_{N-1}^{\mathrm{T}}]^{\mathrm{T}}$ denotes the sequence of disturbances. Variables x_j and u_j with $j \geqslant 0$ are the predicted state and control input respectively and are given by

$$x_j(x, \mathbf{v}, \mathbf{w}) = A_K^j x + \sum_{i=1}^{j} A_K^{i-1} B v_{j-i} + \sum_{i=1}^{j} A_K^{i-1} D w_{j-i},$$

$$u_j(x, \mathbf{v}, \mathbf{w}) = Kx_j + v_j. \tag{3}$$

We consider linear constraints in state and input, $x_k \in X$, $u_k \in U$, where X and U are polyhedral sets. In order to achieve stability, a terminal region constraint defined by a polyhedron, $x_N \in \Omega$, is also taken into account. The min–max constrained predictive controller is defined by the following optimization problem (denoted $P(x)$):

$$\begin{aligned} J^*(x) = \min_{\mathbf{v}} \max_{\mathbf{w} \in W_N} \quad & V(x, \mathbf{v}, \mathbf{w}) \\ \text{s.t.} \quad & x_j(x, \mathbf{v}, \mathbf{w}) \in X, \quad \forall \mathbf{w} \in W_N, \\ & j = 0, \ldots, N-1, \\ & x_N(x, \mathbf{v}, \mathbf{w}) \in \Omega, \quad \forall \mathbf{w} \in W_N, \\ & u_j(x, \mathbf{v}, \mathbf{w}) \in U, \quad \forall \mathbf{w} \in W_N, \\ & j = 0, \ldots, N-1, \end{aligned} \tag{4}$$

where W_N denotes the set of possible disturbance trajectories $\mathbf{w}$ of length N.

$$W_N = \{\mathbf{w} \in \mathbb{R}^{Nn_w} : \|\mathbf{w}\|_\infty \leqslant 1\}.$$

The controller is implemented in a receding horizon scheme. The optimization problem is solved at each sample instant for the current state x. An optimal vector of control correction signals $\mathbf{v}^*$ is obtained and the control input $u_0 = Kx + v_0^* = K_{\mathrm{MPC}}(x)$ is applied. In order to guarantee stability of the closed-loop system, the terminal region Ω and the terminal cost function P must be chosen properly. See [25] for a review on MPC stability issues. In [2], the conditions that must be satisfied to assure robust convergence to a bounded set while assuring robust constraint satisfaction were presented.

3. MPC computation

In this section, the optimization problem that characterizes the proposed min–max MPC control strategy is presented. It is shown that the resulting optimization problem can be posed as a quadratic min–max problem subject to a set of linear constraints that do not depend on the uncertainty.

Let us define the feasible set S_{F} as the pairs $(x, \mathbf{v})$ which satisfy the constraints of problem (4). Taking into account (3), when X, Ω and U are polyhedral regions defined by r_x, r_u and r_Ω linear constraints, respectively, the feasible set S_{F} can be expressed as

$$S_{\mathrm{F}} = \{(x, \mathbf{v}) : G_x x + G_v \mathbf{v} + G_w \mathbf{w} \leqslant m, \quad \forall \mathbf{w} \in W_N\}, \tag{5}$$

where G_x, G_v, G_w and m are matrices of appropriate dimensions with one row for each linear constraint of (4); that is,

$N(r_x+r_u)+r_\Omega$ rows. For the system described (i.e., linear systems with additive uncertainties) it is possible to eliminate the uncertainty from the constraints while still assuring robustness. As each linear constraint must be satisfied for all possible future uncertainty realizations, it can be seen that S_F is equivalent to

$$S_\mathrm{F} = \{(x, \mathbf{v}) | G_x x + G_v \mathbf{v} \leqslant g\},$$

where g is a vector that satisfies

$$e_i^\mathrm{T} g = e_i^\mathrm{T} m - \max_{\|\mathbf{w}\|_\infty \leqslant 1} e_i^\mathrm{T} G_w \mathbf{w} = e_i^\mathrm{T} m - \|G_w^\mathrm{T} e_i\|_1,$$

with e_i the ith column of the identity matrix. Note that W_N is defined as $\|\mathbf{w}\|_\infty \leqslant 1$.

The cost function, $V(x, \mathbf{v}, \mathbf{w})$, is a quadratic convex function of x, $\mathbf{v}$ and $\mathbf{w}$. Taking into account (3), matrices H_x, H_v and H_w can be found such that

$$V(x, \mathbf{v}, \mathbf{w}) = \|H_x x + H_v \mathbf{v} + H_w \mathbf{w}\|_2^2.$$

We conclude that $P(x)$ can be rewritten as

$$J^*(x) = \min_{\mathbf{v}} \max_{\|\mathbf{w}\|_\infty \leqslant 1} \|H_x x + H_v \mathbf{v} + H_w \mathbf{w}\|_2^2 \quad \text{s.t} \quad G_v \mathbf{v} + G_x x \leqslant g. \tag{6}$$

The objective function of the minimization problem is a convex function on $\mathbf{v}$ because it is defined as the point-wise maximum of a convex function. This implies that $P(x)$ is a convex optimization problem. It is well known that convex optimization problems can be solved efficiently if it is possible to evaluate the objective function and a subgradient. Cutting plane methods, the ellipsoid algorithm, bundle methods and gradient methods can be applied to solve problem (4) given a way to evaluate the objective function (see [7,15,12] and the references therein). However, this optimization problem is of high complexity. Defining the max function as

$$V^*(x, \mathbf{v}) = \max_{\|\mathbf{w}\|_\infty \leqslant 1} \mathbf{w}^\mathrm{T} M \mathbf{w} + q^\mathrm{T}(x, \mathbf{v}) \mathbf{w} \tag{7}$$

with

$$M = H_w^\mathrm{T} H_w \geqslant 0, \quad q(x, \mathbf{v}) = 2H_w^\mathrm{T}(H_x x + H_v \mathbf{v}),$$

the min–max problem $P(x)$ can be posed as

$$J^*(x) = \min_{\mathbf{v}} \quad V(x, \mathbf{v}, 0) + V^*(x, \mathbf{v}) \quad \text{s.t.} \quad G_v \mathbf{v} + G_x x \leqslant g, \tag{8}$$

with $V(x, \mathbf{v}, 0) = \|H_x x + H_v \mathbf{v}\|_2^2$.

This optimization problem is of high complexity because the evaluation of the max function is an NP-Hard problem. The maximum of a convex function is found in the boundary of the feasible region, thus the maximum will be attained at least at one of the vertices of the polyhedron W_N (see [4, Theorem 3.4.6]). Without any prior knowledge, to evaluate $V^*(x, \mathbf{v})$, all the vertices have to be explored and 2^{Nn_w} evaluations of a quadratic function are required. This implies that the complexity of evaluating the max function grows exponentially with the prediction horizon ($\sim 2^{Nn_w}$), rendering the computation of the optimal control sequence $\mathbf{v}^*$ a very difficult task.

In this work, we present an efficient way of solving the min–max problem for systems in which all the elements of matrix M are non-negative. This implementation is based on the well known equivalence of quadratic maximization problems with min-cut problems. Taking advantage of this equivalence, $V^*(x, \mathbf{v})$ can be evaluated using a linear programming (LP) minimization problem and hence, the min–max problem is solved using a single QP problem. It is also proposed an approximate formulation of the problem that can be applied to any system.

4. Network flow problems

In this section, network maximum flow and minimum cut problems are presented. These optimization problems are of interest for robust MPC because there exists a direct relation between them and problem (7) (see [28,19]). Network flow problems are classic optimization problems and there are efficient optimization algorithms to solve them (see for example [11,10]). Based on this relation, the min–max problem $P(x)$ can be solved by means of a single quadratic programming (QP) problem.

A network is an information structure made of nodes and arcs that interconnect them. They can be represented as graphs. We consider transport networks, where each arc is characterized by a capacity that defines the maximum flow that can be transported through it. Transport networks are defined by $\mathscr{G}=[\mathscr{N}, C]$ where $\mathscr{N}$ represents the nodes $n_0, \ldots, n_{n+1}$ and C represents the arcs connecting these nodes. For any two nodes n_i and n_j, c_{ij} is the capacity of the arc between both of them. The node n_0 is denoted as the source. The node n_{n+1} is denoted as the sink. The rest of the nodes are denoted as interior nodes. The capacities of the arcs that arrive to or depart from the source ($c_{0,i}$ or $c_{i,0}$) are denoted as source capacities. The capacities of the arcs that arrive to or depart from the sink ($c_{n+1,i}$ or $c_{i,n+1}$) are denoted as sink capacities. The rest of the capacities are denoted as interior capacities. Fig. 1 presents a network where the arcs with null capacity are not shown.

In the maximum flow problem, the unknown is the flow distribution of the network. A flow distribution $F=[f_{ij}]$ defines

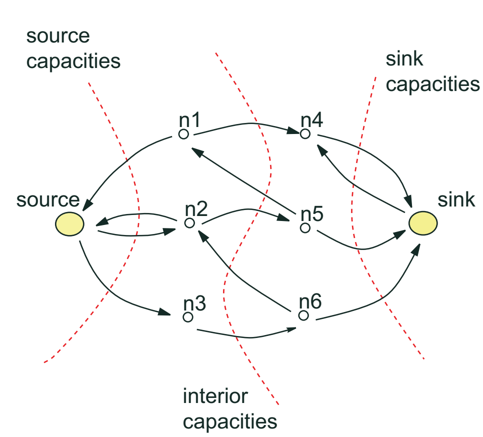

Fig. 1. Example of a network.

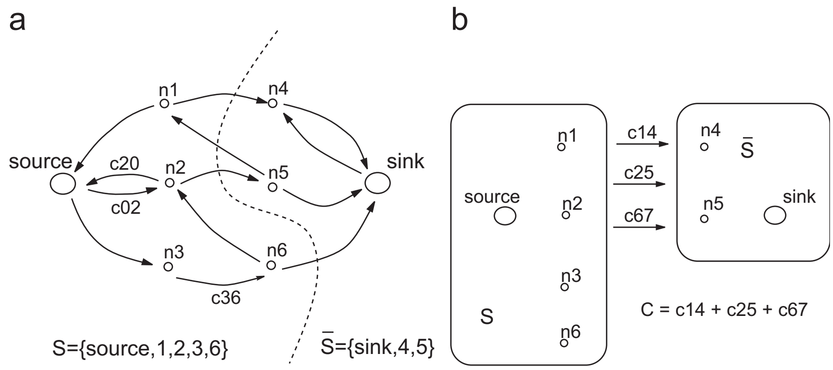

Fig. 2. (a) Example of a cut. (b) Capacity of a cut.

the flow that is transported through a given network. The entry f_{ij} denotes the flow from node n_i to node n_j. It has to be lower than the arc capacity c_{ij}. A feasible flow distribution must also satisfy the flow conservation law. This law states that the flow transported through a network enters only through the source and leaves only from the sink. In this way; for any interior node, the total flow that enters the node must be equal to the total flow that leaves the node. The total flow leaving the source must be equal to the total flow entering the sink. This quantity defines the flow that passes through the network. The max-flow problem is formulated as finding the maximum flow that can be transported from source to sink through a given network.

The max-flow problem can be posed as an LP problem as follows:

$$\begin{aligned} \mathscr{F}^*_{\mathscr{N}} = \max_F \quad & \sum_{i=0}^{n+1} f_{0,i} - \sum_{i=0}^{n+1} f_{i,0} \\ \text{s.t.} \quad & 0 \leqslant f_{ij} \leqslant c_{ij}, \quad \forall i, j, \\ & \sum_{i=0}^{n+1} f_{ij} = \sum_{i=0}^{n+1} f_{ji}, \quad 1 \leqslant j \leqslant n, \\ & \sum_{i=0}^{n+1} f_{0,i} - \sum_{i=0}^{n+1} f_{i,0} = \sum_{i=0}^{n+1} f_{i,n+1} - \sum_{i=0}^{n+1} f_{n+1,i}. \end{aligned} \tag{9}$$

All the capacities are assumed to be positive. The constraints of the problem model the constraint on the maximum flow that can be transported through an arc and the flow conservation law. A flow by definition must be positive or zero. The dual of this problem is finding the minimum cut of the network which can be described as follows:

Let n_0 and n_{n+1} be source and sink respectively for $\mathscr{G}$. A cut separating n_0 and n_{n+1} can be then defined as any node partition $(S, \bar{S})$ where $n_0 \in S$ and $n_{n+1} \in \bar{S}$, $S \cup \bar{S} = \mathscr{N}$, and $S \cap \bar{S} = \emptyset$. Fig. 2(a) shows an example of a cut for the given network. The capacity of $(S, \bar{S})$ is defined as

$$\mathscr{C}_{\mathscr{N}}(S, \bar{S}) = \sum_{n_i \in S} \sum_{n_j \in \bar{S}} c_{ij},$$

that is, the capacity of all the arcs that connect a node of S with a node of $\bar{S}$. As the flow conservation law must hold for a feasible flow distribution, any capacity of a cut of a network is an upper bound of the maximum flow. This is a well-known result and follows from the duality of both problems, see [14,13] and Fig. 2(b). The minimum cut separating source and sink is defined as the cut with minimum capacity.

In the minimum cut problem, the unknown is a cut. All cuts can be represented by a binary vector $[\theta_0, \ldots, \theta_{n+1}]^{\mathrm{T}}$ where $\theta_i = -1$ if $n_i \in S$ and $\theta_i = 1$ if $n_i \in \bar{S}$. By definition, $\theta_0 = -1$ and $\theta_{n+1} = 1$. The capacity of a cut is given by the following expression:

$$\mathscr{C}_{\mathscr{N}}(\theta) = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} c_{ij} \frac{1-\theta_i}{2} \frac{1+\theta_j}{2}.$$

The min-cut problem is formulated as follows:

$$\begin{aligned} \mathscr{C}^*_{\mathscr{N}} = & \min_{\theta_i \in \{-1,1\}} \quad \mathscr{C}_{\mathscr{N}}(\theta) \\ & \text{s.t.} \quad \theta_0 = -1, \quad \theta_{n+1} = 1. \end{aligned} \tag{10}$$

Duality of the min-cut and max-flow problems implies that:

$$\mathscr{F}^*_{\mathscr{N}} = \mathscr{C}^*_{\mathscr{N}}.$$

This is a well-known result in network theory. When all the capacities are non-negative, these problems can be solved efficiently and algorithms are given [14,13]. However, if there are positive and negative capacities, both problems are known to be NP-Hard. In the following section, it is proved that network flow problems are equivalent to the maximization problem (7), and that if a given condition holds, the equivalent networks associated have all the capacities non-negative. In this case, it is possible to solve problem (7) by means of the solution of problem (9).

5. Equivalence of network and min–max problems

In this section, the maximization problem (7) is shown to be equivalent to a min-cut network problem. This is a well known result due to Picard and Ratliff in 1974 (see [28,19]). However, to the best knowledge of the authors, this result has not been applied in the context of min-max MPC. Using this equivalence, $V^*(x, \mathbf{v})$ can be evaluated solving an LP problem.

Theorem 1. *Given a quadratic function defined by* $M \geqslant 0$, *and vector* q, *finding a network* $\mathcal{G} = [\mathcal{N}, C]$ *and a constant* e *which satisfy*:

$$\max_{\|w\|_\infty \leqslant 1} \boldsymbol{w}^{\mathrm{T}} M \boldsymbol{w} + q^{\mathrm{T}} \boldsymbol{w} = e - \mathcal{C}^*_{\mathcal{N}},$$

is always possible. Moreover, if all the elements of M *are non-negative, then it is possible to find a network in which all the capacities are non-negative.*

Proof. A cut can be represented by a binary vector $[\theta_0, \ldots, \theta_{n+1}]^{\mathrm{T}}$ where $\theta_i = -1$ if $n_i \in S$ and $\theta_i = 1$ if $n_i \in \bar{S}$. By definition, $\theta_0 = -1$ and $\theta_{n+1} = 1$. The capacity of a cut is given by

$$4\mathcal{C}_{\mathcal{N}}(\theta) = \sum_{i=0}^{n+1}\sum_{j=0}^{n+1} c_{ij}(1-\theta_i)(1+\theta_j).$$

Let us define $\mathbf{w} = [w_1 \ldots w_n]^{\mathrm{T}}$, where $w_i = \theta_i$ for all $i = 1, \ldots, n$, and $\theta = [-1 \mathbf{w}^{\mathrm{T}} 1]^{\mathrm{T}}$. Taking into account that $\theta_0 = -1$, $\theta_{n+1} = 1$ and $\theta_i = w_i$ the following equalities hold:

$$\begin{aligned}
4\mathcal{C}_{\mathcal{N}}\left(\begin{bmatrix}-1\\ \mathbf{w}\\ 1\end{bmatrix}\right)
&= 4c_{0,n+1} + 2\sum_{j=1}^{n} c_{0,j}(1+w_j) + 2\sum_{i=1}^{n} c_{i,n+1}(1-w_i)\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(1-w_i)(1+w_j)\\
&= 4c_{0,n+1} + 2\sum_{i=1}^{n}(c_{0,i}(1+w_i) + c_{i,n+1}(1-w_i))\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}(1-w_i)(1+w_j)\\
&= 4c_{0,n+1} + 2\sum_{i=1}^{n}(c_{0,i} + c_{i,n+1}) + 2\sum_{i=1}^{n}(c_{0,i} - c_{i,n+1})w_i\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}w_i + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}w_j\\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}w_iw_j.
\end{aligned}$$

Thus,

$$\begin{aligned}
4\mathcal{C}_{\mathcal{N}}\left(\begin{bmatrix}-1\\ \mathbf{w}\\ 1\end{bmatrix}\right) &= 4c_{0,n+1} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} + 2\sum_{i=1}^{n}(c_{0,i}+c_{i,n+1})\\
&\quad + \sum_{i=1}^{n}\left(2(c_{0,i}-c_{i,n+1}) + \sum_{j=1}^{n}(c_{ji}-c_{ij})\right)w_i\\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}w_iw_j,
\end{aligned} \tag{11}$$

and so

$$\begin{aligned}
\mathbf{w}^{\mathrm{T}} M \mathbf{w} + q^{\mathrm{T}}\mathbf{w} &= \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}w_iw_j\\
&\quad + \sum_{i=1}^{n} q_iw_i = e - \mathcal{C}_{\mathcal{N}}\left(\begin{bmatrix}-1\\ \mathbf{w}\\ 1\end{bmatrix}\right),
\end{aligned}$$

if the following equalities are satisfied:

$$4M_{ij} = c_{ij}, \quad i, j = 1, \ldots, n, \tag{12a}$$

$$2q_i = c_{i,n+1} - c_{0,i}, \quad i = 1, \ldots, n, \tag{12b}$$

$$4e = 4c_{0,n+1} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} + 2\sum_{i=1}^{n}(c_{0,i} + c_{i,n+1}). \tag{12c}$$

Note that in these equations it has been taken into account that M is a symmetric matrix. This implies that $c_{ji} - c_{ij} = 0$ for $i, j = 1, \ldots, n$. In order to show that if all the elements of M are non-negative, it is always possible to find a network such that all the capacities are non-negative and (12) is satisfied, consider the following choice of capacities:

$$c_{ij} = 4M_{ij}, \quad i, j = 1, \ldots, n,$$

$$c_{0,i} = \max\{0, -2q_i\}, \quad i = 1, \ldots, n,$$

$$c_{i,n+1} = \max\{0, 2q_i\}, \quad i = 1, \ldots, n,$$

$$c_{i,0} = c_{n+1,i} = 0, \quad i = 1, \ldots, n,$$

$$c_{0,n+1} = c_{n+1,0} = c_{0,0} = c_{n+1,n+1} = 0,$$

$$e = \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} + \frac{1}{2}\sum_{i=1}^{n}(c_{0,i} + c_{i,n+1}). \tag{13}$$

This particular choice of the capacities and the scalar e, guarantees that the constraints of (12) are satisfied and that all the capacities are non-negative. Taking into account that $M \geqslant 0$, the maximum is attained at least at one of the vertices of the polyhedron $\|\mathbf{w}\|_\infty \leqslant 1$ (see [4, Theorem 3.4.6]).

This implies that the following equalities hold:

$$\begin{aligned}\max_{\|\mathbf{w}\|_\infty \leqslant 1} \mathbf{w}^{\mathrm{T}} M \mathbf{w} + q^{\mathrm{T}} \mathbf{w} &= \max_{w_i \in \{-1,1\}} \mathbf{w}^{\mathrm{T}} M \mathbf{w} + q^{\mathrm{T}} \mathbf{w} \\ &= \max_{w_i \in \{-1,1\}} e - \mathscr{C}_{\mathscr{N}}\left(\begin{bmatrix} -1 \\ \mathbf{w} \\ 1 \end{bmatrix}\right) \\ &= e - \min_{w_i \in \{-1,1\}} \mathscr{C}_{\mathscr{N}}\left(\begin{bmatrix} -1 \\ \mathbf{w} \\ 1 \end{bmatrix}\right) \\ &= e - \mathscr{C}^*_{\mathscr{N}}. \qquad \square\end{aligned}$$

Given M and q, there exists an infinite number of capacity distributions C and scalars e such that (12) is satisfied. That is, there is an infinite number of networks such that Theorem 1 holds and that the corresponding solution to the minimum cut network problem provides the solution to the max problem.

Taking into account duality, for a given network with non-negative capacities $\mathscr{F}^*_{\mathscr{N}} = \mathscr{C}^*_{\mathscr{N}}$. This implies that $\mathscr{C}^*_{\mathscr{N}}$ can be evaluated solving (9). It follows that:

$$\begin{aligned}&\max_{\|\mathbf{w}\|_\infty \leqslant 1} \mathbf{w}^{\mathrm{T}} M \mathbf{w} + q^{\mathrm{T}} \mathbf{w} \\ &= \min_{C.e.F} \quad e - \left(\sum_{i=0}^{n+1} f_{0,i} - \sum_{i=o}^{n+1} f_{i,0}\right) \\ &\text{s.t.} \quad 4M_{ij} = c_{ij}, \quad i, j = 1, \ldots, n, \\ &\qquad 2q_i = c_{i,n+1} - c_{0,i}, \\ &\qquad\qquad i = 1, \ldots, n, \\ &4e = 4c_{0,n+1} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} \\ &\qquad + 2\sum_{i=1}^{n}(c_{0,i} + c_{i,n+1}), \\ &0 \leqslant f_{ij} \leqslant c_{ij}, \quad \forall i, j, \\ &\sum_{i=0}^{n+1} f_{ij} = \sum_{i=0}^{n+1} f_{ji}, \quad 1 \leqslant j \leqslant n, \\ &\sum_{i=0}^{n+1} f_{0,i} - \sum_{i=0}^{n+1} f_{i,0} \\ &= \sum_{i=0}^{n+1} f_{i,n+1} - \sum_{i=0}^{n+1} f_{n+1,i}.\end{aligned} \tag{14}$$

In this problem, the number of nodes is fixed by the dimension of the uncertainty vector $\mathbf{w}$, while the capacities of the network are free variables that must satisfy a set of linear constraints. These constraints are defined by (12). A variable e is also added to the optimization problem. For a feasible set of constraints, Theorem 1 holds and $V^*(x, \mathbf{v}) = e - \mathscr{C}^*_{\mathscr{N}}$.

Solving the quadratic maximization problem using a LP problem is a huge advantage. Linear programming is a mature field and there are plenty of efficient solvers available. In Table 1 the computation time for different problems is shown. The results have been obtained from the solution of 200 hundred random problems. Entry T_{mm} is the mean time to evaluate all the vertices, while T_{mf} is the mean time of evaluating problem (14) using GLPK [21]. It can be seen that there is a large difference that grows in an exponential manner with the prediction horizon.

Note that because of duality, any feasible solution of the LP problem gives a higher value than the maximum of the quadratic function, so to evaluate this maximum, a minimization has to be done. This change of sense of the optimization problem allows one to solve the min-max problem, as a single minimization problem, in particular, a QP problem. This formulation is detailed in the following subsection.

5.1. Quadratic formulation

In this paper, the min–max problem is solved using a single quadratic programming (QP) problem. The size (i.e., number of variables and constraints) of this problem is a quadratic function of the prediction horizon. Note that quadratic programming solvers are nowadays a wide-spread technology. The main idea is to solve (7) and (8) in a single optimization problem. Taking into account (14), it can be seen that because the maximization is solved minimizing a linear cost function subject to linear constraints, problems (7) and (8) can be solved simultaneously with a single quadratic problem. Taking into account (14) and the definition of $V(x, \mathbf{v}, 0)$ it follows that $P(x)$ is equivalent to the following QP problem:

$$\begin{aligned}J^*(x) = \min_{\mathbf{v},\mathscr{C},e,F} \quad & \|H_x x + H_v \mathbf{v}\|_2^2 + e - \left(\sum_{i=0}^{n+1} f_{0,i} - \sum_{i=0}^{n+1} f_{i,0}\right) \\ \text{s.t.} \quad & G_v \mathbf{v} + G_x x \leqslant g, \\ & 4M_{ij} = c_{ij}, \quad i, j = 1, \ldots, n, \\ & 2q_i = c_{i,n+1} - c_{0,i}, \quad i = 1, \ldots, n, \\ & 4e = 4c_{0,n+1} + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} \\ &\qquad + 2\sum_{i=1}^{n}(c_{0,i} + c_{i,n+1}), \\ & 0 \leqslant f_{ij} \leqslant c_{ij}, \quad \forall i, j, \\ & \sum_{i=0}^{n+1} f_{ij} = \sum_{i=0}^{n+1} f_{ji}, \quad 1 \leqslant j \leqslant n, \\ & \sum_{i=0}^{n+1} f_{0,i} - \sum_{i=0}^{n+1} f_{i,0} = \sum_{i=0}^{n+1} f_{i,n+1} - \sum_{i=0}^{n+1} f_{n+1,i}.\end{aligned} \tag{15}$$

The size of this QP problem depends in a quadratic manner with the prediction horizon. Eq. (13) states that $c_{ij} = 4M_{ij}$, $\forall i, j = 1, \ldots, n$, $c_{i,0} = c_{n+1,i} = 0$, $\forall i = 1, \ldots, n$ and $c_{0,n+1} =$

Table 1
Mean computation time for evaluating a quadratic maximization problem of dimension n

n	1	3	5	7	9	11
T_{mm} (s)	0	0.0002	0.0003	0.0005	0.0023	0.0064
T_{mf} (s)	0.0013	0.0020	0.0027	0.0050	0.0063	0.0102
n	13	15	20	25	30	35
T_{mm} (s)	0.0335	0.2044	70.7810	–	–	–
T_{mf} (s)	0.0154	0.0256	0.0545	0.0813	0.0913	0.1931

$c_{n+1,0} = c_{0,0} = c_{n+1,n+1} = 0$ provide a feasible set of capacities that satisfies (12). Because the flow must be non-negative, for this choice of capacities the flow variables satisfy $f_{i,0} = f_{n+1,i} = 0$, $\forall i = 1, \ldots, n$ and $f_{0,n+1} = f_{n+1,0} = f_{0,0} = f_{n+1,n+1} = 0$. It follows that the number of variables needed to evaluate the max function can be reduced to $n^2 + 4n + 1$ where n is the dimension of the future uncertainty vector (Nn_w). Note that being able to compute the whole min–max problem as a single minimization problem broadens the class of systems to which min–max MPC can be applied.

6. Generalization of the approach

The proposed implementation of the min–max MPC controller can only be used if matrix M (which describes the quadratic dependence of the value function $V(x, \mathbf{v}, \mathbf{w})$ with respect to $\mathbf{w}$) has only non-negative elements. This necessary condition does not hold in general. This matrix depends on the parameters of the system and the controller. One family of systems where the necessary condition is not too restrictive are SiSo systems. In this case, the elements of M are all non-negative if the system has a constant sign impulse response with respect to the additive uncertainty. In this paper, we propose a modification of the functional that allows us to use this approach for any system while preserving the stability and robustness properties of the original controller. The new functional will be denoted $\tilde{V}(x, \mathbf{v}, \mathbf{w})$ and differs from the original one only in a quadratic term on $\mathbf{w}$.

$$\tilde{V}(x, \mathbf{v}, \mathbf{w}) = V(x, \mathbf{v}, \mathbf{w}) + \mathbf{w}^{\mathrm{T}} S \mathbf{w} = V(x, \mathbf{v}, 0) + \mathbf{w}^{\mathrm{T}}(M + S)\mathbf{w} + q(x, \mathbf{v})^{\mathrm{T}}\mathbf{w}.$$

The modified max function is denoted

$$\tilde{V}^*(x, \mathbf{v}) = \max_{\|\mathbf{w}\|_\infty \leqslant 1} \mathbf{w}^{\mathrm{T}}(M + S)\mathbf{w} + q(x, \mathbf{v})^{\mathrm{T}}\mathbf{w}.$$

The new min–max problem $P(x)$ is stated as

$$\tilde{J}^*(x) = \min_{\mathbf{v}} \quad V(x, \mathbf{v}, 0) + \tilde{V}^*(x, \mathbf{v}) \quad \text{s.t.} \quad G_v \mathbf{v} + G_x x \leqslant g. \tag{16}$$

Note that the feasibility regions of $P(x)$ and $\tilde{P}(x)$ are the same because the two problems are subject to the same set of constraints and so, robust constraint satisfaction of the closed-loop system is guaranteed. In what follows, a procedure to obtain a matrix S which minimizes a bound on the error is presented. In [2] it is proved that the modified control law maintains the robustness and stability properties of the original controller from a qualitative point of view. Using this approach, it is possible to apply an approximate min–max control law on any given system.

Theorem 2. (Alamo et al. [2], Theorem 2). *Compute $S \geqslant 0$ and diagonal matrix $T \geqslant 0$ that solve the following LMI problem*:

$$\min_{S,T} \quad \text{trace } T$$
$$\text{s.t.} \quad 0 \leqslant S \leqslant T$$
$$M_{i,j} + S_{i,j} \geqslant 0, \quad \forall i, \forall j$$

Then making $\sigma = trace\ T$ it results that $\tilde{M} = M + S$ has non-negative elements and:

$$V^*(x, \mathbf{v}) \leqslant \tilde{V}^*(x, \mathbf{v}) \leqslant V^*(x, \mathbf{v}) + \sigma.$$

Proof. As $S \geqslant 0$, it results that

$$\mathbf{w}^{\mathrm{T}} M \mathbf{w} \leqslant \mathbf{w}^{\mathrm{T}}(M + S)\mathbf{w}, \quad \forall \mathbf{w}.$$

This implies that $V^*(x, \mathbf{v}) \leqslant \tilde{V}^*(x, \mathbf{v})$. On the other hand, as $S \leqslant T$, and T is a diagonal matrix it holds

$$\mathbf{w}^{\mathrm{T}}(M + S)\mathbf{w} \leqslant \mathbf{w}^{\mathrm{T}}(M + T)\mathbf{w}, \quad \forall \mathbf{w}.$$

Taking into account that

$$\max_{\|\mathbf{w}\|_\infty \leqslant 1} \mathbf{w}^{\mathrm{T}} T \mathbf{w} = \mathrm{trace}\, \mathrm{T} = \sigma,$$

it follows that, $\tilde{V}^*(x, \mathbf{v}) \leqslant V^*(x, \mathbf{v}) + \sigma$. □

Using this approximation, in [2] is proved that σ is a bound on the error on the original optimization problem, i.e.

$$J^*(x) \leqslant \tilde{J}^*(x) \leqslant J^*(x) + \sigma.$$

Note that a priori bounds on σ cannot be provided. The bound on the approximation error, depends on the parameters of the system and the controller which characterize the elements of matrix M.

7. An illustrative example

Let us consider the linear uncertain system:

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_k,$$

Table 2
Comparison of the proposed implementation with the vertex exploration implementation for different prediction horizons

N	7	8	9	10	11	12
σ	0	0	0.0275	0.0751	0.1953	0.2584
T_{mm}(s)	0.40	0.47	1.06	1.83	3.42	8.68
T_{mf}(s)	0.004	0.004	0.005	0.006	0.012	0.012
Error **v**	0	0	0.0002	0.0008	0.0004	0.0012
Error J^* %	0	0	0.01	0.04	0.09	0.17
N	13	14	15	20	25	30
σ	0.3218	0.3818	0.4411	0.5279	0.7522	0.9765
T_{mm}(s)	16.84	51.98	140.23	–	–	–
T_{mf}(s)	0.012	0.012	0.012	0.0422	0.0984	0.1266
Error **v**	0.0014	0.0039	0.0027	–	–	–
Error J^*%	0.23	0.24	0.25	–	–	–

where both the state and the control action are constrained, namely $\|x_k\|_\infty \leqslant 5$ and $|u_k| \leqslant 5$. The uncertainty is bounded, $\|w_k\| \leqslant \varepsilon$, $\varepsilon = 1$. The objective function is defined by matrices $Q = I$ and $R = 1$. The control gain matrix $K = [-0.4221 - 1.2439]$ corresponds to an LQR control law. The terminal region Ω is chosen as the maximal robust invariant set of the system for K. The terminal cost function is defined by

$$P = \begin{bmatrix} 4.0696 & 3.8641 \\ 3.8641 & 6.6199 \end{bmatrix}.$$

In Table 2, the proposed approximate implementation (16) evaluated using a single QP problem, has been compared with the original min–max controller evaluated using the Matlab function *fmincon* to minimize the max function, which is computed evaluating all possible vertices. The comparison has been done for different prediction horizons over a hundred different feasible states. The results of Table 2 present the mean values over these different experiments. As it is stated in Theorem 2, the difference between the optimum value $J^*(x)$ and the one obtained with the optimal solution of the modified problem is bounded by σ. Note that for $N \leqslant 8$, there is no approximation error. Entry "Error **v**" is the error $|v_0^*(x) - \tilde{v}_0^*(x)|$. Entry "Error J^*" is the relative error in the cost function in percentage. Entry "T_{mm}" is the mean time to evaluate all the vertices, while entry "T_{mf}" is the mean time of evaluating the modified version of problem (15) defined by $M + S$ using Cplex [16]. It can be seen that the computation time of the vertex enumeration algorithm grows in an exponential manner with the prediction horizon. This is not the case for the quadratic formulation. In this case, the size of the QP problem grows polynomially with the prediction horizon.

8. Conclusions

In this paper, an efficient implementation of an MPC robust controller has been presented. If matrix M has only nonnegative elements, the computational burden is polynomial with the control and prediction horizon, while the original problem has an exponential complexity if the maximization is done by means of an extensive vertex search. In fact, the resulting optimization problem is a QP problem, which allows one to use the efficient solvers available and even implement the controller using the explicit solution of the QP problem. This QP problem has a number of variables and constraints that grows, not exponentially, but in a quadratic manner with the prediction horizon. A modified controller has been proposed for those systems that do not satisfy the condition required to solve the problem in polynomial time. This modified controller can be shown to preserve stability. The proposed implementation broadens the family of real plants to which a min–max MPC control can be applied.

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