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Control of a pilot plant using QP based min–max predictive control

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ABSTRACT

The practical implementation of min–max MPC (MMMPC) is limited by the computational burden required to compute the control law. This problem can be circumvented by using approximate solutions or upper bounds of the worst possible case of the performance index. In a previous work, the authors presented a computationally efficient MMMPC control strategy in which a close approximation of the solution of the min–max problem is computed using a quadratic programming problem. In this paper, this approach is validated through its application to a pilot plant in which the temperature of a reactor is controlled. The behavior of the system and the controller are illustrated by means of experimental results.

1. Introduction

In min–max model predictive controllers (MMMPC), the control signal is computed for the worst case of a cost function that considers the effect of process model uncertainties and disturbances in the controller performance (Campo & Morari, 1987). The main drawback of this approach is the computational burden that takes to compute the control signal. This usually involves the solution of a NP-hard min–max problem (Lee & Yu, 1997; Scokaert & Mayne, 1998). As a result, the number of applications of these control strategies is very small, even when there is evidence that they work better than standard predictive controllers in processes with uncertain dynamics (Camacho & Bordons, 2004).

Multi-parametric programming has been applied to show that the MMMPC control law is piecewise affine when a quadratic (Ramirez & Camacho, 2006) or 1-norm based criterion (Bemporad, Borrelli, & Morari, 2003; Kerrigan & Maciejowski, 2004) is used as the cost function. Thus, explicit forms of the control law can be built. Such explicit forms can be evaluated very fast provided that the complexity of the state space partition is moderate, which is the case for many applications. However, if the process model or the controller tuning parameters change, the computation of the controller has to be redone.

A common solution to the computational burden issue is to use an upper bound of the worst case cost instead of computing it explicitly. This upper bound can be computed by using linear matrix inequalities (LMI) techniques such as in Kothare, Balakrishnan, and Morari (1996) and Lu and Arkun (2000). However, the LMI problems have a computational burden that cannot be neglected in certain applications. In Alamo, Ramírez, Muñoz de la Peña, and Camacho (2007) a different approach based on a computationally cheap upper bound of the worst case cost is presented. In that work, the MMMPC strategy computes a close approximation of the exact solution of the min–max problem using quadratic programming (QP). The resulting computational burden using the upper bound of the worst case cost is much lower than the one solving the exact min–max problem and can be compared to that of a standard constrained MPC based on a quadratic cost. Thus, it can be easily implemented in almost any platform capable to run a constrained MPC. Also, stability of the proposed approach is guaranteed.

In this work, the approach presented in Alamo et al. (2007) has been validated by means of its application to a pilot plant. The pilot plant is used to emulate an exothermic chemical reaction with nonlinear dynamics. This process has been used in previous works (Gruber & Bordons, 2007; Gruber, Ramirez, Alamo, Bordons, & Camacho, 2008) as a benchmark system, allowing a direct comparison of the obtained results with the ones of other linear and nonlinear model predictive control strategies. With the MMMPC, based on an identified linear model, controlling a nonlinear process the robustness of the proposed control strategy can be shown by means of experimental results. The computational complexity of the optimization problem is reduced considerably using an upper bound of the worst case cost. The reduction of the complexity of the optimization problem leads to a low computational burden of the control strategy applied to the pilot plant and allows realistic values for the control and prediction horizons (i.e., the parameters on which the computational burden depends). In the experiments, restrictions in the control action and the output have been considered. It is also noteworthy that the computer on which the predictive control

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algorithm is implemented does not have sufficient calculation power to implement a conventional min–max predictive control strategy. Therefore a computationally efficient strategy as that used in this work is a good choice if the use of this type of control is desired.

The results obtained show the validity of the control strategy. Not only good results have been obtained in set point tracking and disturbance rejection, but also, the performance achieved is better than that obtained with a regular predictive controller based on the same type of prediction model.

The paper is organized as follows. Section 2 presents the MMMPC strategy. Section 3 presents the proposed implementation strategy. In Section 4 a detailed description of the used pilot plant is given. The strategy is illustrated by means of experimental results of the pilot plant in Section 5. Finally, Section 6 presents some conclusions.

2. Min–max MPC with bounded additive uncertainties

Consider the following state space model with bounded additive uncertainties:

$$x(t+1) = Ax(t) + Bu(t) + D\theta(t) \tag{1}$$

with $x(t) \in \mathbb{R}^{d_x}$ the state vector, $u(t) \in \mathbb{R}^{d_u}$ the input vector and $\theta(t) \in \{\theta \in \mathbb{R}^{d_\theta} : \|\theta\|_\infty \le \varepsilon\}$ the uncertainty, that is supposed to be bounded. The system is subject to p state and input time invariant constraints $F_u u(t) + F_x x(t) \le g$ where $F_u \in \mathbb{R}^{p\times d_u}$ and $F_x \in \mathbb{R}^{p\times d_x}$. It is assumed a semi-feedback approach in which the control input is given by

$$u(t) = -Kx(t) + v(t) \tag{2}$$

where the feedback matrix K is chosen to achieve some desired property such as nominal stability or LQR optimality without constraints. The MMMPC controller will compute the optimal sequence of correction control inputs $v(t)$. The state equation of system (1) can be rewritten as

$$x(t+1) = A_{CL}x(t) + Bv(t) + D\theta(t) \tag{3}$$

with $A_{CL} = (A - BK)$.

The proposed strategy also works without semi-feedback approach (i.e., $u(t) = v(t)$). All the computational advantages of the strategy remain the same and the procedures described here can be used without any modification. Furthermore if the process is open-loop stable (as in the case of the pilot plant used in this work) the stabilizing conditions, which will be discussed later, can be used without problems.

The cost function is a quadratic performance index

$$V(x,\mathbf{v},\boldsymbol{\theta}) = \sum_{j=0}^{N-1} x(t+j|t)^T Q x(t+j|t) + \sum_{j=0}^{N-1} u(t+j|t)^T R u(t+j|t) + x(t+N|t)^T P x(t+N|t) \tag{4}$$

where $x(t|t) = x$, $x(t+j|t)$ is the prediction of the state for $t+j$ made at t and $u(t+j|t) = -Kx(t+j|t) + v(t+j|t)$. Note that these values depend on the future values of the uncertainty. The sequence of future values of $\theta(t)$ over a prediction horizon N is denoted by $\boldsymbol{\theta} = [\theta(t)^T, \ldots, \theta(t+N-1)^T]^T$, and $\boldsymbol{\Theta} = \{\boldsymbol{\theta} \in \mathbb{R}^{N\cdot d_\theta} : \|\boldsymbol{\theta}\|_\infty \le \varepsilon\}$ is the set of possible uncertainty trajectories. On the other hand, $\mathbf{v} = [v(t|t)^T, \ldots, v(t+N-1|t)^T]^T$ is the control correction sequence. Matrices $Q, P \in \mathbb{R}^{d_x\times d_x}$ and $R \in \mathbb{R}^{d_u\times d_u}$ are symmetric positive definite matrices used as weighting parameters.

Min–max MPC (Campo & Morari, 1987) minimizes the cost function for the worst possible case of the predicted future evolution of the process state or output signal. This is accomplished through the solution of a min–max problem

$$\mathbf{v}^*(x) = \arg\min_{\mathbf{v}} \max_{\theta\in\Theta} V(x,\mathbf{v},\boldsymbol{\theta})$$

$$\text{s.t.}\quad F_u u(t+j|t) + F_x x(t+j|t) \le g$$
$$j = 0,\ldots,N, \quad \forall \boldsymbol{\theta} \in \Theta$$
$$x(t+N|t) \in \Omega, \quad \forall \boldsymbol{\theta} \in \Theta \tag{5}$$

A terminal region constraint $x(t+N|t) \in \Omega$, where Ω is a polyhedron, is included to assure stability of the control law (Mayne, Rawlings, Rao, & Scokaert, 2000).

The predictions $x(t+j|t)$ and control actions $u(t+j|t)$ depend linearly on x, $\mathbf{v}$ and $\boldsymbol{\theta}$. This means that it is possible to find a vector $d \in \mathbb{R}^p$ and matrices G_x, G_v and G_θ (Camacho & Bordons, 2004), such that all the robust linear constraints of problem (5) can be rewritten as:

$$G_x^i x + G_v^i \mathbf{v} + G_\theta^i \boldsymbol{\theta} \le d_i, \quad i = 1,\ldots,p, \ \forall \boldsymbol{\theta} \in \Theta \tag{6}$$

where G_x^i, G_v^i, G_θ^i denote the i-th rows of G_x, G_v and G_θ respectively and d_i is the i-th component of $d \in \mathbb{R}^p$. Denote now $\|G_\theta^i\|_1$ the sum of the absolute values of row G_θ^i. Taking into account that $\max_{\theta\in\Theta} G_\theta^i \boldsymbol{\theta} = \max_{\|\boldsymbol{\theta}\|_\infty\le\varepsilon} G_\theta^i \boldsymbol{\theta} = \varepsilon\|G_\theta^i\|_1$, the robust fulfillment of the constraints is satisfied if and only if $G_x^i x + G_v^i \mathbf{v} + \varepsilon\|G_\theta^i\|_1 \le d_i$, $i = 1,\ldots,p$. Therefore, to guarantee robust constraint satisfaction, the set of linear constraints $G_x x + G_v \mathbf{v} \le d_\varepsilon$ must be satisfied, where the i-th component of d_ε is equal to $d_i - \varepsilon\|G_\theta^i\|_1$. Note that this is a necessary and sufficient condition.

Taking into account (2)–(4), the cost function can be evaluated as a quadratic function:

$$V(x,\mathbf{v},\boldsymbol{\theta}) = \mathbf{v}^T M_{vv}\mathbf{v} + \boldsymbol{\theta}^T M_{\theta\theta}\boldsymbol{\theta} + 2\boldsymbol{\theta}^T M_{\theta v}\mathbf{v} + 2x^T M_{vf}^T \mathbf{v} + 2x^T M_{\theta f}^T \boldsymbol{\theta} + x^T M_{ff} x \tag{7}$$

where the matrices can be obtained from the system and the control parameters (Camacho & Bordons, 2004). Due to the convexity properties of $V(x,\mathbf{v},\boldsymbol{\theta})$, problem (5) is equivalent to

$$\mathbf{v}^*(x) = \arg\min_{\mathbf{v}} \max_{\theta\in\text{vert}(\boldsymbol{\Theta})} V(x,\mathbf{v},\boldsymbol{\theta})$$
$$\text{s.t.}\quad G_x x + G_v \mathbf{v} \le d_\varepsilon \tag{8}$$

where vert($\boldsymbol{\Theta}$) is the set of vertices of $\boldsymbol{\Theta}$ (Camacho & Bordons, 2004).

The terminal region Ω is assumed to satisfy the following conditions:

- **C1**: If $x \in \Omega$ then $A_{CL}x + D\theta \in \Omega$, for every $\theta \in \{\theta \in \mathbb{R}^{d_\theta} : \|\theta\|_\infty \le \varepsilon\}$.
- **C2**: If $x \in \Omega$ then $u(x) = -Kx \in U$, where $U \triangleq \{u : F_u u + F_x x \le g\}$.

Moreover, matrix P that characterizes the terminal cost is assumed to satisfy

- **C3**: $P - A_{CL}^T P A_{CL} > Q + K^T R K$.

The stability of A_{CL} guarantees the existence of a positive definite matrix P satisfying C3.

The maximum cost for a given x and $\mathbf{v}$ is denoted as

$$V^*(x,\mathbf{v}) = \max_{\theta\in\text{vert}(\boldsymbol{\Theta})} V(x,\mathbf{v},\boldsymbol{\theta}) = V(x,\mathbf{v},0) + \max_{\theta\in\text{vert}(\Theta)} \boldsymbol{\theta}^T H\boldsymbol{\theta} + 2\boldsymbol{\theta}^T q(x,\mathbf{v}) \tag{9}$$

where $H = M_{\theta\theta}$, $q(x,\mathbf{v}) = M_{\theta v}\mathbf{v} + M_{\theta f}x$ and $V(x,\mathbf{v},0) = \mathbf{v}^T M_{vv}\mathbf{v} + 2x^T M_{vf}^T\mathbf{v} + x^T M_{ff}x$ is the part of the cost that does not depend on the uncertainty (that is, the nominal cost). With this definition, problem (8) can be rewritten as

$$\mathbf{v}^*(x) = \arg\min_{\mathbf{v}} V^*(x,\mathbf{v})$$
$$\text{s.t.} \quad G_x x + G_v \mathbf{v} \le d_\varepsilon \tag{10}$$

and the system is controlled by $K_{MPC}(x(t)) = -Kx(t) + v^*(t|t)$, where $\mathbf{v}^*(x(t)) = [v^*(t|t)^T, \ldots, v^*(t+N-1|t)^T]^T$.

3. A QP approach to min–max MPC

In this section the main results of Alamo et al. (2007) are presented briefly. In that work, it is shown how the min–max problem (10) can be replaced by a tractable QP problem which provides a close approximation of the solution of the original problem. This can be accomplished with the following steps:

(1) Obtain an initial guess of the solution of (10), denoted $\tilde{\mathbf{v}}^*$. As seen later, this can be achieved by solving a QP problem.
(2) Using $\tilde{\mathbf{v}}^*$, obtain a quadratic function of $\mathbf{v}$ that bounds the worst case cost.
(3) Compute the control law. This involves the solution of a QP problem.

The necessary steps to replace the original min–max problem will be detailed in the following sections.

3.1. Computing $\tilde{\mathbf{v}}^*$

Given H defined as in Eq. (9), denote $T_i = \sum_{j=1}^{N\cdot d_\theta} |H_{ij}|$, where H_{ij} denotes the (i,j)-th component of matrix H. Then, define the diagonal matrix T as

$$T = \mathbf{diag}(T_1, \ldots, T_n) \tag{11}$$

where $T \ge H$ (Alamo et al., 2007). With

$$\tilde{V}(x,\mathbf{v},\boldsymbol{\theta}) = V(x,\mathbf{v},0) + \boldsymbol{\theta}^T T\boldsymbol{\theta} + 2q^T(x,\mathbf{v})\boldsymbol{\theta} \tag{12}$$

the maximum of $\tilde{V}(x,\mathbf{v},\boldsymbol{\theta})$ can be computed as

$$\tilde{V}^*(x,\mathbf{v}) = V(x,\mathbf{v},0) + \|H\|_s\varepsilon^2 + 2\varepsilon\|q(x,\mathbf{v})\|_1 \tag{13}$$

where $\|H\|_s$ denotes the sum of the absolute values of the elements of H. This problem can be casted as a QP problem by making use of slack variables to deal with the 1-norm term. An initial guess of the solution of (10) can be obtained as

$$\tilde{\mathbf{v}}^*(x) = \arg\min_{\tilde{\mathbf{v}}} \tilde{V}^*(x,\tilde{\mathbf{v}})$$
$$\text{s.t.} \quad G_x x + G_v \tilde{\mathbf{v}} \le d_\varepsilon \tag{14}$$

3.2. Obtaining an upper bound of the worst case cost

The upper bound of the maximum will be obtained in the following two steps:

3.2.1. Computing the parameter vector $\alpha(\mathbf{v})$

Note that

$$V^*(x,\mathbf{v}) = \max_{\theta\in\text{vert}(\Theta)} \begin{bmatrix}\boldsymbol{\theta}\\ 1\end{bmatrix}^T \begin{bmatrix} H & q(x,\mathbf{v}) \\ q^T(x,\mathbf{v}) & V(x,\mathbf{v},0)\end{bmatrix} \begin{bmatrix}\boldsymbol{\theta}\\ 1\end{bmatrix} = \max_{\|z\|_\infty\le 1} z^T M(\mathbf{v}) z$$

with

$$z = \begin{bmatrix}\dfrac{\boldsymbol{\theta}^T}{\varepsilon} & 1\end{bmatrix}^T, \quad M(\mathbf{v}) = \begin{bmatrix}\varepsilon^2 H & \varepsilon q(x,\mathbf{v}) \\ \varepsilon q^T(x,\mathbf{v}) & V(x,\mathbf{v},0)\end{bmatrix} \in \mathbb{R}^{n\times n}$$

where $n = N\cdot d_\theta + 1$.

The following procedure provides an upper bound of the worst case cost for a given $\mathbf{v}$. It computes $\alpha(\mathbf{v}) = [\alpha_1(\mathbf{v}), \ldots, \alpha_{n-1}(\mathbf{v})]^T$ and a diagonal matrix $\Gamma(\mathbf{v}) \ge M(\mathbf{v})$ such that its trace is an upper bound of the worst case cost for $\mathbf{v}$ (see Property 1 of Alamo et al., 2007).

Procedure 1. *Computation of $\alpha(\mathbf{v}) = [\alpha_1(\mathbf{v}), \ldots, \alpha_{n-1}(\mathbf{v})]^T$ and $\Gamma(\mathbf{v})$.*

(1) *Let $S^{(0)} = M(\mathbf{v}) \in \mathbb{R}^{n\times n}$.*
(2) *For $k = 1$ to $n-1$*
(3) *Let $M_{sub}^{(k-1)} = [S_{ij}^{(k-1)}]$ for $i,j = k,\ldots,n$.*
(4) *Obtain the partition*

$$M_{sub}^{(k-1)} = \begin{bmatrix} a & b^T \\ b & M_r\end{bmatrix}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}^{n-k}$ and $M_r \in \mathbb{R}^{(n-k)\times(n-k)}$.
(5) *Make $\alpha_k(\mathbf{v}) = \sqrt{\|b\|_1}$.*
(6) *If $\alpha_k(\mathbf{v}) = 0$ then $S^{(k)} = S^{(k-1)}$, else*

$$S^{(k)} = S^{(k-1)} + \begin{bmatrix}\mathbf{0}_{k-1,1}^T & \alpha_k(\mathbf{v}) & \dfrac{-b^T}{\alpha_k(\mathbf{v})}\end{bmatrix}^T \begin{bmatrix}\mathbf{0}_{k-1,1}^T & \alpha_k(\mathbf{v}) & \dfrac{-b^T}{\alpha_k(\mathbf{v})}\end{bmatrix}$$

(7) *end for*
(8) *Make $\Gamma(\mathbf{v}) = S^{(n-1)}$.*

Note that in the previous procedure, $\mathbf{0}_{m,n}$ denotes a $(m\times n)$ matrix of zeros. Property 1 of Alamo et al. (2007) shows that the trace of $\Gamma(\mathbf{v})$ constitutes an improved upper bound of $V^*(x,\mathbf{v})$. That is, $V^*(x,\mathbf{v}) \le \text{trace}(\Gamma(\mathbf{v})) \le \tilde{V}^*(x,\mathbf{v})$.

Property 1. *Matrices $S^{(0)}, S^{(1)}, \ldots, S^{(n-1)}$, obtained by means of Procedure 1 satisfy:*

(i) *$S^{(k)}$ is a partially diagonalized matrix. That is, there is a diagonal matrix $T^{(k)} \in \mathbb{R}^{k\times k}$ such that $S^{(k)} = \mathbf{diag}(T^{(k)}, M_{sub}^{(k)})$.*
(ii) *$S^{(n-1)} = \Gamma(\mathbf{v})$ is a diagonal matrix.*
(iii) *$V^*(x,\mathbf{v}) \le \text{trace}(\Gamma(\mathbf{v}))$.*
(iv) *$\|S^{(k)}\|_s \le \|S^{(k-1)}\|_s$.*
(v) *$\text{trace}(\Gamma(\mathbf{v})) \le \tilde{V}^*(x,\mathbf{v})$, $\forall\mathbf{v}$.*

Proof. See Alamo et al. (2007). □

This procedure is the foundation to obtain a QP problem that provides a solution with a worst case cost that is close to the optimal worst case cost but with the advantage of the lower computational burden of a QP problem (see Section 3.2.2).

3.2.2. Obtaining the bound as a quadratic function on $\mathbf{v}$

The diagonalization process shown in Procedure 1 can be used to obtain a matrix denoted by $\hat{\Gamma}(\mathbf{v})$, which allows one to obtain a bound of the maximum that can be computed as a quadratic function of $\mathbf{v}$. This is achieved by means of the following procedure:

Procedure 2. *Obtaining the matrix $\hat{\Gamma}(\mathbf{v})$.*

(1) *Obtain $\tilde{\mathbf{v}}^*$ from the QP problem defined in* (14).
(2) *Compute $\alpha(\tilde{\mathbf{v}}^*)$ by* Procedure 1.
(3) *Let $\hat{S}^{(0)}(\mathbf{v}) = M(\mathbf{v}) \in \mathbb{R}^{n\times n}$.*
(4) *For $k = 1$ to $n-1$.*
(5) *Let $\hat{M}_{sub}(\mathbf{v}) = [\hat{S}_{ij}^{(k-1)}(\mathbf{v})]$ for $i,j = k,\ldots,n$.*
(6) *Obtain the partition*

$$\hat{M}_{sub}(\mathbf{v}) = \begin{bmatrix} a(\mathbf{v}) & b^T(\mathbf{v}) \\ b(\mathbf{v}) & M_r(\mathbf{v})\end{bmatrix}$$

where $a(\mathbf{v}) \in \mathbb{R}$.

(7) *If* $\alpha_k(\tilde{\mathbf{v}}^*) = 0$ *then* $\hat{S}^{(k)}(\mathbf{v}) = \hat{S}^{(k-1)}(\mathbf{v})$, *else*

$$\hat{S}^{(k)}(\mathbf{v}) = \hat{S}^{(k-1)}(\mathbf{v}) + \left[\mathbf{0}_{k-1,1}^T \;\; \alpha_k(\tilde{\mathbf{v}}^*) \;\; \frac{-b^T(\mathbf{v})}{\alpha_k(\tilde{\mathbf{v}}^*)}\right]^T \left[\mathbf{0}_{k-1,1}^T \;\; \alpha_k(\tilde{\mathbf{v}}^*) \;\; \frac{-b(\mathbf{v})^T}{\alpha_k(\tilde{\mathbf{v}}^*)}\right]$$

(8) *end for*
(9) *Make* $\hat{\Gamma}(\mathbf{v}) = \hat{S}^{(n-1)}(\mathbf{v})$.

Theorem 1. *Denote* $\hat{V}^*(x, \mathbf{v}) = \mathrm{trace}\,(\hat{\Gamma}(\mathbf{v}))$. *Then*

(1) $\hat{\Gamma}(\tilde{\mathbf{v}}^*) = \Gamma(\tilde{\mathbf{v}}^*)$.
(2) $\hat{V}^*(x, \mathbf{v})$ *is a quadratic function on* $\mathbf{v}$.
(3) $V^*(x, \mathbf{v}) \le \hat{V}^*(x, \mathbf{v})$.

Proof. See Alamo et al. (2007). □

Denote $\hat{V}^*(x, \mathbf{v}) = \mathrm{trace}\,(\hat{\Gamma}(\mathbf{v}))$. Theorem 1 of Alamo et al. (2007) shows that $\hat{V}^*(x, \mathbf{v})$ is a quadratic function on $\mathbf{v}$ and also an upper bound of the original worst case cost $V^*(x, \mathbf{v})$.

3.3. Computing the control law

The value of the control signal is obtained by solving the following QP optimization problem:

$$\hat{\mathbf{v}}^*(x) = \arg\min_{\hat{\mathbf{v}}} \hat{V}^*(x, \hat{\mathbf{v}})$$
$$\text{s.t.} \quad G_x x + G_v \hat{\mathbf{v}} \le d_\varepsilon \qquad (15)$$

and the system is controlled by $\hat{K}_{MPC}(x(t)) = -Kx(t) + \hat{v}^*(t|t)$, where $\hat{v}^*(t|t)$ is the first element of $\hat{\mathbf{v}}^*(x)$.

The computational burden of the proposed strategy based on an upper bound of the worst case cost is considerably lower than that of the exact MMMPC (Alamo et al., 2007). This computational complexity results mostly from the two QP problems that have to be solved to obtain an initial guess and to compute the solution of the min–max problem. Note that both QP problems of the proposed strategy have the complexity of a standard constrained MPC using a quadratic cost function. For the computation of the exact worst case cost $V^*(x, \mathbf{v})$ the cost function has to be evaluated for all $2^{N \cdot d_\theta}$ vertices of Θ. The optimization of the exact min–max problem is a well known NP-hard problem and allows only the use of small prediction horizons due to its high computational burden.

4. Process description

A real process represented by a pilot plant has been chosen for the application of the proposed algorithm. The process has been studied previously by several authors (Gruber et al., 2008; Szeifert, Chovan, & Nagy, 1995) and has been used as a benchmark for control purposes (Ramírez, Limón, Ortega, & Camacho, 1999).

4.1. Laboratory process

The pilot plant (see Fig. 1) is used to emulate exothermic chemical reactions based on temperature changes as done in Santos, Afonso, Castro, Oliveira, and Biegler (2001). The main elements of the pilot plant are the reactor, the heat exchanger, the cooling jacket and the valve to manipulate the flow rate through the cooling jacket. The plant structure with the mentioned main elements is given in the schematic diagram in Fig. 2.

The cooling jacket is used to reduce the caloric energy of the reactor content. The heat dissipation can be regulated by the valve v_8 manipulating the flow rate F_j through the cooling jacket. Fig. 3 shows the static relation between the flow rate F_j and the opening of valve v_8 obtained from experimental data. The cooling fluid, water, circulating through the cooling jacket is taken from a tank with a capacity of $1\,m^3$. After circulating through the jacket the cooling fluid returns to the tank. To maintain the temperature of the cold water constant the tank has an auxiliary cooler controlled by a thermostat which maintains the temperature T_{T2} near to a desired value in an interval of approximately 1°.

The reactant is supplied to the reactor by the feed $F_{f,in}$ to keep the chemical reaction active. Before entering the reactor, the feed passes through a heat exchanger in order to reduce the

Fig. 1. Pilot plant used to apply the proposed MMMPC.

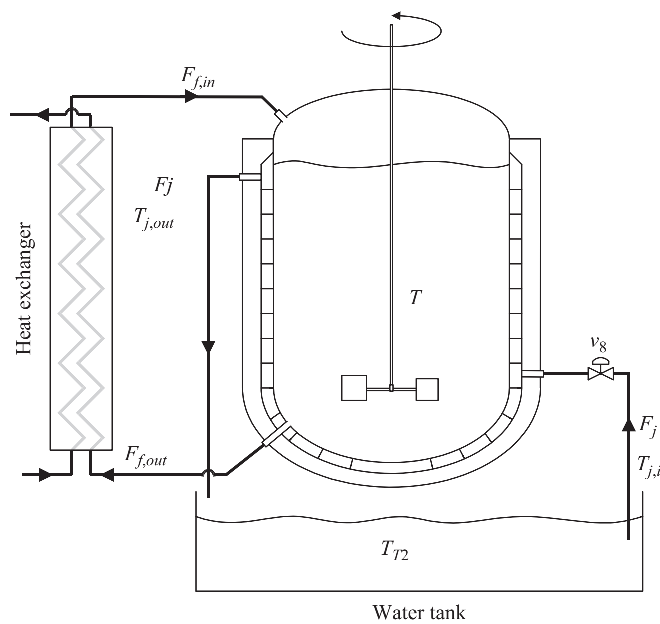

Fig. 2. Block diagram of the pilot plant with its main elements.

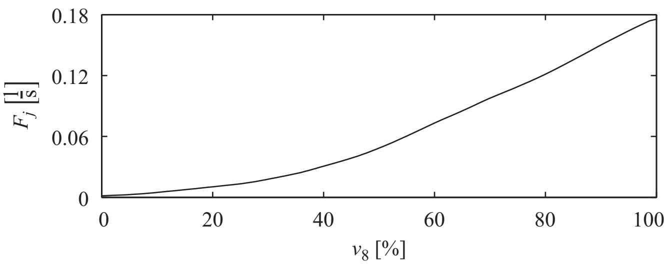

Fig. 3. Static relation between the flow rate F_j and the opening of valve v_8.

temperature difference between the feed and the reactor content. The outflow $F_{f,out}$ is used to keep the volume of the reactor content constant. As a consequence, as feed and outflow have the same flow rate and nearly the same temperature, the two flows hardly provoke temperature changes in the interior of the reactor.

To emulate exothermic reactions, the reactor possesses an electrical resistance in order to supply caloric energy. The energy to be supplied by the 14.4 kW electrical resistance is calculated with a mathematical model of the reaction. The use of a resistance has the advantage that no chemical reaction takes place in the reactor, instead the reaction is emulated on basis of temperature changes, as done in Santos et al. (2001).

4.2. Mathematical model

Although it is not necessary to have a mathematical model for the design of the min–max predictive controller, this section shows the process model to emphasize its nonlinear character. The mathematical model also justifies the way to emulate the heat generated by the chemical reaction with the aid of the resistance.

The emulated chemical reaction, representing a refinement process, was used previously in Lee, Lee, and Kim (2000) and Gruber and Bordons (2007). Considering identical flow rates for the feed and the outflow, i.e. $F_f = F_{f,in} = F_{f,out}$, the reactor volume V and the mass M are constant. The temperature changes of the reactor content can be defined as

$$\frac{dT}{dt} = -\frac{F_j}{V}(T_{j,out} - T_{j,in}) + \frac{(-\Delta H)V}{MC_p}\, k_0\, e^{-E/(RT)} C_A^2 \qquad (16)$$

where the first term considers the heat dissipation by the cooling jacket and the second term denotes the generated heat by the exothermic chemical reaction. Note that the second term is used to calculate the heat to be supplied by the electric resistance in the reactor tank in order to emulate the chemical reaction based on temperature changes. The variables F_j, $T_{j,in}$ and $T_{j,out}$ represent the flow rate through the cooling jacket and the temperature of the cooling fluid entering and leaving the cooling jacket, respectively. C_A is the concentration of the reactant in the reactor content. It has been assumed that the feed neither supplies nor removes caloric energy from the reactor as the feed passes through an heat exchanger and enters the reactor nearly with the temperature of the reactor content.

The reactant concentration C_A in the plant reactor is calculated by

$$\frac{dC_A}{dt} = \frac{F_f}{V}(C_{A,in} - C_A) - k_0 e^{-E/(RT)} C_A^2 \qquad (17)$$

where the first term represents changes in the reactant concentration due to the feed and the outflow. The second term considers the reduction of the concentration as a result of the reactant consumption by the chemical reaction. $C_{A,in}$ denotes the reactant concentration in the feed. The model parameters and the variables used with constant values are shown in Table 1.

Table 1
Model parameters and constant variables of the chemical reaction.

	Value	Unit
Parameter		
k_0	1.2650×10^{17}	l/(mol s)
C_p	4.18	kJ/(K kg)
ΔH	−105.57	kJ/mol
E/R	13 550	K
Variable		
V	25	l
M	25	kg
$C_{A,in}$	1.2	mol/l
F_f	0.05	l/s
$T_{j,in}$	291.15	K

The chemical reaction is nonlinear in the dynamics of the temperature and the concentration due to the exponential function and the quadratic terms of the concentration in the model Eqs. (16) and (17). Furthermore, the relation between the opening of the valve v_8 and the flow rate F_j through the cooling jacket (see Fig. 3) adds some static nonlinearity to the model.

5. Experimental results

The strategy described in Section 3 has been applied to the chemical reaction process described in Section 4. In this section the experimental results will be exposed and discussed. An input–output model with integrated bounded additive uncertainty has been used in the experiments:

$$A(z^{-1})y(t) = z^{-d}B(z^{-1})u(t-1) + \frac{\theta(t)}{\Delta} \qquad (18)$$

with $\Delta = 1 - z^{-1}$, $\theta(t) \in \{\theta \in \mathbb{R}^{d_y} : \|\theta\|_\infty \leq \varepsilon\}$, and d_y the dimension of $y(t)$. The use of this type of prediction models results in a control law without error in steady state. The main difference between using the algorithm of Section 3 for a state space model and the given input–output model with bounded additive uncertainties is the method used to find the matrices of the cost function (7) (see Camacho & Bordons, 2004). Besides that, the algorithm can be applied as described in Section 3.

In the following sections the control system in the pilot plant will be described and the necessary steps to obtain a prediction model will be presented. Finally, experimental results will be exposed.

5.1. Description of the control system

The sensors and actuators in the plant are connected to a PMC-10 control unit. The PMC-10 is connected by ARCnet to a personal computer that runs the SCADA (supervisory control and data acquisition) system *Simatic-IT*. The control algorithm has been implemented directly in *Matlab* and the communication with *Simatic-IT* is done using the *OPC* protocol (*OLE for process control*). Both *Simatic-IT* and the controller run on the same personal computer, based on a Pentium II processor at 300 MHz. This computer does not have enough computational power to solve exactly the min–max problem of a typical MMMPC, but can compute the control action using the proposed strategy.

5.2. Identification of the prediction model

A *PRMSS* (*pseudo-random multilevel step sequence*) has been applied to the recirculation valve with the objective of collecting data for the parameter identification of the prediction model. The periods of the *PRMSS* have been chosen sufficiently long to observe the reaction of the pilot plant to changes in the input (see Fig. 4). It can be seen that the temperature of the tank reaches steady state in each step in something more than 2 h, although the variations in steady state are of several degrees. The reactant concentration also suffers variations in steady state. It can be observed that the input–output gain is negative and clearly variable (greater gain for low openings of v_8). A first order transfer function model with delay is proposed as prediction model. It has to be mentioned that the proposed low order prediction model cannot describe correctly the nonlinear dynamic behavior of the used process. Therefore the used process is a good candidate to be controlled by a control strategy considering uncertainties and disturbances.

Using the data of Fig. 4 the system delay was approximated with $t_d = 31.25$ s. Taking in account the response time of the system, the sampling period has been chosen to $T_s = 60$ s. The delay of the system was rounded to 1 sampling time in order to avoid approximations of the time delay, e.g. Padé approximation. With the experimental data (see Fig. 4) a least squares identification has been carried out and the following model has been identified:

$$y(t) = 0.939y(t-1) - 0.0597u(t-2) \tag{19}$$

Thereby, the following input–output prediction model with integrated bounded additive uncertainty was obtained:

$$y(t+1) = 0.939y(t) - 0.0597u(t-1) + \frac{\theta(t)}{\Delta} \tag{20}$$

5.3. Experimental results of the controller

The proposed control strategy was applied to the pilot plant described in Section 4.1 using (20) as a prediction model. For the implementation of the MMMPC a prediction horizon of $N = 15$ and a control horizon of $N_u = 12$ were used. Note that the use of different prediction and control horizons ($N \neq N_u$) in the cost function (4) requires minor changes in the matrices M_{vv}, $M_{\theta v}$ and M_{vf} in the quadratic cost function (7) as well as in the matrix G_v in the considered min–max problem (8).[1] In the implementation of the proposed control strategy the terminal constraint and the terminal cost have not been considered. With a prediction horizon of $N = 15$, including approximately one time constant of the process, the terminal constraint is not active for the region of interest. Also, the prediction horizon is sufficiently large and therefore, the effect of not including a terminal cost can be neglected. For a formal study when it is possible to disregard the terminal constraint and terminal cost see Mayne et al. (2000), Hu and Linnemann (2002) and Limon, Alamo, Salas, and Camacho (2006).

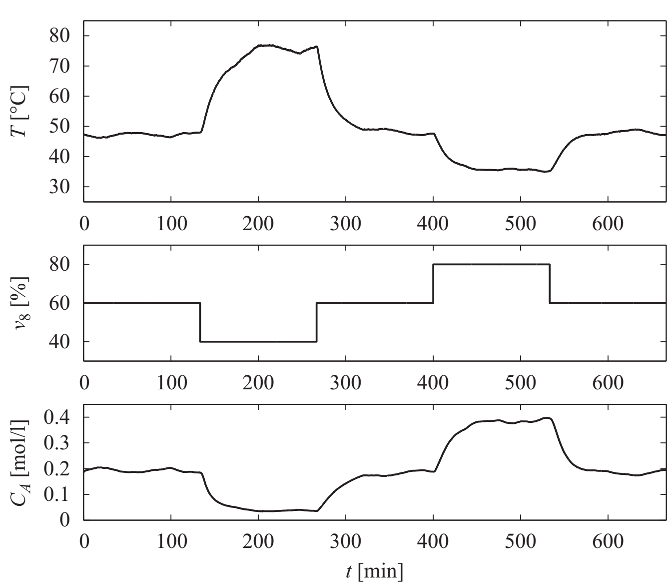

Fig. 4. Experiment for the model identification. From top to bottom: tank temperature (T), valve opening (v_8) and reactant concentration (C_A).

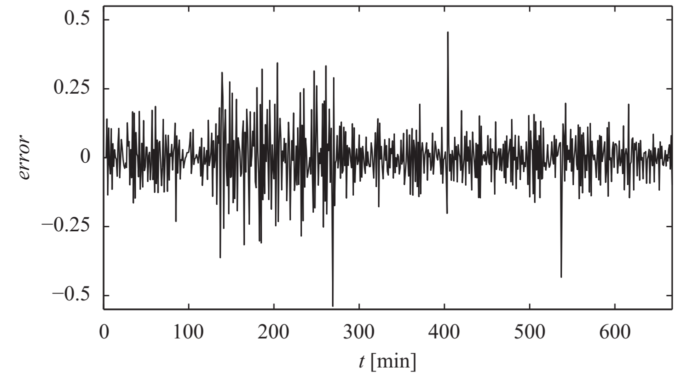

Fig. 5. One step ahead prediction error during the experiment for the model identification.

The weighting factor for the control effort has been chosen equal to $R = 2$. Based upon the one step ahead prediction error (see Fig. 5) the parameter ε has been chosen to $\varepsilon = 0.25$. As a result, in 97% of the samples the one step ahead prediction error is bounded by the chosen value. In order to verify the goodness of fit of the identified model a second set of experimental data has been used to calculate the one step ahead prediction error. Fig. 6 shows the tank temperature and the one step ahead prediction error of the prediction model (20). It can be seen from the figure that the one step ahead prediction error is bounded by $\varepsilon = 0.25$ nearly throughout the whole experiment, only in a few samples the prediction error exceeds the bound. Therefore the verification confirms the election of $\varepsilon = 0.25$ as a valid choice.

Due to the varying delay of real process a correction in the prediction of $y(t+1)$ has been used. With the Smith like predictor the predicted output at time $t+1$ using the nominal model, $\hat{y}_n(t+1|t)$, is corrected as

$$\hat{y}(t+1|t) = \hat{y}_n(t+1|t) + (\hat{y}_n(t|t) - y(t)) \tag{21}$$

being $y(t)$ the real process output at time t. This simple correction improves the performance of predictive controllers in the case of time delay systems (Normey-Rico & Camacho, 2007).

Finally, in order to restrict the system input and output in the experiments, the following constraints have been used:

$$30 \leq \hat{y}(t+j|t) \leq 70, \quad j = 2, \ldots, 16, \ \forall \boldsymbol{\theta} \in \text{vert}(\boldsymbol{\Theta})$$

$$5 \leq u(t+j|t) \leq 100, \quad j = 0, \ldots, 11$$
$$-20 \leq \Delta u(t+j|t) \leq 20, \quad j = 0, \ldots, 11 \tag{22}$$

[1] Defining the variable $\sigma = N - N_u$ (the difference of the two horizons), the necessary adjustment of the matrices M_{vf}, G_v, $M_{\theta v}$ and M_{vv} due to different prediction and control horizons leads to the elimination of the last σ rows of M_{vf}, the last σ columns of G_v and $M_{\theta v}$ and the last σ rows and columns of M_{vv}.

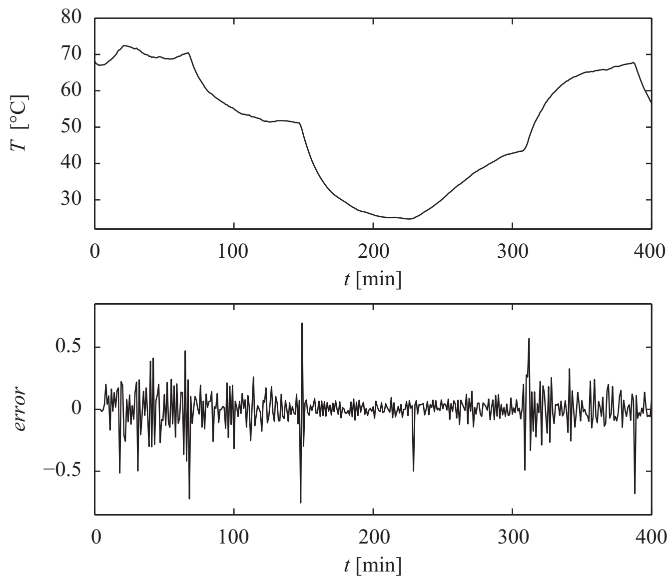

Fig. 6. Experiment to verify the goodness of fit of the prediction model (20). From top to bottom: tank temperature (T), one step ahead prediction error.

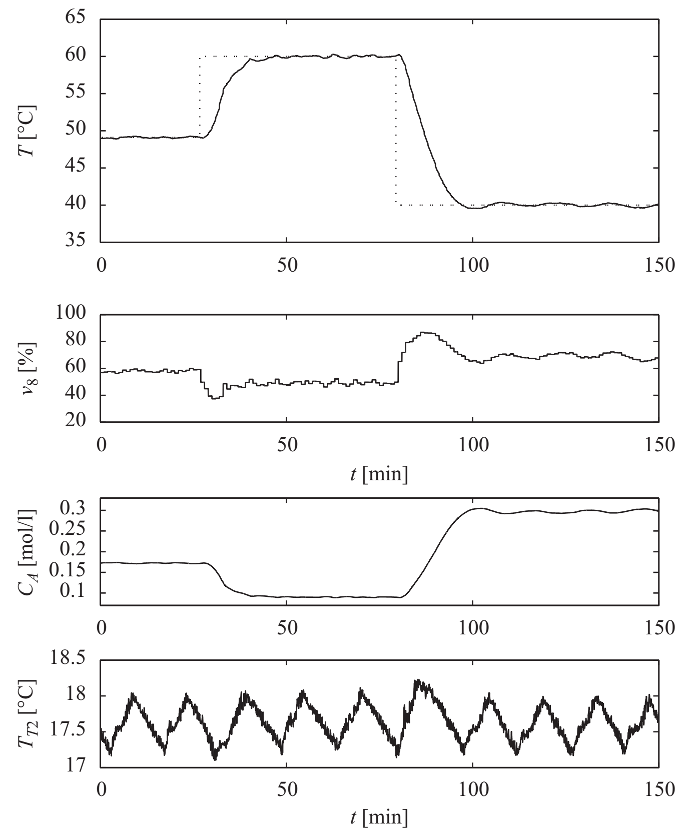

Fig. 7. Reference tracking experiment. From top to bottom: tank temperature (T), valve opening (v_8), reactant concentration (C_A) and cold water temperature (T_{T2}).

Note that in the output restrictions the effect of the uncertainty has to be considered.

In order to analyze the system behavior, several experiments with reference changes and disturbance rejection have been made using the proposed control strategy. Fig. 7 shows the results of the tracking experiment with references different enough to result in control actions in a large interval. After the first reference change no overshoot appears in spite of a quite fast controller reaction. After the second reference change a small overshoot (of about $-0.5\,^{\circ}$C), justified by the nonlinear process behavior, can be observed. In steady state the controller shows small changes in the control action necessary to stabilize the output on the reference in presence of variations in the generated heat and the cold water temperature.

In the second experiment the disturbance rejection capabilities of the proposed MMMPC strategy were proven by means of an additive disturbance in the input of the system. The results of the experiment can be seen in Fig. 8 where u denotes the necessary input signal for a given setpoint calculated by the min–max model predictive controller and v_8 represents the effective opening of the valve measured directly in the pilot plant. In normal operation mode and in absence of errors in the integrated valve controller the variables u and v_8 have the same values, i.e. $v_8 = u$. In the experiment a discrepancy between u and v_8 has been introduced so that it acts as an input disturbance (like an error in the integrated valve controller). With a disturbance $\Delta v_8 = -15\%$ the measured opening of the valve has the value $v_8 = u + \Delta v_8$. After the application of the input disturbance in $t = 70$ min the effective opening of the valve v_8 is too low for the given setpoint and leads to a rising temperature. The proposed MMMPC strategy reacts rapidly to the increasing error in the temperature and rejects the effect of the disturbance in approximately 20 min. After the disappearance of the disturbance in $t = 101$ min the controlled system shows the same behavior and reaches steady state in about 20 min. Neither the temperature nor the control action show oscillations after the disappearance of the disturbance.

Fig. 9 shows the experimental results applying an additive disturbance in the feeding F_f. In $t = 70$ min a change in the feed flow rate of $\Delta F_f = 0.0125$ l/s, which corresponds to an error of 25%, has been applied. With an increasing error, the controller reduces the opening of the valve and reaches a compensation of the divergence after 15 min. In this experiment an overshoot of $-0.50\,^{\circ}$C can be observed. The oscillation in the temperature and the control action is quite small and seems acceptable due to the strong disturbance.

The reference tracking experiment was repeated with a linear constrained predictive controller (GPC) to allow the comparison between the proposed strategy and a standard MPC method. The GPC is based on the same linear model (20) as the MMMPC strategy and was tuned with the same parameters as the MMMPC (prediction horizon $N = 15$, control horizon $N_u = 12$ and weighting factor for the control effort $R = 2$). As a consequence both control strategies are based on similar cost functions to be optimized.[2] Furthermore, the GPC was implemented with the same constraints (22) as the MMMPC. It can be observed in the results (see Fig. 10) that the process controlled by the GPC exhibits significant oscillations in the temperature and the control action after the reference changes. The comparison of the results shows that the MMMPC stabilizes the temperature more efficiently and with fewer oscillations in the opening of the valve than the standard GPC. Although both controllers are based on the same linear prediction model, the MMMPC obtains better results controlling the nonlinear process due to the explicit consideration of uncertainties and perturbations in the optimization problem.

[2] A fair comparison between predictive controllers can only be carried out when the cost functions are the same or at least close to be equal. Note that the tuning parameters (including the uncertainty bounds in the MMMPC) define the cost function. Therefore for both MMMPC and GPC the same linear model, the same horizons and the same tuning parameters have been used (with the exception of the uncertainty bounds that are not present in GPC). As a result, the cost functions to be minimized by the MMMPC and GPC are as similar as possible.

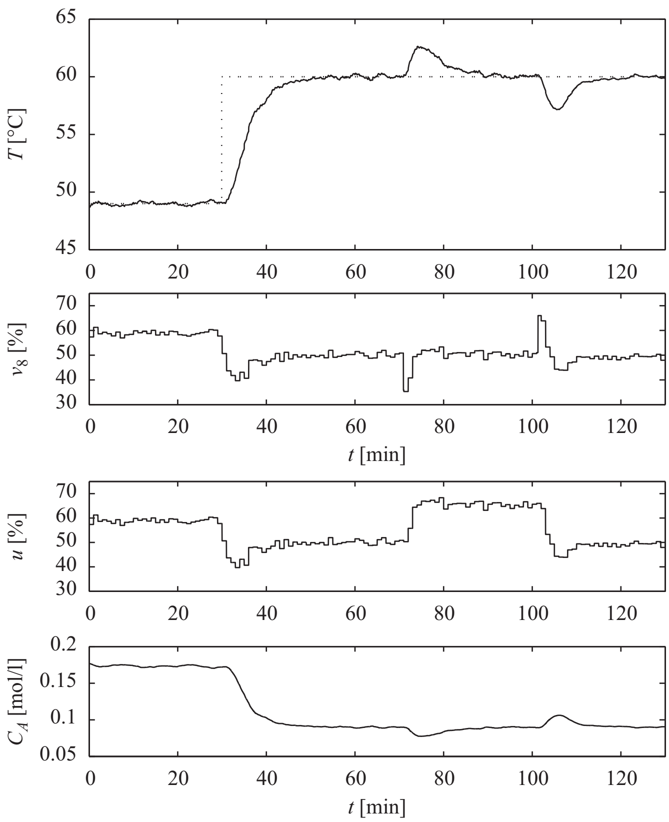

Fig. 8. Experiment with input disturbance rejection. From top to bottom: tank temperature (T), valve opening (v_8), controller output (u) and reactant concentration (C_A).

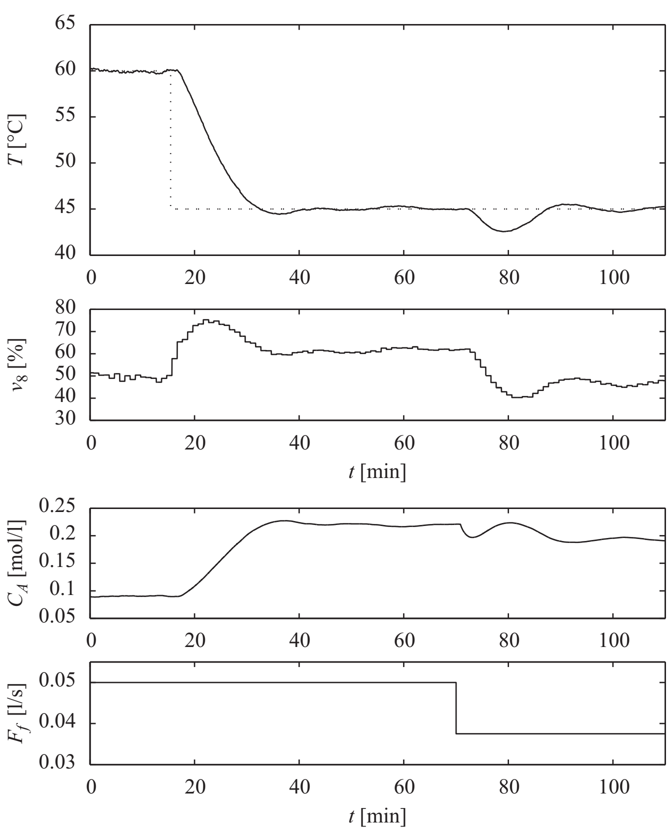

Fig. 9. Experiment with disturbance rejection in the feed flow rate. From top to bottom: tank temperature (T), valve opening (v_8), reactant concentration (C_A) and feed flow rate (F_f).

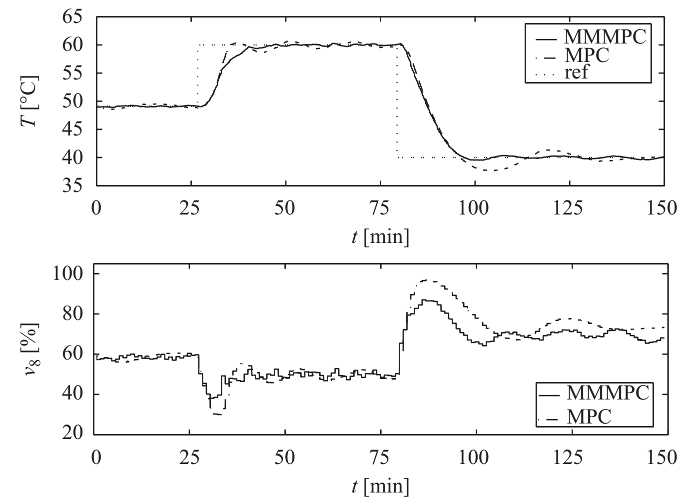

Fig. 10. Reference tracking results of the MMMPC and the GPC. From top to bottom: tank temperature (T), valve opening (v_8).

Finally, it is important to mention that the calculation of the control signal took place without problems within the chosen sampling time (60 s). During the experiments the average computation time was 5.64 s, with a maximum of 9.90 s and a minimum of 1.86 s. These computation times are much smaller than those required to compute the exact solution (e.g., using the same computer, the average time for $N = N_u = 12$ was 324.63 s).

6. Conclusions

In this paper an MMMPC based on an tractable QP problem was applied to a pilot plant. The results showed a good system behavior and the stabilization of the plant temperature around the operation point. After reference changes the controller quickly compensates deviations. Furthermore, the MMMPC showed its capacity to compensate errors caused by the disturbances.

The application to a process shown in this work joins the small number of MMMPC applications reported in specialized literature. The low computational requirements of the proposed control strategy allowed the use of appropriate sampling times and realistic prediction and control horizons. Thereby it is shown that the use of the proposed strategy allows the application of this kind of controllers to a larger number of processes.

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