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A gradient-based strategy for the one-layer RTO + MPC controller

Teodoro Alamo [a], Antonio Ferramosca [b,*], Alejandro H. González [b], Daniel Limon [a], Darci Odloak [c]

[a] Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla, Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain
[b] Institute of Technological Development for the Chemical Industry (INTEC), CONICET-Universidad Nacional del Litoral (UNL), Güemes 3450, 3000 Santa Fe, Argentina
[c] Department of Chemical Engineering, University of São Paulo, Av. Prof. Luciano Gaulberto, trv 3 380, 61548 São Paulo, Brazil

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ABSTRACT

In the process industries model predictive controllers (MPC) have the task of controlling the plant ensuring stability and constraints satisfaction, while an economic cost is minimized. Usually the economic objective is optimized by an upper level Real Time Optimizer (RTO) that passes the economically optimal setpoints to the MPC level. The drawback of this structure is the possible inconsistence/unreachability of those setpoints, due to the different models employed by the RTO and the MPC, as well as their different time scales. In this paper an MPC that explicitly integrates the RTO structure into the dynamic control layer is presented. To overcome the complexity of this one-layer formulation a gradient-based approximation is proposed, which provides a low-computational-cost suboptimal solution.

1. Introduction

Model predictive control (MPC) is one of the most successful advanced control techniques in the process industries. MPC theoretical background has been widely investigated in the last two decades, showing how MPC is a control technique capable to provide stability, robustness, constraint satisfaction and tractable computation for linear and for nonlinear systems [6,24].

Recently, researchers are focused on improving the economic performance of MPC. In this context, the study of the hierarchical control structure, typical in process industries, has a certain relevance [11]: at the top of this structure, an economic schedule and planner determines the whole plant production (level, quality, etc.). The outputs of this layer are sent to a Real Time Optimizer (RTO), which is devoted to compute the stationary setpoints according to economic criteria. This optimizer is usually based on a complex nonlinear stationary model of the plant and so has a sampling time different from other layers. Then, the setpoints computed by the RTO are sent to the MPC control level which calculates the control actions necessary for the plant to reach those setpoints, taking into account a simplified dynamic model of the plant and the variable constraints. One well-known drawback of this hierarchical control structure is that the communication between the economic/stationary and the dynamic layers may be inconsistent, producing in this way problems that go from unreachability of the setpoints to poor economic performances. As a result, a proper strategy to unify these (probably competing) objectives becomes highly desired from an operating point of view.

A way to reduced inconsistency is the so-called two-layer structure: an extra optimization level – the Steady State Target Optimizer (SSTO) – is added in between the RTO and the MPC to decide the best admissible target for the MPC, according to a local approximation of the RTO cost function, and using the same simplified model used in the MPC layer [20,21,23]. In the two-layer framework, another approach is represented by the Dynamic Real Time Optimizer (D-RTO) [5,26,9], which solves a dynamic economic optimization and delivers target trajectories (instead of target steady state) to the MPC layer.

Another solution is the so-called one-layer strategy: the idea is to merge the RTO layer with the MPC layer, by designing controllers that integrates the RTO economic cost function as part of the MPC cost as in [1,27], or controllers based on a general (economic) cost function, as

* Corresponding author. Tel.: +54 342 4559174.
E-mail addresses: alamo@us.es (T. Alamo), ferramosca@santafe-conicet.gov.ar (A. Ferramosca), alejgon@santafe-conicet.gov.ar (A.H. González), dlm@us.es (D. Limon), odloak@usp.br (D. Odloak).

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in [8,4,3,16,17,10,14,12]. The main problem of this strategy is that the economic objectives are usually represented by a complex nonlinear function that turn the one-layer optimization cost also nonlinear and difficult to solve.

In order to reduce the computational burden, an approximation of the RTO function can be considered, as economic cost function: for instance, in [7], the gradient of the economic objective function is included in the controller cost function, in order to obtain a computational low-cost strategy. This solution allows one to solve the resulting control/optimization problem as a single QP problem, and the results are promising from both, theoretic and practical points of view. This idea has been then extended in [2], in order to obtain a stable formulation. However, instead of applying to the system the optimal solution of an approximated problem, the applied control action is the convex combination of an arbitrary feasible solution and an approximated solution. In this way, a suboptimal MPC strategy is obtained, which ensures recursive feasibility and convergence to the optimal steady state in the economic sense, with a reduced computational cost.

Following the ideas presented in [2], the main objective of this paper is to propose a stable MPC controller that efficiently incorporates the stationary-economic objectives into a single control formulation. To do that, a practical framework is selected in order to obtain a flexible and intuitive formulation. For instance, by means of the use of a velocity model in the input (Δu instead of u) it is possible to include extra integrating modes to the system in such a way that: (i) the predicted steady state of the system, although it is achieved at $j\to\infty$, can be easily characterized, and (ii) input increment constraints (which, in real problems, are important constraints) can be easily included into the formulation.

The paper is organized as follows. In Section 2 the control problem is stated. In Sections 3–5 the proposed one-layer MPC formulation is presented. In Section 6 the control algorithm is introduced, and a stability proof is given. In Section 7 the properties of the proposed controller are presented. In Section 8 an illustrative example is given. Finally, in Section 9 some conclusions are drawn.

2. Problem statement

Consider a system described by a linear time-invariant discrete time model

$$x^+ = Ax + B\Delta u \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\Delta u \in \mathbb{R}^m$ is the current control increment and x^+ is the successor state. The solution of this system for a given sequence of control inputs $\mathbf{\Delta u} = \{\Delta u(0), \ldots, \Delta u(j-1)\}$ and an initial state x is denoted as $x(j) = \phi(j; x, \mathbf{\Delta u})$, $j \in \mathbb{I}_{\geq 1}$, where $x = \phi(0; x, \mathbf{\Delta u})$. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $\Delta u(k)$ respectively. Notice that, $\Delta u(k) = u(k) - u(k-1)$.

The system is subject to hard constraints on state and input:

$$x(k) \in X, \quad \Delta u(k) \in \Delta U \tag{2}$$

for all $k \geq 0$, where $X \subset \mathbb{R}^n$ and $\Delta U \subset \mathbb{R}^m$.

It is assumed that the following holds.

Assumption 1. The pair (A, B) is controllable and the state is measured at each sampling time.

Assumption 2. The set X is convex and closed, ΔU is convex and compact and both sets contain the origin in their interior.

Remark 3. It is worth to remark that in (1), the input appears in the incremental form, and so constraints are considered not only on the input, but on the increments as well. This is an important feature in real process, aimed to avoid aggressive control moves, and gives a better description of the whole system. Notice also that, the input constraints $u \in U$ are part of the state constraints, as it will be better explained in the next section.

2.1. Dynamic system decoupling (Jordan decomposition)

It is assumed in this work that matrix A has $n_{ss} = m$ integrating eigenvalues, n_{un} (pure) unstable eigenvalues and n_{st} stable eigenvalues. This assumption is easily accomplished by using an appropriated velocity model, as discussed in [15]. Therefore, matrix A in (1) can be decomposed into its stationary, unstable and stable modes using the Jordan decomposition, as follows:

$$A = W\Lambda V = \begin{bmatrix} W_{ss} & W_{un} & W_{st} \end{bmatrix} \begin{bmatrix} \Lambda_{ss} & 0 & 0 \\ 0 & \Lambda_{un} & 0 \\ 0 & 0 & \Lambda_{st} \end{bmatrix} \begin{bmatrix} V'_{ss} \\ V'_{un} \\ V'_{st} \end{bmatrix} = A_{ss} + A_{un} + A_{st},$$

where $T \triangleq WV = I_n$, and Λ is a block diagonal matrix (Jordan canonical form), in which Λ_{ss}, Λ_{un} and Λ_{st} are upper triangular matrices. The columns of the linear maps $T_{ss} \triangleq W_{ss}V'_{ss}$, $T_{un} \triangleq W_{un}V'_{un}$ and $T_{st} \triangleq W_{st}V'_{st}$ span the (complementary) stationary, unstable and stable subspaces or manifolds of the state space, $\mathcal{W}_{ss}$, $\mathcal{W}_{un}$ and $\mathcal{W}_{st}$, respectively; and they trivially satisfy $T_{ss} + T_{un} + T_{st} = T = I_n$.

Since $\mathcal{W}_{ss} \oplus \mathcal{W}_{un} \oplus \mathcal{W}_{st} \equiv \mathbb{R}^n$, every state can be decomposed as $x = x_{ss} + x_{un} + x_{st}$, where $x_{ss} = T_{ss}x$ belongs to $\mathcal{W}_{ss}$, $x_{un} = T_{un}x$ belongs to $\mathcal{W}_{un}$ and $x_{st} = T_{st}x$ belongs to $\mathcal{W}_{st}$. Furthermore, given that $\mathcal{W}_{ss} \cap \mathcal{W}_{un} \cap \mathcal{W}_{st} \equiv \{0\}$, if $x \in \mathcal{W}_{ss}$ then $x_{st} = x_{un} = 0$, and so on for the others subspaces. As it is known, $\mathcal{W}_{ss} \subseteq \mathbb{R}^n$, $\mathcal{W}_{un} \subseteq \mathbb{R}^n$ and $\mathcal{W}_{st} \subseteq \mathbb{R}^n$ are invariant subspaces of the state space under the transformation A.

Taking into account the model decomposition presented above, it is assumed that the original state constraint set is given by the decoupled set:

$$X \subseteq X_{ss} \oplus X_{un} \oplus X_{st}$$

where $X_{ss} \subseteq \mathcal{W}_{ss}$ (notice that this set includes the input constraints $u \in U$, since the control move $u(k-1)$ is part of the state vector), $X_{un} \subseteq \mathcal{W}_{un}$ and $X_{st} \subseteq \mathcal{W}_{st}$ are closed convex sets. Furthermore, X_{st} is assumed to be a contractive set in $\mathcal{W}_{st}$.

It can be shown (after some algebra) that the relation between the integrating states x_{ss} and the inputs u is given by $x_{ss} = T_{ss}x = T_{ss}Bu$. As a result, the set X_{ss} is directly related to the input constraint set U.

Remark 4. In case of original integrating systems (i.e., systems that already have integrating modes) there are several options to obtain the proposed decomposition. If the number of original integrating modes is smaller than m, then they can be completed up to number of m, by adding velocity input variables in that states that are not affected by the original integrating modes. If the original integrating modes are exactly m, then, under simple assumptions regarding the algebraic and geometric multiplicity (that real system almost always fulfill) the required decomposition is made directly, without including Δu variables. Finally, if the number of original integrating modes is greater than m, then, it is clear that unstable (double integrating) modes appear.

2.2. Steady state characterization and economic optimum

If we consider the joint variable $(x, \Delta u)$, the state and input equilibrium subspace, associated to model (1), is given by

$$\mathcal{V}_{ss} = \mathcal{N}([(A-I)B]) \subseteq \mathbb{R}^{n+m},$$

where $\mathcal{N}$ is the null space operator. Because of the velocity form of model (1) it can be shown that the steady state input set is given by the origin, $\Delta U_{ss} = \{0\}$. So, the equilibrium subspace can be defined in $\mathbb{R}^n$, and is given by:

$$\mathcal{N}(A-I) = \mathcal{W}_{ss}.$$

We define now the set of admissible stationary states as

$$X_{ss} = \{x \in X \mid x \in \mathcal{W}_{ss}\},$$

which is a convex set in the equilibrium subspace.

Now, taking into account the economic objectives, let us consider the following definition:

Definition 1. The optimal steady state, x_s, satisfy

$$x_s = arg \min_x f_{eco}(x, p)$$
$$s.t. \, x \in X_{ss}$$

where $f_{eco}(x, p)$ defines an economic cost function and p is a parameter that takes into account prices, costs or production goals. Notice that, since according to model (1) we are considering velocity models, system input u is part of the state vector.

Assumption 3. The economic cost function $f_{eco}(x, p)$ is convex in x and twice differentiable.

In addition, according to most real cases, it is assumed that $f_{eco}(x, p)$ is nonlinear and its evaluation takes a significant computation time, provided that it is based on complex stationary models of the real plant.

3. The one layer economic MPC strategy

As it was previously mentioned in the introduction, the objective of the work is to design a controller that directly account for stationary economic objectives. This can be done, by designing a one-layer RTO-MPC controller. Such a solution was first proposed in [27], which presents an industrial application of a one-layer MPC in which the nonlinear economic RTO cost function $f_{eco}(x, p)$ is added to the MPC cost function. The drawback of this solution is clearly the resulting highly nonlinear MPC control problem. In [7], the gradient of the economic objective function is included in the controller cost function, in order to obtain a computational low-cost strategy. However, convergence and stability are not proved.

In this work, we proposed a new one-layer MPC that also resorts to the idea of using the gradient of the RTO cost function in the MPC cost. Moreover, the controller formulation takes into account the results published in [15,19,13].

First of all, consider the one-layer MPC controller, with the following cost function:

$$V_N(x, p; \Delta \mathrm{u}) = V_N^{dyn}(x, \Delta \mathrm{u}) + V_{ss}(x_{ss}, p) \tag{3}$$

where $V_N^{dyn}(x, \Delta \mathrm{u}) = \sum_{j=0}^{N-1} \|x_j - x_{ss}\|_Q^2 + \|\Delta u_j\|_R^2 + \sum_{j=N}^{\infty} \|x_j - x_{ss}\|_Q^2$, $Q > 0$ and $R > 0$ are penalization matrices of appropriate dimension, and $V_{ss}(x_{ss}, p) = f_{eco}(x_{ss}, p)$.

For any current state x, the optimization problem $P_N(x, p)$ to be solved is given by:

Problem $P_N(x, p)$

$$\begin{aligned}
\min_{\Delta \mathrm{u}} \quad & V_N(x, p; \Delta \mathrm{u}) \\
s.t. \quad & \\
& x_0 = x, \\
& x_{j+1} = Ax_j + B\Delta u_j, \quad j \in \mathbb{I}_{0:N-1} \\
& x_j \in X, \;\; \Delta u_j \in \Delta U, \quad j \in \mathbb{I}_{0:N-1} \\
& \Delta u_j = 0, \quad j \in \mathbb{I}_{N:\infty} \\
& T_{ss}x_N = x_{ss}, \quad T_{un}x_N = 0,
\end{aligned} \tag{4}$$

In this optimization problem, x and p are the parameters, while the sequence $\mathbf{\Delta u}=\{\Delta u(0), \ldots, u(N-1)\}$ is the optimization variable. The control law is given by $\kappa_N(x, p)=\Delta u^0(0\,;x)$, where $\Delta u^0(0\,;x)$ is the first element of the solution sequence $\mathbf{\Delta u}^0(x)$.

Remark 3. Notice that in (3), the first term is a pure dynamic term (since the state x_{ss} defines only an admissible equilibrium point in X_{ss}, which is not penalized) while the second one is a pure stationary term (since it only penalizes the admissible equilibrium, x_{ss}, according to the economic objectives). It is worth remarking that x_{ss} represents the stationary part of the terminal state x_N. Notice that x_{ss} is not a parameter neither assume a constant value, since it directly depends on the value of $\mathbf{\Delta u}$.

Given that the predicted input increments are null at $j=N, N+1, \ldots$, and the constraint $T_{un}x_N=0$ forces the unstable state component to be zero beyond time N, then the predicted state tends asymptotically to the steady state $T_{ss}x_N$. So, the constraint $T_{ss}x_N=x_{ss}$ only defines the steady state x_{ss} to be used in the steady state cost. A rank condition necessary to ensure that constraint $T_{un}x_N=0$ can be fulfilled is presented next:

Assumption 4. For a given system (A, B), the control horizon N is such that

$$rank(Co_N)\geq n_{un}$$

where $Co_j=[A^{j-1}BA^{j-2}\ldots B]$ is the j-controllability matrix of system (A, B).

Since the unstable component of the states are zeroed, and no control action is taken beyond N, then the second infinite summation term of the cost $V_N(x, p, \mathbf{\Delta u})$ converges. The infinite cost $V_N(x, p\,; \mathbf{\Delta u})$ can be written as:

$$V_N(x, p; \Delta \mathrm{u}) = \sum_{j=0}^{N-1} \|x_j - x_{ss}\|_Q^2 + \|\Delta u_j\|_R^2 + \|x_N - x_{ss}\|_P^2 + V_{ss}(x_{ss}, p)$$

where P is the solution to the Lyapunov equation $P = A_{st}^T P A_{st} + Q$, with $A_{st} = W_{st}\Lambda_{st}V_{st}^T$.

Remark 4. The domain of attraction of the controller derived from the iterative solution of Problem $P_N(x, p)$ is given by the states that can be steered in N steps to the equilibrium-stable subspace of $\mathbb{R}^n$, $\mathcal{W}_{ss-st} = \mathcal{W}_{ss} \oplus \mathcal{W}_{st}$, fulfilling the constraints along the path. This set is the N-step controllable set from X to $X_{ss-st}=X_{ss}\oplus X_{st}$, and will be denoted as $\mathcal{X}_N$. Notice that, since X_{ss-st} is an invariant (contractive) set, so is the N-step controllable set to it.

Given that the economic cost is generally based on a complex nonlinear model, problem (4) is not easy to solve, mainly when large dimension processes are considered. So, in many real applications the computational power may not be sufficient to solve this one-layer problem within the sample time of the control system. On the other hand, it is known that to ensure convergence and recursive feasibility a suboptimal solution of (4) could be sufficient. In this context, instead of directly solve the complex one-layer problem, the convex combination of an easy-to-obtain feasible solution and an approximated optimal solution could be used to obtain a decreasing cost. Following the ideas presented in [7], the gradient of the economic cost $f_{eco}(x, p)$ – instead of the cost itself – is used to construct the approximated cost and the approximated optimal solution. Then, by means of an appropriate algorithm, the so-obtained suboptimal solution can be used to implement a control strategy that ensures convergence and recursive feasibility, and needs to solve only simple QP problems.

4. A feasible and an approximated optimal solution to the original economic optimization problem

Let us consider a feasible solution, $\hat{\Delta \mathrm{u}}$, to problem (4). This feasible solution is a sequence of control inputs which is associated to a corresponding (infinite) sequence of states:

$$\hat{\Delta \mathrm{u}} = \{\hat{\Delta u}_0, \hat{\Delta u}_1, \ldots, \hat{\Delta u}_{N-1}\}$$

$$\hat{\mathbf{x}} = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{ss}\}.$$

Although the proper feasible solution to problem (4) is the input sequence $\hat{\Delta \mathrm{u}}$, we will name it as $(\hat{\mathbf{x}}, \hat{\Delta \mathrm{u}})$ in order to make explicit the associated state sequence. The usual way to obtain a feasible solution to (4), at a given sample time k, is by using the shifted solution of the same problem at time $k-1$. As will be shown later, this choice not only gives an easy-to-obtain feasible solution, but allows to prove the closed-loop convergence of the proposed strategy. For the initial sample time, $k=0$, when a $(k-1)$-solution is not available, given an initial condition $x\in\mathcal{X}_N$, the feasible solution can be obtained by solving the following reduced-cost optimization problem:

Problem $P_N^{dyn}(x)$

$$\begin{aligned}
&\min_{\Delta \mathrm{u}} \quad V_N^{dyn}(x; \Delta \mathrm{u}) \\
&s.t. \\
&\qquad x_0 = x, \\
&\qquad x_{j+1} = Ax_j + B\Delta u_j, \quad j\in\mathbb{I}_{0:N-1} \\
&\qquad x_j \in X,\ \Delta u_j \in \Delta U, \quad j\in\mathbb{I}_{0:N-1} \\
&\qquad \Delta u_j = 0, \quad j\in\mathbb{I}_{N:\infty} \\
&\qquad T_{ss}x_N = x_{ss}, \quad T_{un}x_N = 0,
\end{aligned} \tag{5}$$

Notice that the feasibility of $(\hat{\mathbf{x}}, \hat{\Delta \mathrm{u}})$ follows from the trivial fact that the constraints of problems $P_N^{dyn}(x)$ and $P_N(x, p)$ are the same.

Let us consider now an *approximated optimal solution* to (4), which can be obtained by solving the following approximated problem:
Problem $P_N^{app}(x,p)$

$$\begin{aligned}
\min_{\Delta u} \quad & V_N^{app}(x,p;\Delta u) \\
s.t. \quad & \\
& x_0 = x, \\
& x_{j+1} = Ax_j + Bu_j, \quad j \in \mathbb{I}_{0:N-1} \\
& x_j \in X,\ \Delta u_j \in \Delta U, \quad j \in \mathbb{I}_{0:N-1} \\
& \Delta u_j = 0, \quad j \in \mathbb{I}_{N:\infty} \\
& T_{ss}x_N = x_{ss}, \quad T_{un}x_N = 0,
\end{aligned} \tag{6}$$

where the approximated cost is given by

$$V_N^{app}(x,p;\Delta u) = V_N^{dyn}(x,\Delta u) + \nabla V_{ss}(\hat{x}_{ss},p)\left[x_{ss} - \hat{x}_{ss}\right] \tag{7}$$

and $\nabla V_{ss}(\hat{x}_{ss},p)$ represents the gradient of V_{ss} w.r.t. x, evaluated at the point $\hat{x}_{ss}$.

As it can be seen, this approximated optimal solution try to optimize problem (4) by means of a simplified version of it. In some sense, this solution is similar to the one obtained by the method presented in [7]. However, it should be noted that the direct application of this solution into the MPC scheme does not guarantee either a stable, or a recursive feasible closed-loop.

Let us denote the optimal solution to problem (6), $P_N^{app}(x,p)$ (which we named *approximated optimal solution*) as

$$\Delta u^* = \{\Delta u_0^*, \Delta u_1^*, \ldots, \Delta u_{N-1}^*\}$$
$$\mathbf{x}^* = \{x_1^*, x_2^*, \ldots, x_{ss}^*\}$$

Next, a Lemma relating the costs $V_N^{dyn}(x;\hat{\Delta u})$ and $V_N^{app}(x,p;\Delta u^*)$ is stated:

Lemma 1. *If* $(\hat{\mathbf{x}},\hat{\Delta u}) \neq (\mathbf{x}^*,\Delta u^*)$, *then*

$$V_N^{dyn}(x;\hat{\Delta u}) > V_N^{dyn}(x;\Delta u^*) + \nabla V_{ss}(\hat{x}_{ss},p)\left[x_{ss}^* - \hat{x}_{ss}\right].$$

Proof. Since $(\hat{\mathbf{x}},\hat{\Delta u})$ is a feasible solution to problem (4), it is also feasible to problem (6). The cost $V_N^{app}(x,p;\Delta u)$ corresponding to this solution is $V_N^{app}(x,p;\hat{\Delta u}) = V_N^{dyn}(x;\hat{\Delta u})$. On the other hand, since $(\hat{\mathbf{x}},\hat{\Delta u}) \neq (\mathbf{x}^*,\mathbf{\Delta u}^*)$, $(\mathbf{x}^*,\mathbf{\Delta u}^*)$ is optimal, and $V_N^{app}(x,p;\Delta u)$ is convex in $(\mathbf{x},\mathbf{\Delta u})$; then $(\mathbf{x}^*,\mathbf{\Delta u}^*$ will produce a smaller cost than $(\hat{\mathbf{x}},\hat{\Delta u})$ – since otherwise, the solution will be exactly $(\hat{\mathbf{x}},\hat{\Delta u})$ – i.e.,

$$V_N^{app}(x,p;\Delta u^*) < V_N^{dyn}(x;\hat{\Delta u}),$$

and therefore,

$$V_N^{dyn}(x;\Delta u^*) + \nabla V_{ss}(\hat{x}_{ss},p)\left[x_{ss}^* - \hat{x}_{ss}\right] < V_N^{dyn}(x;\hat{\Delta u}).$$

■

This results will be useful later, to prove the main result of this work.

5. Improving the economic MPC cost

The *exact costs* $V_N(x,\mathbf{\Delta u})$ corresponding to the solutions $(\hat{\mathbf{x}},\hat{\Delta u})$ and $(\mathbf{x}^*,\mathbf{\Delta u}^*)$ are given, respectively, by:

$$\hat{V} = V_N(x,p;\hat{\Delta u}) = \sum_{j=0}^{\infty}\|\hat{x}_j - \hat{x}_{ss}\|_Q^2 + \|\hat{\Delta u}_j\|_R^2 + V_{ss}(\hat{x}_{ss},p)$$

and

$$V^* = V_N(x,p;\Delta u^*) = \sum_{j=0}^{\infty}\|x_j^* - x_{ss}^*\|_Q^2 + \|\Delta u_j^*\|_R^2 + V_{ss}(x_{ss}^*,p)$$

Consider now a parameterized family of feasible solutions, given by the convex combination of the *feasible solution* and the *approximated optimal solution*:

$$\Delta u(\lambda) = \{(1-\lambda)\hat{\Delta u}_1 + \lambda\Delta u_1^*, \ldots, (1-\lambda)\hat{\Delta u}_{N-1} + \lambda\Delta u_{N-1}^*\} = (1-\lambda)\hat{\Delta u} + \lambda\Delta u^*$$

$$\mathbf{x}(\lambda) = \{(1-\lambda)\hat{x}_1 + \lambda x_1^*, \ldots, (1-\lambda)\hat{x}_{ss} + \lambda x_{ss}^*\} = (1-\lambda)\hat{\mathbf{x}} + \lambda\mathbf{x}^*.$$

with $\lambda \in [0,1]$.

Define also the following performance indexes:

$$V(\lambda)=\sum_{j=0}^{\infty}\|x_j(\lambda)-x_{ss}(\lambda)\|_Q^2+\|\Delta u_j(\lambda)\|_R^2+V_{ss}(x_{ss}(\lambda),p)$$

which is the exact one-layer cost $V_N(x, p\,; \mathbf{\Delta u})$ parameterized in λ, and

$$V_g(\lambda)=\sum_{j=0}^{\infty}\|x_j(\lambda)-x_{ss}(\lambda)\|_Q^2+\|\Delta u_j(\lambda)\|_R^2+\hat{V}_{ss}(\hat{x}_{ss},p)+\nabla V_{ss}(\hat{x}_{ss},p)\left[x_{ss}(\lambda)-\hat{x}_{ss}\right]$$

which is the exact one-layer cost $V_N(x, p\,; \mathbf{\Delta u})$, with the economic cost $V_{ss}(x, p)$ replaced by its first-order Taylor approximation.

Lemma 2. *If* $(\hat{\mathbf{x}}, \hat{\Delta \mathrm{u}}) \neq (\mathbf{x}^*, \Delta \mathrm{u}^*)$, *then*

$$V_g(1)<V_g(0). \tag{8}$$

Proof. Consider the difference

$$V_g(0)-V_g(1)=V_N^{dyn}(x;\hat{\Delta \mathrm{u}})+V_{ss}(\hat{x}_{ss},p)-V_N^{dyn}(x;\Delta \mathrm{u}^*)-V_{ss}(\hat{x}_{ss},p)-\nabla V_{ss}(\hat{x}_{ss},p)\left[x_{ss}^*-\hat{x}_{ss}\right]=V_N^{dyn}(x;\hat{\Delta \mathrm{u}})$$
$$-\left(V_N^{dyn}(x;\Delta \mathrm{u}^*)+\nabla V_{ss}(\hat{x}_{ss},p)\left[x_{ss}^*-\hat{x}_{ss}\right]\right)$$

Then from Lemma 1, we have $V_g(0) - V_g(1) > 0$, and inequality (8) holds.

■

Next, the main result of this work will be presented.

Theorem 1. *The following hold*

(i) *The pair* $(\mathbf{x}(\lambda), \mathbf{\Delta u}(\lambda))$, *for every* $\lambda \in [0, 1]$, *provides a feasible solution to problem* (4).
(ii) *If* $(\hat{\mathbf{x}}, \hat{\Delta \mathrm{u}}) \neq (\mathbf{x}^*, \Delta \mathrm{u}^*)$, *then there exists a* $\tilde{\lambda} \in (0, 1]$ *such that*

$$V(\tilde{\lambda})<V(0)=\hat{V}.$$

Proof. Taking into account that the constraints to problem (4) are convex on $(\mathbf{x}(\lambda), \mathbf{\Delta u}(\lambda))$ and that both the *feasible solution* $(\hat{\mathbf{x}}, \hat{\Delta \mathrm{u}})$ and the *approximated optimal* one $(\mathbf{x}^*, \mathbf{\Delta u}^*)$ are feasible for this problem, it results that any convex combination of them results in a feasible solution. This proves the first claim of the Theorem.

We now proceed to prove the second claim. The convexity of $V_g(\lambda)$ with respect to λ implies that

$$V_g(\lambda)\leq(1-\lambda)V_g(0)+\lambda V_g(1)$$

Now, consider a point between $x_{ss}(\lambda)$ and $\hat{x}_{ss}$, which can be parameterized with a parameter $\theta \in [0, 1]$ as

$$x_{ss}(\theta)=(1-\theta)x_{ss}(\lambda)+\theta\hat{x}_{ss}$$

Since $V_{ss}(\cdot\,, p)$ is twice differentiable, one can affirm that for every $\lambda \in [0, 1]$ and $\theta \in [0, 1]$, there exists the Hessian $H(\lambda, \theta) = H(x_{ss}(\theta))$; and, by the *mean value theorem*, it follows that

$$V_{ss}(x_{ss}(\lambda),p)=V_{ss}(\hat{x}_{ss},p)+\nabla V_{ss}(\hat{x}_{ss},p)\left[x_{ss}(\lambda)-\hat{x}_{ss}\right]+\frac{1}{2}\left[x_{ss}(\lambda)-\hat{x}_{ss}\right]^T H(\lambda,\theta)\left[x_{ss}(\lambda)-\hat{x}_{ss}\right]$$

for every $\lambda \in [0, 1]$ and for a $\theta \in [0, 1]$. With the last equality, we now have that for every $\lambda \in [0, 1]$ and for some $\theta \in [0, 1]$,

$$V(\lambda)=V_g(\lambda)+\lambda^2\left(\frac{1}{2}\right)\left[x_{ss}^*-\hat{x}_{ss}\right]'H(\lambda,\theta)\left[x_{ss}^*-\hat{x}_{ss}\right] \tag{9}$$

Furthermore, since $x_{ss}(\theta)$ is a point between $x_{ss}(\lambda)$ and $\hat{x}_{ss}$, and $x_{ss}(\lambda)$, a point between x_{ss}^* and $\hat{x}_{ss}$, then an upper bound for the second term in (9) can be computed as:

$$\rho=\max_{\lambda\in[0,1]}\left(\frac{1}{2}\right)\left[x_{ss}^*-\hat{x}_{ss}\right]'H(\lambda)\left[x_{ss}^*-\hat{x}_{ss}\right]$$

where $H(\lambda) = H(x_{ss}(\lambda))$.

Then

$$V(\lambda)\leq V_g(\lambda)+\lambda^2\rho\leq(1-\lambda)V_g(0)+\lambda V_g(1)+\lambda^2\rho=V_g(0)-\lambda(V_g(0)-V_g(1)-\lambda\rho)$$

Since $V_g(0) = \hat{V}$, hence

$$V(\lambda)\leq\hat{V}-\lambda(\hat{V}-V_g(1)-\lambda\rho)$$

Since it is assumed that $V_g(1) < V_g(0) = \hat{V}$, we obtain that $\hat{V} - \lambda(\hat{V} - V_g(1) - \lambda\rho)$ is positive for λ smaller than

$$\tilde{\lambda} = \min\left\{\frac{\hat{V} - V_g(1)}{\rho}, 1\right\}, \tag{10}$$

which implies that

$$V(\lambda) < \hat{V}, \quad \forall \lambda \leq \tilde{\lambda}. \tag{11}$$

This means that for every $\lambda \in [0, \tilde{\lambda}]$, the pair $(\mathbf{x}(\lambda), \mathbf{\Delta u}(\lambda))$ provides not only a feasible solution to the original problem, but also an improved *exact one-layer cost* when compared with the one corresponding to the feasible solution $(\hat{\mathbf{x}}, \hat{\Delta \mathrm{u}})$. □

Remark 5. From a practical point of view, it is not necessary to explicitly compute the maximization (10). In fact, one can heuristically search for a value of λ that gives a cost $V(\lambda)$ smaller than $\hat{V}$. What Theorem 1 ensures, is that this value of λ does exist.

Remark 6. Notice that although $(\mathbf{x}^*, \mathbf{\Delta u}^*)$ is a feasible solution to problem (4), it does not necessarily produce a decrement of the *exact one-layer cost* $V_N(x, p\,; \mathbf{\Delta u})$ respect to the feasible solution $(\hat{\mathbf{x}}, \hat{\Delta \mathrm{u}})$. Since the cost corresponding to the feasible solution (as will be shown in the next section) is closely related with the past cost, the decrement of the exact economic cost is necessary to apply both, the classical convergence proof and the recursive feasibility of an MPC strategy.

6. Proposed algorithm

Based on the results presented in Section 5, an iterative algorithm will be proposed now to obtain an MPC policy. The algorithm is as follows:

Algorithm 1. At each sample time k,

1. compute the *feasible solution* $(\hat{\mathbf{x}}, \hat{\Delta \mathrm{u}})$ to problem (4), using the shifted solution applied to the system at the sample time $k-1$. If the current time is $k=0$, compute the *feasible solution* $(\hat{\mathbf{x}}, \hat{\Delta \mathrm{u}})$ by solving the reduced problem (5).
2. compute the gradient of the economic cost function $V_{ss}(x, p)$ w.r.t. x, $\nabla V_{ss}(x, p)$.
3. compute the *approximated optimal solution* to problem (4), $(\mathbf{x}^*, \mathbf{\Delta u}^*)$, by solving the approximated problem (6).
4. compute the value of the parameter $\tilde{\lambda}$, as in (10). Also, the value of $\tilde{\lambda}$ can be computed heuristically, in such a way that condition (11) holds.
5. if $V(\tilde{\lambda}) > V(1)$, then take $\tilde{\lambda} = 1$.
6. from the solution $(\mathbf{x}^0, \Delta \mathrm{u}^0) \triangleq (\mathbf{x}(\tilde{\lambda}), \Delta \mathrm{u}(\tilde{\lambda}))$, take the first input action of the sequence $\mathbf{\Delta u}^0$ to implement the implicit MPC control law, $\kappa_N(x, p) \triangleq \Delta u^0(0; x)$.

Remark 7. Notice that in the last step of the Algorithm 1, and provided that the sample time of the process is large enough, the solution $(\mathbf{x}(\tilde{\lambda}), \Delta \mathrm{u}(\tilde{\lambda}))$ can be iteratively improved, within the current sample time, to obtain a better approximation to the optimum.

6.1. Stability of the proposed controller

In this section, asymptotical stability of the proposed controller will be proved.

Theorem 2. *Consider that Assumptions 1–4 hold, and consider a given parameter p for the economic cost $V_{ss}(x, p) = f_{eco}(x, p)$. Then, for any initial state $x \in \mathcal{X}_N$, the system controlled by the MPC control law derived from the application of Algorithm 1 at each time step k is stable and fulfills the constraints throughout the time. Furthermore, the closed-loop system converges asymptotically to a steady state x_s that satisfy*

$$x_s = \arg\min_{x \in X_{ss}} f_{eco}(x, p).$$

Proof. *Feasibility:*

The feasibility follows directly from the fact that the set $\mathcal{X}_N$ is a control invariant set for system (A, B).

Convergence and stability:

Consider a state $x \in \mathcal{X}_N$, at a given time k. Consider also the solution defined in Algorithm 1, for this state, $\mathbf{\Delta u}^0(x) = \{\Delta u^0(0\,; x), \ldots, \Delta u^0(N-1\,; x)\}$, and the corresponding state sequence $\mathbf{x}^0(x) = \{x^0(1\,; x), \ldots, x^0(N\,; x)\}$, where $x^0(N; x) = x^0_{ss}(x)$ and

$$x^0_{ss}(x) \in X_{ss}. \tag{12}$$

Now, consider the state $x^+ = Ax + B\Delta u^0(0\,; x) = x^0(1\,; x)$, at time $k+1$, which is obtained by implementing the control law of step 6 of Algorithm 1, and define the following feasible solution to problem $P_N(x^+, p)$ at time $k+1$,

$$\tilde{\Delta \mathrm{u}} = \{\Delta u^0(1; x), \ldots, \Delta u^0(N-1; x), 0\},$$

which is a sequence made by shifting one step ahead the sequence $\mathbf{\Delta u}^0(x)$ and adding a null control action. This solution has an associated state sequence, $\tilde{\mathbf{x}} = \{x^0(2; x), \ldots, x^0_{ss}(x), x^0_{ss}(x)\}$, where the additional state is given by $x^0_{ss}(x) = Ax^0_{ss}(x)$.

Now, following standard arguments in MPC literature [24], two consecutive cost functions will be compared. The cost function corresponding to $\mathbf{\Delta u}^0(x)$ is given by

$$V_N(x, p; \Delta \mathrm{u}^0(x)) = V_N^{dyn}(x; \Delta \mathrm{u}^0(x)) + V_{ss}(x^0_{ss}(x), p). \tag{13}$$

On the other hand, the cost function, at $k+1$, corresponding to $\tilde{\Delta u}$, is given by

$$V_N(x^+, p; \tilde{\Delta u}) = V_N^{dyn}(x^+; \tilde{\Delta u}) + V_{ss}(x_{ss}^0(x), p).$$

If we compare now these costs, we have:

$$\begin{aligned} V_N(x^+, p; \tilde{\Delta u}) - V_N(x, p; \Delta u^0(x)) &= V_N^{dyn}(x^+; \tilde{\Delta u}) - V_N^{dyn}(x; \Delta u^0) = -\|x - x_{ss}^0(x)\|_Q^2 - \|\Delta u^0(0; x)\|_R^2 + \|x_{ss}^0(x) - x_{ss}^0(x)\|_P^2 \\ &= -\|x - x_{ss}^0(x)\|_Q^2 - \|\Delta u^0(0; x)\|_R^2 \end{aligned} \tag{14}$$

Now, from Algorithm 1, step 3, the approximated optimal solution at time $k+1$ is given by $\mathbf{\Delta u}^*(x^+)$, and then, by step 6 of the Algorithm, we obtain $\Delta u^0(x^+) = (1-\lambda)\tilde{\Delta u} + \lambda \Delta u^*(x^+)$.

By Theorem 1, we have that the cost corresponding to the solution $\mathbf{\Delta u}^0(x^+)$, $V_N(x, p; \mathbf{\Delta u}^0(x^+))$, is such that $V_N(x^+, p; \Delta u^0(x^+)) < V_N(x^+, p; \tilde{\Delta u})$. So, from (14), it follows that

$$V_N(x^+, p; \Delta u^0(x^+)) - V_N(x, p; \Delta u^0(x)) \le -\|x - x_{ss}^0(x)\|_Q^2 - \|\Delta u^0(0; x)\|_R^2 \tag{15}$$

Since Q and R are positive definite, (15) implies that there exists a $\mathcal{K}$-function α such that:

$$V_N(x^+, p; \Delta u^0(x^+)) - V_N(x, p; \Delta u^0(x)) \le -\alpha(\|x - x_{ss}^0(x)\|) \tag{16}$$

This is not sufficient to prove convergence to the economically optimal steady state x_s. In fact, what the decreasing cost guarantees is that, roughly speaking, the two sequences x and $x_{ss}^0(x)$ get closer to each other (but they could still be wandering around into X_{ss}).

So, we need to show now that these two sequences converge to the same point, that is $x \to x_s$ and $x_{ss}^0(x) \to x_s$, as $k \to \infty$.

To this aim, let us define the function $\Phi(x, p) = V_N(x, p; \mathbf{\Delta u}^0(x)) - V_{ss}(x_s, p)$. Define also $e(x) = x - x_{ss}^0(x)$. Notice that function $\Phi(x, p)$ is defined on $\mathcal{X}_N$, is such that $\Phi(x = x_s, p) = 0$, and is positive away from $x = x_s$, due to the nonnegativity of the stage cost and the definition of $V_{ss}(x, p)$. Due to this facts, there exists a $\mathcal{K}-$function α such that $\Phi(x, p) \ge \alpha(\| e(x) \|)$, for all $x \in \mathcal{X}_N$. Moreover, from (16) we have that $\Phi(x^+, p) - \Phi(x, p) \le \alpha(\| e(x) \|)$, for all $x \in \mathcal{X}_N$.

From Lemma 4 in the Appendix section, it follows that

$$\alpha(\|e(x)\|) \ge \alpha(\alpha_e(\|x - x_s\|)) = \alpha_\Phi(\|x - x_s\|)$$

where α_Φ is a $\mathcal{K}-$function. Then, we can conclude that

(i) $\Phi(x, p) \ge \alpha_\Phi(\| x - x_s \|)$, for all $x \in \mathcal{X}_N$.
(ii) $\Phi(x^+, p) - \Phi(x, p) \le -\alpha_\Phi(\| x - x_s \|)$, for all $x \in \mathcal{X}_N$.
(iii) Since $\mathcal{X}_N$ is compact, $\Phi(x_s, p) = 0$, and $\Phi(x, p)$ is continuous in $x = x_s$, then there exists a $\mathcal{K}$-function β_Φ such that $\Phi(x, p) \le \beta_\Phi(\| x - x_s \|)$, for all $x \in \mathcal{X}_N$, [24].

Hence $\Phi(x, p)$ is a Lyapunov function and x_s is an asymptotically stable equilibrium point for the closed-loop system, that is, there exists a $\mathcal{KL}$-function ϑ such that

$$\|x(k) - x_s\| \le \vartheta(\|x(0) - x_s\|, k)$$

for all $x(0) \in \mathcal{X}_N$.

■

7. Properties of the proposed controller

The proposed controller provides the following main properties:

- *Solution of just one QP*. The main drawback of including the RTO cost into the MPC controller, is that the control problem requires the solution of a NLP problem, at each sampling time, due to the highly nonlinearity of function f_{eco}. In the proposed controller, since the economic objective function is included in the MPC cost function by means of its gradient, the optimization problem results in just one QP.
- *Not need to compute the Hessian of V_O*. If an heuristic procedure is used to compute $\tilde{\lambda}$ – like bisection methods or backtracking – such that equation (11) holds, then there is not need to compute the Hessian of V_O, which can be computationally demanding. Another possible strategy is to estimate the value of $\tilde{\lambda}$ by estimating the value of the Hessian using the gradient in a Quasi-Newton fashion [22].
- *Feasibility under any change of the economic objective*. Notice that the proposed controller cost function is formulated as distance to any feasible steady state, and the terminal constraints require that the system at the end of the horizon converges to *any* steady state, while the unstable modes are zeroed. As a consequence, the controller guarantees feasibility under any change of the economic objective function.
- *Convergence to the economic objective*. It has been proved that the proposed controller ensures convergence to the point that minimizes $f_{eco}(x, p)$. Since this function is in effect the economic RTO function, the controller ensures convergence to the economic objective.
- *Enlargement of the domain of attraction*. The domain of attraction of the proposed controller, that is set $\mathcal{X}_N$, is a feasible set of initial x such that one can reach *any* feasible steady state – that is, set X_{ss} – with N admissible inputs. In standard MPC formulation, the domain of attraction is a feasible set of initial x such that one can reach *the optimal* steady state x_s with N admissible inputs. Define this last set as $\mathcal{X}_N(x_s)$. Since $\{x_s\} \subset X_{ss}$, hence $\mathcal{X}_N(x_s) \subset \mathcal{X}_N$. Then, the set $\mathcal{X}_N$ is larger, and, in some applications much larger, than the one provided by

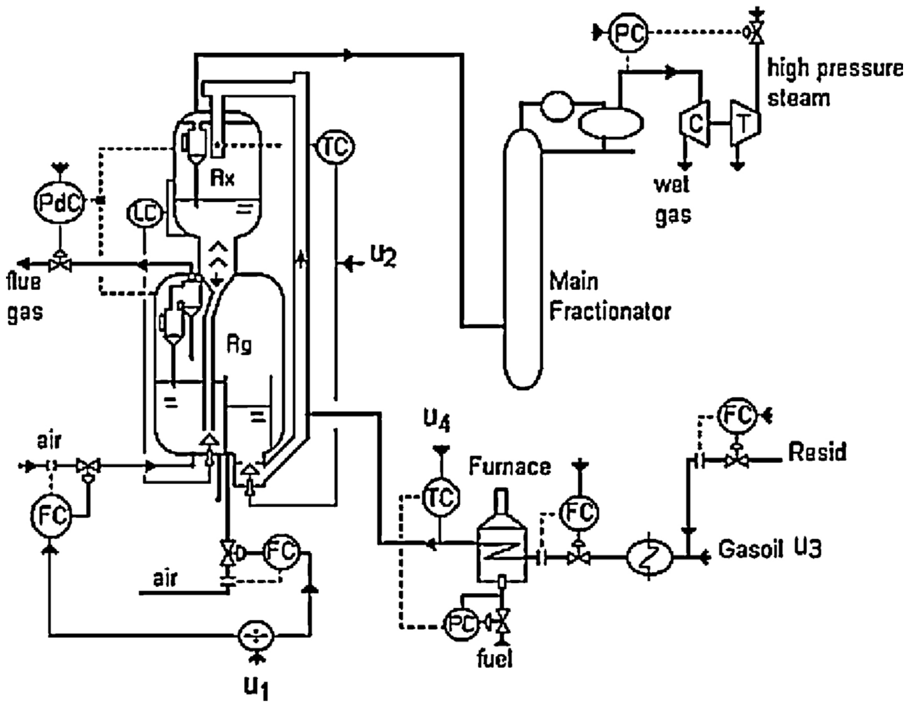

Fig. 1. Scheme of the FCC system [7].

the standard MPC formulation. Notice that, due to the particular choice of a velocity model, the proposed controller provides the largest possible domain of attraction [15].

8. Example

The properties of the proposed controller have been tested in a simulation example, on the fluid catalytic cracking unit (FCC) studied in [27] and [7]. This system is suitable to test the proposed controller, since it represents an industrial problem, that is controlling a multivariable system, subject to constraints, while an economic objective is optimized.

A complete description of the FCC system is given in [18]. In the simplified version considered here (see Fig. 1), the output to be controlled are the temperature in the dilute phase of the regenerator y_1 (C), the temperature in the dense phase of the regenerator y_2 (C), the conversion of the cracking reaction y_3 (%), the riser temperature y_4 (C). The inputs (that in the mode-decoupled model are implicitly included into the state vector) are the total air flow-rate of the two stage catalyst regenerator u_1 (ton/h), the valve opening of the regenerated catalyst u_2 (%), the gasoil feed flow-rate u_3 (m^3/h), the temperature of the feed u_4 (C).

The transfer function of the simplified model is given by:

$$G(s)=\begin{bmatrix} \frac{0.0084}{s^2-1.887+0.8937} & \frac{-0.0014}{s^2-1.969+0.9709} & \frac{-0.0034}{s^2-1.883+0.8889} & \frac{0.0052}{s^2-1.885+0.8939} \\ \frac{0.0082}{s^2-1.91+0.9163} & \frac{-0.0016}{s^2-1.967+0.9689} & \frac{-0.0025}{s^2-1.916+0.9203} & \frac{0.0039}{s^2-1.917+0.9235} \\ \frac{0.0034}{s^2-1.2+0.2277} & \frac{0.0059}{s^2-1.228+0.2554} & \frac{-0.0273}{s^2-0.6782-0.0797} & \frac{0.0026}{s^2-1.046+0.0718} \\ \frac{0.0211}{s^2-1.331+0.3611} & \frac{0.0102}{s^2-1.536+0.5515} & \frac{-0.0612}{s^2-0.7404-0.1309} & \frac{0.0368}{s^2-0.7301-0.2055} \end{bmatrix} \quad (17)$$

By discretizing (17) with a sample time $T_s = 1$ min, we get a discrete-time state-space model with integral modes of the form:

$$x^+ = Ax + B\Delta u$$
$$y = Cx$$

The economic objective is to maximize the production of liquefied petroleum gas (LPG). This function is a nonlinear convex function of the feed properties and the process operating condition [27] and is given by $f_{LPG} = u_3 \times LPGV$, where $LPGV$ is the volumetric yield of LPG. Since maximizing a convex function is equivalent to minimize a concave function, we cannot use f_{LPG} for the algorithm proposed here. However, another way to maximize LPG production is to minimize light cycle oil (LCO) production [7], which is also a nonlinear convex function of the feed properties and the process operating condition (in the next, f_{LCO}).

The system has the following constraints on the inputs: $u_{max} = (228, 98, 406, 235)'$ and $u_{min} = (200, 50, 400, 234.9)'$. Notice that input u_4 has been kept inside a very tight range to represent the case of shut off gasoil heating furnace, which is a very common condition in an FCC unit.

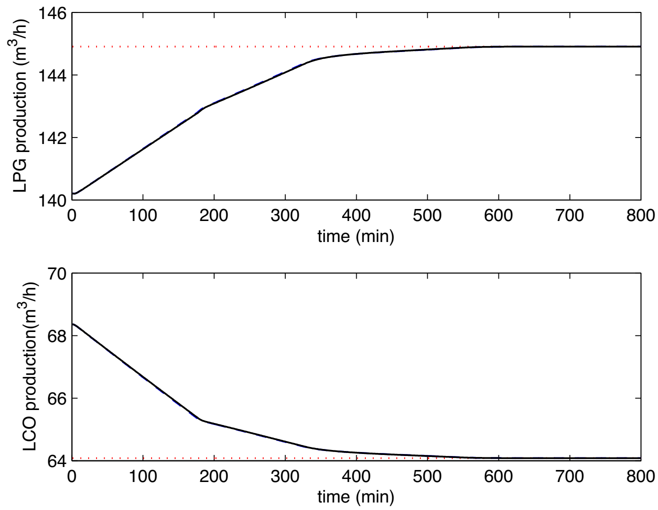

Fig. 2. Time evolution of the LPG and LCO productions.

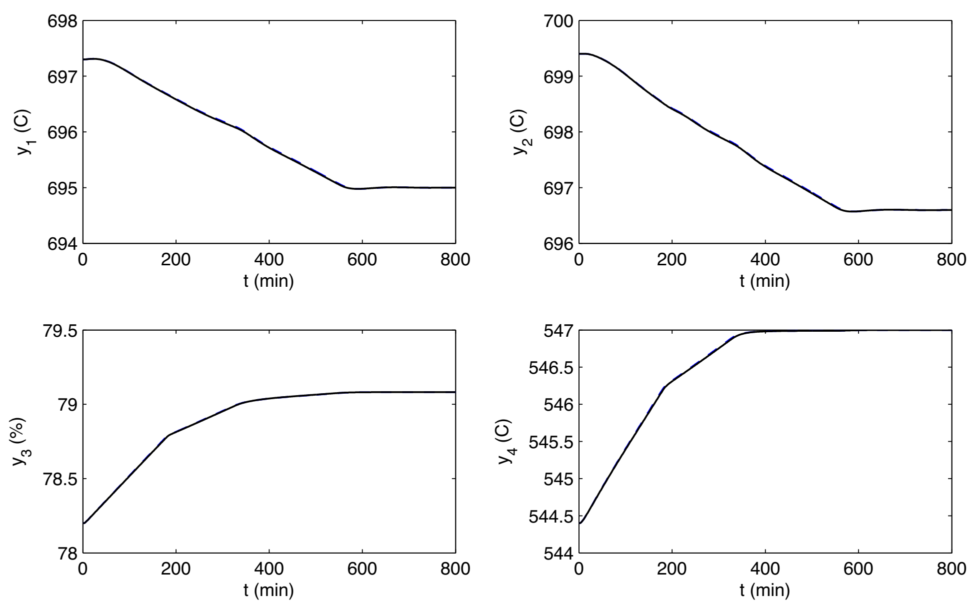

Fig. 3. Evolution of the outputs.

A zone control strategy has been adopted, in such a way that the outputs are required to lie into the zone given by y_{max} = (705, 725, 95, 547)′ and y_{min} = (695, 695, 60, 540)′. The sampling time is T_s = 1 min. The initial steady state is given by y_{ss} = (697.3, 699.4, 78.2, 544.4)′ and u_{ss} = (220.7, 85, 404, 234.9)′.

The first step of this simulation, has been to solve the classic RTO problem (see Definition 1). In this way, we have been able to obtain the optimal value of the economic cost function, $f^*_{LCO} = 64.0810$, which corresponds to an optimal LPG production of $f^*_{LPG} = 144.9068$, and to optimal output and input given respectively by $y^*_s = (695.00, 696.5981, 79.0817, 547.00)'$ and $u^*_s = (218.7965, 88.1039, 400, 234.9)'$.

Then, simulations have been run, in order to compare the performance of the proposed controller with the one of a one-layer MPC with the RTO nonlinear cost built into the cost function.

As for the controllers setup, the weighting matrices of the two MPC controllers have been taken as $Q = C'Q_yC$, where $Q_y = diag(0.2, 0.1, 0.1, 1)$, and $R = diag(5, 5, 5, 5)$. Matrix P is taken as the solution of the Lyapunov equation $P = A'_{st}PA_{st} + Q$. An horizon N = 3 has been considered.

The results of the simulation are presented in Figs. 2–4. Fig. 2 shows the production of LPG and LCO. The values provided by the proposed controller are plotted in black solid line, while the values provided by the one-layer MPC of problem (4) are drawn in blue dashed line. The optimal values are plotted in red dotted line. See how the two evolutions are very similar, and most of all, they converge to the optimal values f^*_{LCO} and f^*_{LPG}.

Figs. 3 and 4 show the time evolution of outputs and inputs, respectively. In these figures, as well as in the previous one, the evolutions of output and input provided by the proposed controller are drawn in black solid line, while the one relative to problem (4) are plotted in blue dashed line.

Notice that these evolutions are very similar. Notice also that, in order to maximize the production of LPG, input u_3 and u_4 are pushed by the controller to their minimum bounds, respectively (represented in Fig. 4 by a red dotted line). This indicates that all four degrees of freedom are used in order to maximize the LPG production, while constraints are always fulfilled.

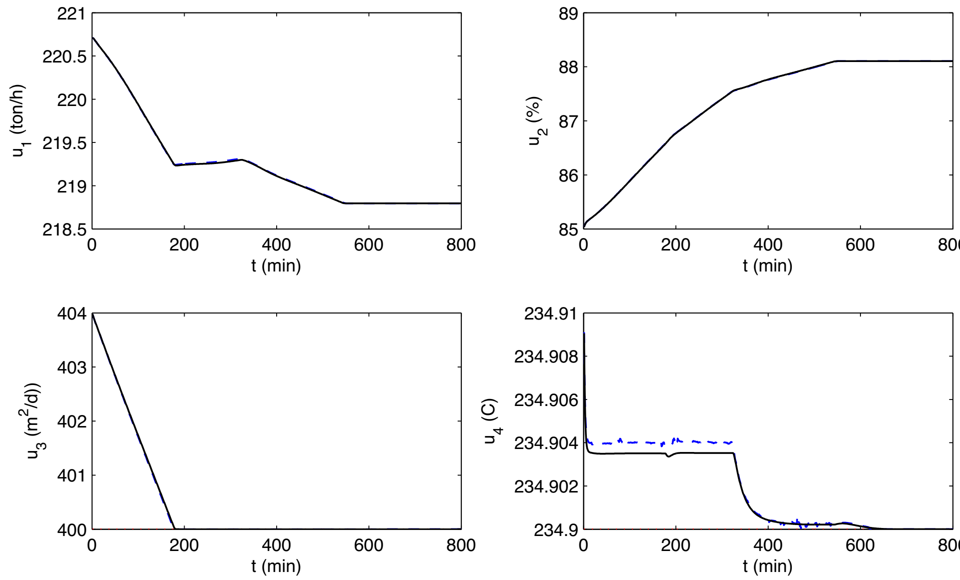

Fig. 4. Evolution of the inputs.

Table 1
Execution time and performance comparison.

	Av. exec. time (s)	Φ	Φ_t
Proposed controller	0.0253	16.4423	647.8234
Problem (4)	0.0896	16.3596	644.2742

Another comparison between the two controllers has been made by calculating the average time of execution of the optimizations, and the performance of both controllers. The performance indexes used for this comparison have been an average measure of the distance to the optimal output and input:

$$\Phi = \frac{1}{T}\sum_{k=0}^{T} \|y(k) - y_s^*\|_{Q_y}^2 + \|u(k) - u_s^*\|_R^2$$

where T is the duration of the simulation, and a transient profit, given by:

$$\Phi_t = \sum_{k=0}^{T} f_{LCO}(k) - f_{LCO}^*$$

The results are given in Table 1.

As it was expected, the average execution time of the proposed controller is smaller than the one provided by the one-layer MPC integrating the nonlinear f_{eco}. In particular, notice that, even if the proposed controller performance indexes are worse (loss of 0.51% and 0.55% respectively), its execution time is less than one third of problem $P_N(x, p)$ execution time, which is a great advantage is case of a higher dimensions system.

9. Conclusions

This paper has presented a one-layer MPC controller that integrates an economic objective by including the gradient of the RTO cost function into the same MPC cost. In virtue of this solution, the control problem can be solved by means of just one QP problem. Moreover, instead of applying to the plant the solution to the approximated problem, a convex combination with a previously obtained solution is considered. This way, recursive feasibility, asymptotic stability and convergence to the economical optimum have been proved.

Two meaningful practical properties of the proposed controller are: (i) since the controller has been formulated as an Infinite Horizon MPC, then it ensures the maximal possible domain of attraction, and (ii) by means of a velocity model in the input, it has been possible to include integrating modes in such a way that, the steady state of the model can be always characterized, and input increment constraints can be added to the control problem.

Appendix A. Technical lemmas

In this section, the technical lemmas used to prove Theorem 2 are presented.

Lemma 3. *Consider system (1) subject to constraints (2). Consider that Assumptions 1–4 hold, and consider a given parameter p for the economic cost $V_{ss}(x, p) = f_{eco}(x, p)$. Consider that the state of the system x is such that $x = x_{ss} \in X_{ss}$ and that this point is a fixed point, that is the solution to (4) is such that $x^+ = x_{ss}$. Let $\bar{x}_s$ be given by*

$$\bar{x}_s = arg \min_{x \in X_{ss}} f_{eco}(x, p)$$

Then $x = x_{ss} = \bar{x}_s$.

Proof. Consider that, since $x = x_{ss}$, the cost function value is given by $V_N(x, p\,; \mathbf{\Delta u}^0(x)) = V_{ss}(x_{ss}, p)$. Assume now that $x_{ss} \neq \bar{x}_s$.

Let us define

$$x_s(\beta) = (1-\beta)x_{ss} + \beta \bar{x}_s \tag{A.1}$$

with $\beta \in [0, 1]$. Since both $\bar{x}_s$ and x_{ss} are in X_{ss}, and this set is convex, then a convex combination of these points, $x_s(\beta)$, is also in X_{ss}. Furthermore, since by Assumption 3 V_{ss} is convex in x, we have that for a given value of p,

$$V_{ss}(x_s(\beta), p) \leq (1-\beta)V_{ss}(x_{ss}, p) + \beta V_{ss}(\bar{x}_s, p).$$

and by optimality of $\bar{x}_s$ we have $V_{ss}(x_{ss}, p) > V_{ss}(\bar{x}_s, p)$, and so

$$V_{ss}(x_s(\beta), p) \leq V_{ss}(x_{ss}, p),$$

for every $\beta \in [0, 1]$.

Since the system is not at the optimal point $\bar{x}_s$, it should be more convenient to move towards that point, than to remain in x_{ss}. That is, the cost to move the system to $x_s(\beta)$ has to be better (smaller) than the cost to keep the system at x_{ss}. So, let us consider the following control sequence:

$$\begin{aligned} \Delta \mathrm{u}(\beta) &= \begin{bmatrix} T_{un}Co_N \\ T_{ss}Co_N \end{bmatrix}^{\dagger} (\mathrm{x}_s - x_{ss})\beta, \quad \text{for some } \beta \in [0, 1] \\ &= H\beta \end{aligned}$$

where $\dagger$ is the pseudo-inverse operator and, for simplicity, we assume that $rank(Co_N) = n_{un} + m$. For some $\beta \in [0, 1]$, this sequence is a feasible sequence that produces the following state sequence:

$$x_j(\beta) = x_{ss} + [Co_j \quad 0_{n,(N-j)\cdot m}]\Delta \mathrm{u}(\beta) = x_{ss} + [Co_j \quad 0_{n,(N-j)\cdot m}]H\beta, \quad \text{for } j \in \mathbb{I}_{0:N}$$

This state sequence fulfills the constraints of problem $P_N(x_{ss}, p)$ and tends asymptotically to the stationary value $x_s(\beta)$, defined in (A.1). The cost $V_N^{dyn}(x_{ss}, \Delta \mathrm{u}(\beta))$ corresponding to this control and state sequences is given by

$$V_N(x_{ss}, p; \Delta \mathrm{u}(\beta)) = V_N^{dyn}(x_{ss}, \Delta \mathrm{u}(\beta)) + V_{ss}(x_s(\beta), p) = \|[MH - (\bar{x}_s - x_{ss})]\beta\|_Q^2 + \|H\beta\|_R^2 + \|(Co_N H - (\bar{x}_s - x_{ss}))\beta\|_P^2 + V_{ss}(x_s(\beta), p)$$

where

$$M = \begin{bmatrix} Co_0 & \mathbf{0}_{n,N\cdot m} \\ Co_1 & \mathbf{0}_{n,(N-1)\cdot m} \\ \vdots & \\ Co_{N-1} & \mathbf{0}_{n,m} \end{bmatrix}$$

The last cost can be re-written as:

$$V_N(\beta) = \beta^2(\|[MH - (\bar{x}_s - x_{ss})]\|_Q^2 + \|H\|_R^2 + \|(Co_N H - (\bar{x}_s - x_{ss}))\|_P^2) + V_{ss}(x_s(\beta), p).$$

This cost trivially satisfies $V_N(0) = V_{ss}(x_{ss}, p)$. Now, let us consider the derivative of $V_N(\beta)$ w.r.t β,

$$\frac{\partial V_N(\beta)}{\partial \beta} = 2\beta(\|[MH - (\bar{x}_s - x_{ss})]\|_Q^2 + \|H\|_R^2 + \|(Co_N H - (\bar{x}_s - x_{ss}))\|_P^2) + g'(\bar{x}_s - x_{ss})$$

where $g' \in \partial V_{ss}(x_s(\beta), p)$, defining $\partial V_{ss}(x_s(\beta), p)$ as the subdifferential of $V_{ss}(x_s(\beta), p)$.

If we now evaluate this derivative at $\beta = 0$, we have

$$\left.\frac{\partial V_N(\beta)}{\partial \beta}\right|_{\beta=0} = \bar{g}'(\bar{x}_s - x_{ss})$$

where $\bar{g}' \in \partial V_{ss}(x_{ss}, p)$, defining $\partial V_{ss}(x_{ss}, p)$ as the subdifferential of $V_{ss}(x_{ss}, p)$. From convexity and taking into account the optimality of $\bar{x}_s$, we get that

$$\bar{g}'(\bar{x}_s - x_{ss}) \leq V_{ss}(\bar{x}_s, p) - V_{ss}(x_{ss}, p) < 0$$

This means that a β does exist, such the cost corresponding to move the system from x_{ss} to $x_s(\beta)$ is smaller that the one corresponding to remain in the stationary state x_{ss}. But this contradicts the assumption that x_{ss} is a fixed point for the closed-loop system, that is, the solution to (4) – $P_N(x, p)$ – at $x = x_{ss}$ is still x_{ss}. So, it has to be that $x_{ss} = \bar{x}_s$, which proves the Lemma.

■

Lemma 4. *Consider system (1) subject to constraints (2). Consider that Assumptions 1–-4 hold. Let x_s be the optimal steady state defined in Definition 1. For all $x \in \mathcal{X}_N$ and $x_{ss}^0 \in X_{ss}$, define the function $e(x) = x - x_{ss}^0$. Then, there exists a $\mathcal{K}$-function α_e such that*

$$\|e(x)\| \geq \alpha_e(\|x - x_s\|) \tag{A.2}$$

Proof. Notice that, due to convexity, $e(x)$ is a continuous function [24]. Moreover, let us consider these two cases.

1 $\|e(x)\| = 0$ iff $x = x_s$. In fact, (i) if $e(x) = 0$, then $x = x_{ss}^0$, and from Lemma 3, this implies that $x_{ss}^0 = x_s$; (ii) if $x = x_s$, then by optimality $x_{ss}^0 = x_s$, and then $x = x_{ss}^0$.
2 $\|e(x)\| > 0$ for all $\|x - x_s\| > 0$. In fact, for any $x \neq x_s$, $\|e(x)\| \neq 0$ and moreover $\|x - x_s\| > 0$. Then, $\|e(x)\| > 0$.

Then, since $\mathcal{X}_N$ is compact, in virtue of [25, Chapter 5, Lemma 6, p. 148], there exists a $\mathcal{K}$-function α_e such that $\|e(x)\| \geq \alpha_e(\|x - x_s\|)$.

■

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