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Brief paper

On the computation of convex robust control invariant sets for nonlinear systems[☆]

M. Fiacchini[a,*], T. Alamo[b], E.F. Camacho[b]

[a] LAAS-CNRS Toulouse, 7 Avenue de Colonel Roche, 31077 Toulouse Cedex 4, France

[b] Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain

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ABSTRACT

In this paper we provide a method to compute robust control invariant sets for nonlinear discrete-time systems. A simple criterion to evaluate if a convex set in state space is a robust control invariant set for a nonlinear uncertain system is presented. The criterion is employed to design an algorithm for computing a polytopic robust control invariant set. The method is based on the properties of DC functions, i.e. functions which can be expressed as the difference of two convex functions. Since the elements of a wide class of nonlinear functions have DC representation or, at least, admit an arbitrarily close approximation, the method is quite general. The algorithm requires relatively low computational resources.

1. Introduction

Part of the importance of invariant sets in control is due to the implicit stability properties of these regions of the state space. An invariant set is a region of the state space such that the trajectory generated by the dynamical system remains confined in the set if the initial condition lies within it, see Blanchini and Miani (2008). Particularly relevant is the property of robust (control) invariance of a set, useful in the context of stability and constraints satisfaction for uncertain systems. Also the issue of convergence of model predictive control strategies, see Camacho and Bordons (2004), is strongly related to invariance. Many results on invariance have been obtained in previous years, for both linear, see for instance Gilbert and Tan (1991), Kolmanovsky and Gilbert (1998) and Raković, Kerrigan, Kouramas, and Mayne (2005), and nonlinear systems, see Alamo, Cepeda, Fiacchini, and Camacho (2009), Bravo, Limon, Alamo, and Camacho (2005) and da Silva and Tarbouriech (1999). The problem of stability of nonlinear uncertain systems has been recently addressed in Chesi, Garulli, Tesi, and Vicino (2009).

In this paper we present a method for computing a convex robust control invariant set for discrete-time nonlinear uncertain systems, exploiting the properties of DC functions. Since many nonlinear functions are expressible as DC functions, or at least, can be arbitrarily closely approximated by one of them, the proposed procedure is widely applicable.

The paper is organized as follows: Section 2 presents the problem statement. In Section 3 a brief review on DC functions is given. Conditions for control invariance for nonlinear systems are formulated in Section 4. In Section 5 computational issues are considered. In Section 6 the results are compared with other methods via a numerical example.

Notation. For every $n \in \mathbb{N}$, define $\mathbb{N}_n = \{x \in \mathbb{N} : 1 \leq x \leq n\}$. Given $A \in \mathbb{R}^{n\times m}$, A_i with $i \in \mathbb{N}_n$ denotes its i-th row. Given a set D, $\text{co}(D)$ denotes its convex hull, $\text{int}(D)$ denotes its interior, ∂D its boundary and, for every $\alpha \geq 0$, define the set $\alpha D = \{\alpha x : x \in D\}$. The operators ∇_x and ∇_u denote the differential with respect to x and u respectively.

2. Problem statement

Consider the nonlinear discrete-time time-invariant system

$$x^+ = f(x, u) + w, \tag{1}$$

where $x \in X \subseteq \mathbb{R}^n$ is the state, $x^+ \in \mathbb{R}^n$ is the successor, $u \in U \subseteq \mathbb{R}^m$ is the control and $w \in W \subset \mathbb{R}^n$ is the unknown but bounded uncertainty, that can be a function of x and u and other terms representing noises and exogenous disturbances.

Assumption 1. Assume that sets $X \subseteq \mathbb{R}^n$, $U \subseteq \mathbb{R}^m$ and $W \subset \mathbb{R}^n$ are closed, convex and contain the origin in their interiors. Assume also that W is compact.

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* Corresponding author. Tel.: ++33 05 61 33 63 07; fax: +33 0 561336936.
E-mail addresses: fiacchini@laas.fr (M. Fiacchini), alamo@cartuja.us.es (T. Alamo), eduardo@cartuja.us.es (E.F. Camacho).

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The formal definition of DC function follows.

Definition 1. A function $\alpha : \mathbb{R}^p \to \mathbb{R}$, defined on $D \subseteq \mathbb{R}^p$ convex, is a DC function if there exist $\beta, \gamma : \mathbb{R}^p \to \mathbb{R}$ convex on D and such that $\alpha(x) = \beta(x) - \gamma(x)$ for all $x \in D$.

We will refer to $\alpha : \mathbb{R}^p \to \mathbb{R}^q$ as a DC function if $\alpha_j(\cdot)$ is DC for all $j \in \mathbb{N}_q$. Similarly, $\beta : \mathbb{R}^p \to \mathbb{R}^q$ is convex if $\beta_j(\cdot)$ is convex for all $j \in \mathbb{N}_q$. We say that $f(\cdot,\cdot)$, as in (1), is a DC function with respect to variables $x \in X$ and $u \in U$, meaning that $f(x,u) = g(x,u) - h(x,u)$ where $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are convex in $(x,u) \in X \times U$.

Assumption 2. Assume that $f(\cdot,\cdot)$ in (1) is a DC function defined on $X \times U$ and differentiable at $(0,0)$. Denote $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ the convex functions such that $f(x,u) = g(x,u) - h(x,u)$, for all $(x,u) \in X \times U$ and assume $g(0,0) = h(0,0) = 0$.

Below we recall general definitions of a robust control invariant set and a λ-contractive set for a generic nonlinear system, see Blanchini and Miani (2008).

Definition 2. A set Ω is a robust control invariant set for the system $x^+ = f(x,u,w)$ and constraints $x \in X$ and $u \in U$ if $\Omega \subseteq X$ and for all $x \in \Omega$ there exists a $u(x) \in U$ such that $f(x,u(x),w) \in \Omega$, for all $w \in W$.

In absence of uncertainty the related set is called a control invariant set. Hence, a set Ω is a robust control invariant set for the system if and only if there exists an admissible control law $u = u(x) \in U$ defined for all $x \in \Omega$ such that every trajectory of the controlled system (1) starting within Ω remains inside it regardless of the uncertainty realization.

Definition 3. A convex compact set Ω with $0 \in \text{int}(\Omega)$ is λ-contractive for system $x^+ = f(x,u,w)$ and constraints $x \in X$ and $u \in U$ if $\Omega \subseteq X$ and for all $x \in \Omega$ there exists a $u(x) \in U$ such that $f(x,u(x),w) \in \lambda\Omega$, for all $w \in W$, with $\lambda \in [0,\ 1]$.

Analogous definitions can be given for autonomous systems. Clearly, λ-contractiveness induces (robust) control invariance. For $\Omega \subseteq X$ polytopic, i.e. $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\}$, the condition for λ-contractiveness is the existence of $u(x) \in U$ such that $Hf(x,u(x),w) \leq \lambda 1$, for all $x \in \Omega$ and $w \in W$.

3. Brief overview on DC functions

We recall some important properties of DC functions. First, it is worth mentioning that the set of DC functions defined on a compact convex set of $\mathbb{R}^n$ is dense in the set of continuous functions of this set. Then, every function defined on a compact convex set admits a DC approximation arbitrarily close. Moreover, any twice differentiable function is DC representable. In fact, suppose $f : D \to \mathbb{R}$ satisfies $\frac{\partial^2}{\partial x^2} f(x) \geq -2aI$, for all $x \in D$ with $a > 0$. Then $f(x) = g_c(x) - h_c(x)$, with $g_c(x) = f(x) + ax^Tx$ and $h_c(x) = ax^Tx$ is a DC representation of $f(x)$. See Adjiman and Floudas (1996) for methods to obtain appropriate values of a. Some properties of DC functions, see Horst and Thoai (1999), follow.

Property 1. *DC functions satisfy the following properties:*

- *Every function $f : \mathbb{R}^n \to \mathbb{R}$ whose second partial derivatives are continuous everywhere is DC.*
- *Let D be a compact convex subset of $\mathbb{R}^n$. Then, every continuous function on D is the limit of a sequence of DC functions which converges uniformly on D.*
- *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a DC function and $g : \mathbb{R} \to \mathbb{R}$ convex. Then, the composite function $(g \circ f)(x) = g(f(x))$ is DC.*

If $f(\cdot)$ is a DC function, also the following functions are DC:

- *Any affine combination of DC functions.*
- *The pointwise maximum and minimum of DC functions.*
- *Functions $|f(x)|$, $\max\{0, f(x)\}$ and $\min\{0, f(x)\}$.*
- *The product of DC functions.*

Finally, note that there exist infinitely many DC representations for every DC function $f(x) = g(x) - h(x)$, obtained by adding the same convex function to $g(\cdot)$ and $h(\cdot)$, for instance.

4. Control invariance condition for nonlinear systems

The results presented in this section constitute the main contributions of this paper. Sufficient conditions for a set to be a control invariant set for a DC system are given.

Definition 4. Let Assumptions 1 and 2 hold. Given the DC function $f(\cdot,\cdot) : X \times U \to \mathbb{R}^n$ as in (1) and $c \in \mathbb{R}^n$, define $F(\cdot,\cdot,\cdot) : X \times U \times \mathbb{R}^n \to \mathbb{R}$ as the function

$$F(x,u,c) = \sum_{j \in b_+} c_j(g_j(x,u) - h_j^L(x,u)) + \sum_{j \in b_-} c_j(g_j^L(x,u) - h_j(x,u)), \quad (2)$$

where $g_j^L(x,u) = \nabla_x g_j(0,0)x + \nabla_u g_j(0,0)u$ and $h_j^L(x,u) = \nabla_x h_j(0,0)x + \nabla_u h_j(0,0)u$, for $j \in \mathbb{N}_n$, and $b_+ = b_+(c) = \{j \in \mathbb{N}_n : c_j \geq 0\}$ and $b_- = b_-(c) = \{j \in \mathbb{N}_n : c_j < 0\}$.

Notice that $b_+(c)$ and $b_-(c)$ are the set of indices of non-negative and negative elements of c, respectively.

Property 2. *Let Assumptions 1 and 2 hold. Given the DC function $f(\cdot,\cdot) : X \times U \to \mathbb{R}^n$ as in (1), for every $c \in \mathbb{R}^n$, function $F(\cdot,\cdot,c)$ defined in (2) is convex in $(x,u) \in X \times U$.*

Proof. Function $F(\cdot,\cdot,c)$ is the sum of elements composed by the sum of a convex term and a linear one, then convex. □

In the following it is proved that, for any $c \in \mathbb{R}^n$, the function $F(\cdot,\cdot,c)$ provides an upper bound of the function $c^T f(\cdot,\cdot)$.

Property 3. *Let Assumptions 1 and 2 hold. Given the DC function $f(\cdot,\cdot) : X \times U \to \mathbb{R}^n$ as in (1), for every $c \in \mathbb{R}^n$ we have that $c^T f(x,u) \leq F(x,u,c)$, for all $(x,u) \in X \times U$.*

Proof. Being $g_j^L(\cdot,\cdot)$ and $h_j^L(\cdot,\cdot)$ the linearizations at $(0,0)$ of convex functions, then $g_j^L(x,u) \leq g_j(x,u)$ and $h_j^L(x,u) \leq h_j(x,u)$ for all $j \in \mathbb{N}_n$ and $(x,u) \in X \times U$. Thus, $c_j(h_j^L(x,u) - h_j(x,u)) \leq 0$ if $j \in b_+$ and $c_j(g_j(x,u) - g_j^L(x,u)) \leq 0$ if $j \in b_-$ for all $j \in \mathbb{N}_n$ and $(x,u) \in X \times U$. From this and (2), we have

$$c^T f(x,u) - F(x,u,c) = \sum_{j \in b_+} c_j \left(h_j^L(x,u) - h_j(x,u)\right) + \sum_{j \in b_-} c_j \left(g_j(x,u) - g_j^L(x,u)\right) \leq 0$$

for all $c \in \mathbb{R}^n$ and for any $(x,u) \in X \times U$. □

If $g_j(\cdot,\cdot)$ and $h_j(\cdot,\cdot)$ are convex but not differentiable at $(0,0)$, the linear functions $g_j^L(\cdot,\cdot)$ and $h_j^L(\cdot,\cdot)$ can be obtained by means of the subgradient of $g_j(\cdot,\cdot)$ and $h_j(\cdot,\cdot)$ at $(0,0)$.

Property 4. *Let Assumptions 1 and 2 hold. Given the DC function $f(\cdot,\cdot) : X \times U \to \mathbb{R}^n$ as in (1), for every $c^k \in \mathbb{R}^n$ and $\theta^k \geq 0$, with $k \in \mathbb{N}_{n_k}$, such that $c = \sum_{k=1}^{n_k} \theta^k c^k$, we have*

$$F(x,u,c) \leq \sum_{k=1}^{n_k} \theta^k F(x,u,c^k), \quad \forall (x,u) \in X \times U. \quad (3)$$

Proof. Given $j \in \mathbb{N}_n$, denote $d = d(j) = [c_j^1, c_j^2, \ldots, c_j^{n_k}]^T$ and define $d^+ = \sum_{k \in b_+(d)} \theta^k c_j^k$ and $d^- = \sum_{k \in b_-(d)} \theta^k c_j^k$. We have that

$d^+ \geq 0$, $d^- \leq 0$, by definition. Suppose that $c_j \in b_+(c)$, the case of $c_j \in b_-(c)$ is similar. If we prove

$$c_j\left(g_j(x,u) - h_j^L(x,u)\right) \leq d^+\left(g_j(x,u) - h_j^L(x,u)\right) + d^-\left(g_j^L(x,u) - h_j(x,u)\right), \quad (4)$$

that is, if the j-th term of the left-hand side of (3) is smaller than or equal to the j-th term of the right-hand side, the property is proved. Since $c_j = d^+ + d^-$, $g_j(x,u) - g_j^L(x,u) \geq 0$, $h_j(x,u) - h_j^L(x,u) \geq 0$ by convexity, and $d^- \leq 0$, we have

$$c_j\left(g_j(x,u) - h_j^L(x,u)\right) \leq (d^+ + d^-)\left(g_j(x,u) - h_j^L(x,u)\right) - d^-\left(g_j(x,u) - g_j^L(x,u)\right) - d^-\left(h_j(x,u) - h_j^L(x,u)\right),$$

which is equivalent to (4). Then the property is proved. □

We recall here the definition of support function of a set.

Definition 5. Given a set $\Gamma \subseteq \mathbb{R}^n$, the support function of Γ evaluated at $c \in \mathbb{R}^n$ is defined as: $\phi_\Gamma(c) = \sup_{x\in\Gamma} c^T x$.

If Γ is bounded then its support function is finite for any $c \in \mathbb{R}^n$. If Γ is convex and compact, it can be expressed as $\Gamma = \{x \in \mathbb{R}^n : c^T x \leq \phi_\Gamma(c), \forall c \in \mathbb{R}^n\}$ and if Γ is a polytope, $\Gamma = \{x \in \mathbb{R}^n : Hx \leq b\}$, then $x \in \Gamma$ if and only if $H_i x \leq b_i = \phi_\Gamma(H_i^T)$, for all $i \in \mathbb{N}_{n_h}$, see Rockafellar (1970). The following property provides the necessary and sufficient condition for λ-contractiveness (and then for robust control invariance) for system (1) in terms of support functions.

Property 5. *Let Assumptions 1 and 2 hold. Given $\lambda \in [0,1]$, a convex, compact set $\Omega \subseteq X$ is a λ-contractive set for system (1) and constraints $x \in X$ and $u \in U$ if and only if there exists a control law $u = u(x) \in U$ such that*

$$c^T f(x, u(x)) \leq \lambda\phi_\Omega(c) - \phi_W(c), \quad \forall x \in \Omega,\ \forall c \in \mathbb{R}^n. \quad (5)$$

Proof. By definition, $\Omega \subseteq X$ is a λ-contractive set for system (1) if there exists a control law $u = u(x) \in U$ such that $x^+ = f(x, u(x)) + w \in \lambda\Omega$, for all $x \in \Omega$, and $w \in W$. Since Ω is a convex, compact set, this is equivalent to

$$c^T(f(x,u(x)) + w) \leq \phi_{\lambda\Omega}(c), \quad \forall x \in \Omega,\ \forall w \in W, \forall c \in \mathbb{R}^n. \quad (6)$$

Since $\phi_{\lambda\Omega}(c) = \lambda\phi_\Omega(c)$, then (6) holds if and only if $c^T f(x,u(x)) \leq \lambda\phi_\Omega(c) - \sup_{w\in W} c^T w$, for all $x \in \Omega$ and any $c \in \mathbb{R}^n$, which, in turn, is equivalent to (5). □

Notice that condition (5) is given by an infinite number of non-convex constraints, for all $x \in \Omega$ and all $c \in \mathbb{R}^n$. It will be shown that, if Ω is a polytope, a condition for invariance can be posed as a finite number of convex constraints involving only its vertices. Recall that the condition for invariance for nonlinear systems cannot, in general, be restricted to the boundary of the set, see Blanchini and Miani (2008).

4.1. Control invariance for polytopic Ω

First we consider a sufficient condition for a polytope $\Omega \subseteq X$ to be λ-contractive for the deterministic nonlinear system

$$x^+ = f(x,u), \quad (7)$$

where $f(\cdot,\cdot)$ is the DC dynamical function of (1). Then the result will be used to provide a sufficient condition for robust control invariance of a polytope for the uncertain nonlinear system (1). In what follows, given a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$, its n_v vertices are denoted $v^j \in \mathbb{R}^n$, for $j \in \mathbb{N}_{n_v}$, and n_h are the rows of H, i.e. $H \in \mathbb{R}^{n_h\times n}$.

Property 6. *Let Assumptions 1 and 2 hold. Given $\lambda_n \in [0,1]$ and a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$, if there exist control actions defined at the vertices, $u^j = u(v^j) \in U$, for all $j \in \mathbb{N}_{n_v}$, such that*

$$F(v^j, u^j, H_i^T) \leq \lambda_n, \quad \forall j \in \mathbb{N}_{n_v},\ \forall i \in \mathbb{N}_{n_h}, \quad (8)$$

then Ω is a λ-contractive set for system (7) and constraints $x \in X$ and $u \in U$. Moreover, there exists $u(x) \in U$ defined on Ω such that for any $x_0 \in \Omega$ the trajectory $\{x_k\}_{k\in\mathbb{N}}$ generated by (7) with $u_k = u(x_k)$, satisfies $x_k \in \lambda_n^k\Omega$, for all $k \in \mathbb{N}$.

Proof. From Property 3, it follows that

$$F(x,u,H_i^T) \leq \lambda\phi_\Omega(H_i^T) = \lambda, \quad \forall x \in \Omega,\ \forall i \in \mathbb{N}_{n_h} \quad (9)$$

implies (5) with $W = \{0\}$, and then λ-contractiveness of Ω. In general the inverse is not true, the condition is only sufficient. We prove that there exists $u^j \in U$ defined at vertices v^j, for $j \in \mathbb{N}_{n_v}$ such that (8) is satisfied if and only if there exists $\bar{u}(x) \in U$ defined on Ω such that (9) is fulfilled. Necessity is trivial, since $v^j \in \Omega$ for all $j \in \mathbb{N}_{n_v}$. Sufficiency has to be proved. Since any $x \in \Omega$ can be expressed as the convex combination of the vertices then there exist $\theta^j(x) \geq 0, j \in \mathbb{N}_{n_v}$, such that $x = \sum_{j=1}^{n_v}\theta^j(x)\,v^j$ and $\sum_{j=1}^{n_v}\theta^j(x) = 1$. Moreover $u(x) = \sum_{j=1}^{n_v}\theta^j(x)\,u^j$ is admissible, i.e. $u(x) \in U$, from convexity of U. Consider $\epsilon \in [0,1]$. From the convexity of $F(\cdot,\cdot,H_i^T)$, for any $H_i^T \in \mathbb{R}^n$ and (8), we have that

$$F(\epsilon v^j, \epsilon u^j, H_i^T) - \epsilon\lambda_n \leq \max_{\epsilon\in[0,1]}\left\{F(\epsilon v^j, \epsilon u^j, H_i^T) - \epsilon\lambda_n\right\}$$
$$= \max\left\{F(0,0,H_i^T) - 0; F(v^j, u^j, H_i^T) - \lambda_n\right\} \leq 0,$$

that means that $F(\epsilon v^j, \epsilon u^j, H_i^T) \leq \epsilon\lambda_n$, for all $j \in \mathbb{N}_{n_v}$ and $i \in \mathbb{N}_{n_h}$, for any $\epsilon \in [0,1]$. Consider $\bar{x} \in \epsilon\Omega$ and notice that there exists $x \in \Omega$ such that $\bar{x} = \epsilon x = \sum_{j=1}^{n_v}\theta^j(x)\epsilon v^j$, by definition. Define $u(\bar{x}) = \epsilon u(x) = \sum_{j=1}^{n_v}\theta^j(x)\epsilon u^j$, clearly $u(\bar{x}) \in U$. From Property 3 and convexity of function $F(\cdot,\cdot,H_i^T)$, for any $H_i^T \in \mathbb{R}^n$, it follows that if $\bar{x} \in \epsilon\Omega$ then

$$H_i f(\bar{x}, u(\bar{x})) \leq F\left(\sum_{j=1}^{n_v}\theta^j(x)\epsilon v^j, \sum_{j=1}^{n_v}\theta^j(x)\epsilon u^j, H_i^T\right)$$
$$\leq \sum_{j=1}^{n_v}\theta^j(x)F\left(\epsilon v^j, \epsilon u^j, H_i^T\right)$$
$$\leq \sum_{j=1}^{n_v}\theta^j\epsilon\lambda_n = \epsilon\lambda_n.$$

This with $\epsilon = 1$ implies (9) and then λ-contractiveness. Moreover, we have that $\bar{x} \in \epsilon\Omega$ implies $f(\bar{x}, u(\bar{x})) \in \epsilon\lambda_n\Omega$, for all $\epsilon \in [0,1]$, thus $x_0 \in \Omega$ implies $x_k \in \lambda_n^k\Omega$. □

The following corollary can be employed to enlarge a (robust) control invariant set.

Corollary 1. *Let Assumptions 1 and 2 hold. Consider a polytopic set $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$, and admissible control actions defined at the vertices $u^j = u(v^j) \in U$ for all $j \in \mathbb{N}_{n_v}$ such that the condition (8) is fulfilled. Given $\bar{x} \in X$, define $\bar{\Omega} = \mathrm{co}(\Omega \cup \bar{x}) = \{x \in \mathbb{R}^n : \bar{H}x \leq 1\}$, where $\bar{H} \in \mathbb{R}^{n_{\bar{h}}\times n}$ and $n_{\bar{h}} \in \mathbb{N}$. If there exists $\bar{u} = \bar{u}(\bar{x}) \in U$ such that $F(\bar{x}, \bar{u}, \bar{H}_i^T) \leq \lambda_n$, for every $i \in \mathbb{N}_{n_{\bar{h}}}$, then $\bar{\Omega}$ is a λ-contractive set for system (7) and constraints $x \in X$ and $u \in U$.*

Proof. Consider $\bar{\Omega}$ as candidate λ-contractive set in Property 6. If $\bar{x} \in \Omega$, then $\bar{\Omega} = \Omega$, trivial. Consider $\bar{x} \notin \Omega$. We check condition (8) for $\bar{\Omega}$ and all its vertices, that are given by $\bar{x}$ and a subset of the vertices of Ω. Point $\bar{x}$ fulfills it by assumption. Consider any vertex of Ω. Since $\Omega \subseteq \bar{\Omega}$ then $a_i = \max_x\{\bar{H}_i x : x \in \Omega\} \leq 1$, for every $i \in \mathbb{N}_{n_{\bar{h}}}$. Since strong duality holds, see Boyd and Vandenberghe (2004), we have that $a_i = \min_{\theta^i \in \mathbb{R}^{n_h}} \left\{ \sum_{k=1}^{n_h} \theta_k^i : \bar{H}_i = \sum_{k=1}^{n_h} \theta_k^i H_k,\ \theta^i \geq 0 \right\}$, which means that the dual optimizer, denote it $\bar{\theta}^i \in \mathbb{R}^{n_h}$, is such that $\bar{H}_i = \sum_{k=1}^{n_h} \bar{\theta}_k^i H_k$ and $\sum_{k=1}^{n_h} \bar{\theta}_k^i = a_i \leq 1$, for all $i \in \mathbb{N}_{n_{\bar{h}}}$. From Property 4, for all v^j and u^j, $j \in \mathbb{N}_{n_v}$, and for all $i \in \mathbb{N}_{n_{\bar{h}}}$, we have that $F(v^j, u^j, \bar{H}_i^T) \leq \sum_{k=1}^{n_k} \bar{\theta}_k^i F(v^j, u^j, H_k^T) \leq a_i \lambda_n \leq \lambda_n$, since vertices of Ω satisfy (8). The result is proved. □

Property 6 provides a condition to determine if a polytope is a control invariant set for system (7), Corollary 1 permits to determine an enlarged control invariant set. These results are extended to the uncertain system (1). No proof is given since it is analogous to those of Property 6 and Corollary 1.

Property 7. *Let Assumptions 1 and 2 hold. Consider a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$, and the uncertain DC system (1). If there exist control actions defined at the vertices $u^j = u(v^j) \in U$ for all $j \in \mathbb{N}_{n_v}$, such that*

$$F(v^j, u^j, H_i^T) \leq \lambda_w - \phi_W(H_i^T), \quad \forall j \in \mathbb{N}_{n_v},\ \forall i \in \mathbb{N}_{n_h}, \tag{10}$$

for a $\lambda_w \in [0, 1]$, then Ω is a λ-contractive set for system (1) and constraints $x \in X$, $u \in U$. Moreover, given any $\bar{x} \in X$ and denoting $\bar{\Omega} = \mathrm{co}(\Omega \cup \bar{x}) = \{x \in \mathbb{R}^n : \bar{H}x \leq 1\}$ with $\bar{H} \in \mathbb{R}^{n_{\bar{h}}}$, if there exists $\bar{u} = u(\bar{x}) \in U$ satisfying $F(\bar{x}, \bar{u}, \bar{H}_i^T) \leq \lambda_w - \phi_W(\bar{H}_i^T)$, for all $i \in \mathbb{N}_{n_{\bar{h}}}$, also the set $\bar{\Omega}$ is a λ-contractive set for the system (1) and constraints $x \in X$, $u \in U$.

Notice that a λ-contractive set Ω for the deterministic system (7) with contracting factor λ_n, is also a λ-contractive set for the uncertain system (1), with contraction factor λ_w if W is such that $\max_{i \in \mathbb{N}_{n_h}} \phi_W(H_i^T) \leq \lambda_w - \lambda_n$.

5. Practical issues on design

The first issue to be tackled in order to apply the results shown in the previous section is how to define the potential control invariant set Ω. Once a suitable guess for Ω is given, the sufficient condition for control invariance can be applied. One possible choice is to select, as initial guess of Ω, a (robust) invariant set for the linear system obtained linearizing the nonlinear one, using for instance results in Blanchini and Miani (2008), Gilbert and Tan (1991) and Kolmanovsky and Gilbert (1998). An algorithmic procedure yielding an invariant set for a deterministic autonomous DC system is provided in Fiacchini, Álamo, and Camacho (2007). We propose here a procedure to obtain λ-contractive polytopes for nonlinear systems, based on (10). The approach leads to a convex optimization problem. Afterward, an enlarging method which permits one to generate a sequence of nested λ-contractive polytopes is illustrated. The enlarging method is characterized by a greater computational burden and then can be applied only to relatively low dimensional systems. Once a control λ-contractive set Ω is computed, many approaches can be considered in order to obtain the control law which makes the set λ-contractive in closed-loop. From a practical point of view, it is sufficient to define a control law at the vertices, $u^j = u(v^j)$, since any proper convex combination of $u(v^j)$, $j \in \mathbb{N}_{n_v}$, ensures invariance of Ω.

5.1. Computation of robust λ-contractive polytope

Consider a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\}$, as the initial guess determining the geometric shape of the λ-contractive set. The objective is to determine the maximal $\alpha \geq 0$, such that the set $\alpha\Omega$ is a λ-contractive set for the nonlinear system (1). Recall that, for every $\alpha \geq 0$, we have that $\alpha\Omega = \{x \in \mathbb{R}^n : Hx \leq \alpha\}$ and its vertices are αv^j, for $j \in \mathbb{N}_{n_v}$. The maximal value of α can be obtained by solving n_v convex optimization problems in $1 + m$ variables. Notice that the complexity grows with the number of vertices of the polytope Ω.

Algorithm 1 Computing a λ-contractive set for system (1).

Given the system (1) and the polytope Ω:

for $j = 1, \ldots, n_v$ solve

$$\alpha^j = \max_{\gamma^j > 0,\ u^j \in U} \gamma^j, \quad \text{s.t.} \quad F(\gamma^j v^j, u^j, H_i^T) \leq \lambda_w \gamma^j - \phi_W(H_i^T),\ i \in \mathbb{N}_{n_h},$$

end

$\alpha = \min\{\alpha^j : j \in \mathbb{N}_{n_v}\}$,

return $\alpha\Omega$.

5.2. Enlarging method

Once a control invariant or λ-contractive set Ω has been obtained, a first enlarging method, based on Corollary 1, can be designed. Random points $\bar{x}$ in the state space are generated: if for $\bar{x} \in X$ there exists a $\bar{u} \in U$ fulfilling the hypothesis of Corollary 1, then the new control invariant set is obtained as the convex hull of the current control invariant set and point $\bar{x}$. It has to be pointed out that the enlargement step often requires a considerable computational effort. Notice in fact that simply the selection of points $\bar{x} \in X$ such that $\bar{x} \notin \Omega$ can be non-trivial for high dimensional problems. We propose an alternative procedure based on the following corollary.

Corollary 2. *Let Assumptions 1 and 2 hold. Consider a polytope $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\} \subseteq X$, with $H \in \mathbb{R}^{n_h \times n}$, and $\lambda \in [0, 1]$, such that hypothesis of Property 7 holds for Ω, and, given $\hat{x} \in X$, define the set $\hat{\Omega} = \mathrm{co}(\Omega \cup \hat{x})$. If there exists a $\hat{u} = \hat{u}(\hat{x}) \in U$ such that $F(\hat{x}, \hat{u}, H_i^T) \leq \lambda_w - \phi_W(H_i^T)$, for every $i \in \mathbb{N}_{n_h}$, then $\hat{\Omega}$ is a robust λ-contractive set for system (1) and constraints $x \in X$ and $u \in U$.*

Proof. Consider the non-trivial case of $x \notin \Omega$. We prove that $\hat{\Omega}$ satisfies the hypothesis of Property 7. Denote $\hat{H} \in \mathbb{R}^{n_{\hat{h}} \times n}$ the matrix such that $\hat{\Omega} = \{x \in \mathbb{R}^n : \hat{H}x \leq 1\}$. The set of $n_{\hat{v}}$ vertices of $\hat{\Omega}$ is composed by $\hat{x}$ and a subset of vertices of Ω and then every vertex of $\hat{\Omega}$ satisfies (10). We prove that satisfaction of condition (10) with H_i^T, for every $i \in \mathbb{N}_{n_h}$, also implies fulfillment with $\hat{H}_j^T$, for all $j \in \mathbb{N}_{n_{\hat{h}}}$. As proved for Corollary 1, $\Omega \subseteq \hat{\Omega}$ implies the existence of $\theta^j \geq 0$ such that $\hat{H}_j^T = \sum_{k=1}^{n_h} \theta_k^j H_k^T$, $\sum_{k=1}^{n_h} \theta_k^j \leq 1$ for all $j \in \mathbb{N}_{n_{\hat{h}}}$. From this and Property 4 we have that for vertex $\hat{v}^i$ of $\hat{\Omega}$ there exists $\hat{u}^i \in U$ such that $F(\hat{v}^i, \hat{u}^i, \hat{H}_j^T) \leq \lambda_w - \sum_{k=1}^{n_h} \theta_k^j \phi_W(H_k^T)$, for all $j \in \mathbb{N}_{n_{\hat{h}}}$, and every $k \in \mathbb{N}_{n_{\hat{v}}}$. From $\phi_W(\alpha\eta) = \alpha\phi_W(\eta)$ for all $\alpha \geq 0$ and $\phi_W(\eta + \gamma) \leq \phi_W(\eta) + \phi_W(\gamma)$ for all $\eta, \gamma \in \mathbb{R}^n$, see Schneider (1993), we have that $\phi_W(\hat{H}_j^T) \leq \sum_{k=1}^{n_h} \theta_k^j \phi_W(H_k^T)$ for all $j \in \mathbb{N}_{n_{\hat{h}}}$, then $F(\hat{v}^i, \hat{u}^i, \hat{H}_j^T) \leq \lambda_w - \phi_W(\hat{H}_j^T)$. Hence the hypothesis of Property 7 holds for $\hat{\Omega}$. □

The following convex optimization problem can be solved to determine a point $\hat{x} \in X$ to enlarge the λ-contractive set $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\}$. Given $c \in \mathbb{R}^n$, the optimizer of

$$\max_{\hat{x} \in X, \hat{u} \in U} c^T \hat{x} \quad \text{s.t.} \quad F(\hat{x}, \hat{u}, H_i^T) \leq \lambda_w - \phi_W(H_i^T), \quad i \in \mathbb{N}_{n_h}, \tag{11}$$

is such that $\hat{\Omega} = \mathrm{co}(\Omega \cup \hat{x})$ is a λ-contractive set for the system (1) and such that $\Omega \subseteq \hat{\Omega}$. Thus, the iterative resolution of problem (11) provides a sequence of nested control λ-contractive polytopes Ω_k. Although a random component is still present, in the choice $c \in \mathbb{R}^n$, with this enlarging method point $\hat{x}$ lies in the complement of Ω or on its boundary.

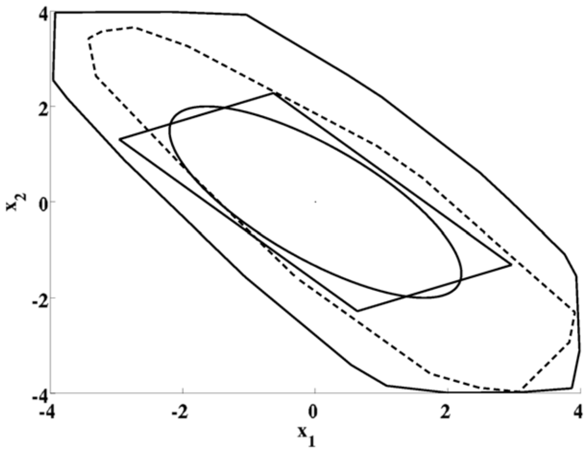

Fig. 1. Comparison: invariant ellipsoid, invariant parallelogram and Ω_{60} and robust control invariant set (dashed).

6. Numerical example

To illustrate the presented method, we apply it to an example proposed in a continuous-time version in Chen, Ballance, and O'Reilly (2001), where ellipsoidal invariant sets are considered. The same system, discretized, has been used by Cannon, Deshmukh, and Kouvaritakis (2003) to test their results on computation of control invariant parallelogram. The example allows us to compare the results illustrated in this paper with different methods. Consider the system

$$x_{k+1} = \begin{bmatrix} 1 & T \\ T & 1 \end{bmatrix} x_k + T \left\{ \mu \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1-\mu) \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} x_k \right\} u_k$$

with $T = 0.01$, $\mu = 0.9$ and the constraints on input and state are $U = \{u \in \mathbb{R} : |u| \leq 2\}$, $X = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 4\}$. The system considered is deterministic. We set $\lambda_n = 1$ to obtain a control invariant set. Algorithm 1 has been applied to obtain $\Omega_0 = \alpha\Omega$, then the enlarging method has been applied to generate a sequence Ω_k of nested control invariant sets, with $k \in \mathbb{N}_{60}$. Fig. 1 presents a comparison between the ellipsoidal invariant set proposed in Chen et al. (2001), the parallelogram provided in Cannon et al. (2003) and the control invariant set Ω_{60}. Finally, a robust control invariant set is also computed. If additive uncertainty for the continuous-time system is bounded by $W = \{w \in \mathbb{R}^n : \|w\|_\infty \leq 0.4\}$, the set depicted in dashed line in Fig. 1 is a robust control invariant set.

7. Conclusions

In this paper a condition for a convex set to be a control invariant set for a nonlinear uncertain system is provided. Such a condition is employed to design an algorithm for computing a contractive polytopic invariant set for nonlinear systems. The method overcomes the main problem, often computationally unmanageable, of the computation of control invariant sets for nonlinear systems, although improvements in the numerical implementation deserve further analysis.

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M. Fiacchini was born in Italy in 1977. He received the Laurea degree in Computer Science Engineering in 2004 from the University of Florence and the Ph.D. in Control Engineering from University of Seville in 2010. He was working as visiting Ph.D. student at ETH of Zurich (Switzerland). He is currently post-doctoral researcher at LAAS–CNRS, Toulouse, France. He has participated in several Spanish and European Projects, and his current research interests are in Model Predictive Control, robust control, set-theory in control and invariant sets.

T. Alamo was born in Spain in 1968. He received his M.Eng. degree in Telecommunications Engineering from the Polytechnic University of Madrid (Spain) in 1993 and his Ph.D. in Telecommunications Engineering from the University of Seville in 1998. From 1993 to 2000 he was assistant professor of the Department of System Engineering and Automatic Control of the University of Seville. Since 2001, he has been associate professor and since 2010 he is full professor in the same department. He was a Researcher in the Ecole Nationale Supérieure des Télécommunications (Telecom Paris) from 1991 to 1993 and he has participated in several European Projects. He is the author or coauthor of more than 120 publications including book chapters, journal papers, conference proceedings and educational books. He has carried out reviews for various conferences and technical journals. His current research interests are in model predictive control, randomized algorithms, robust control, identification, control of constrained systems, invariant sets and convex optimization.

E.F. Camacho received his doctorate in Electrical engineering from the University of Seville where he is now a full professor of the Department of System Engineering and Automatic Control. He has written the books: "Model Predictive Control in the Process industry" (1995), "Advanced Control of Solar Plants" (1997) and "Model Predictive Control" (1999), (2004 second edition) published by Springer-Verlag, "Control e Instrumentación de Procesos Quimicos" published by Ed. Sintesis and "Control of Dead-time Processes" to be published by Springer-Verlag (2007). He has authored and co-authored more than 200 papers in journals and international conferences.

He has served on various IFAC technical committees and chaired the IFAC publication Committee from 2002–2005. He is an elected member of the Board of Governors of the IEEE/CSS. He is the President of the European Union Control Association (EUCA) for the period 2006–2007 and chairs the IFAC Policy Committee.

He has carried out review and editorial work for various technical journals and many conferences. At present he is one of the editors of the IFAC journal, Control Engineering Practice, and has been associate editor of the European Journal of Control until 2006 when he was promoted to editor at large. He chaired the IEEE CSS International Affais Committee 2003–2006. He was Publication Chair for the IFAC World Congress b'02 and General Chair of the joint 44th IEEE Control and Decision Conference (CDC) and European Control Conference (ECC) held in 2005 in Sevilla.