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Brief paper

Min–max MPC using a tractable QP problem ☆

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Abstract

Min–max model predictive controllers (MMMPC) suffer from a great computational burden that is often circumvented by using approximate solutions or upper bounds of the worst possible case of a performance index. This paper proposes a computationally efficient MMMPC control strategy in which a close approximation of the solution of the min–max problem is computed using a quadratic programming problem. The overall computational burden is much lower than that of the min–max problem and the resulting control is shown to have a guaranteed stability. A simulation example is given in the paper.

Keywords: Robust control; Uncertain systems; Min–max; Model predictive control

1. Introduction

Min–max model predictive controllers (MMMPC) have been limited to a narrow field of applications due to their great computational burden. The computation of the control signal to be applied implies the minimization of the worst case of a performance index. Solving this problem can be very time consuming because it is an NP-hard problem (Lee & Yu, 1997; Scokaert & Mayne, 1998; Veres & Norton, 1993).

A common solution to the computational burden issue is to use an upper bound of the worst case cost instead of computing it explicitly. This upper bound can be efficiently computed by using linear matrix inequalities (LMI) techniques such as in Kothare, Balakrishnan, and Morari (1996), Lu and Arkun (2000) and Wan and Kothare (2003). The LMI problems have a computational burden that cannot be neglected in certain applications. In Ramirez, Alamo, Camacho, and de la Peña (2006) a different approach based on a computationally cheap upper bound of the worst case cost is presented. The computational burden is much lower than that of the exact MMMPC but it is still much higher than that of a conventional constrained MPC. Moreover, stability results were not provided.

This paper proposes a different strategy in which the min–max problem is replaced by a quadratic programming (QP) problem that provides a close approximation to the solution of the original min–max problem. The computational burden is much lower than that of the min–max problem and also lower than that of Ramirez et al. (2006). Moreover, it is comparable to that of a standard constrained MPC based on a quadratic cost. Thus, it can be easily implemented in almost any platform capable to run a constrained MPC. Also, stability of the proposed approach is guaranteed.

The paper is organized as follows: Section 2 presents the MMMPC strategy. Section 3 presents the proposed implementation strategy. Robust stability of the proposed controller is shown in Section 4. The strategy is illustrated by means of a simulation example in Section 5. Finally, Section 6 presents some conclusions.

2. Min–Max MPC with bounded additive uncertainties

Consider the following state-space model with bounded additive uncertainties (Camacho & Bordóns, 2004):

$$x(t+1) = Ax(t) + Bu(t) + D\theta(t+1) \tag{1}$$

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with $x(t) \in \mathbb{R}^{\dim x}$ the state vector, $u(t) \in \mathbb{R}^{\dim u}$ the input vector and $\theta(t) \in \{\theta \in \mathbb{R}^{\dim \theta} : \|\theta\|_\infty \leqslant \varepsilon\}$ the uncertainty, that is supposed to be bounded. The system is subject to p state and input time invariant constraints $F_u u(t) + F_x x(t) \leqslant g$ where $F_u \in \mathbb{R}^{p\times \dim u}$ and $F_x \in \mathbb{R}^{p\times \dim x}$. It is assumed a semi-feedback approach (Rossiter, Kouvaritakis, & Rice, 1998) which reduces the conservativeness of the open-loop MMMPC controllers without increasing the computational burden. In this approach the control input is given by

$$u(t) = -Kx(t) + v(t), \tag{2}$$

where the feedback matrix K is chosen to achieve some desired property such as nominal stability or LQR optimality without constraints. The MMMPC controller will compute the optimal sequence of correction control inputs $v(t)$. The state equation of system (1) can be rewritten as

$$x(t+1) = A_{\mathrm{CL}} x(t) + Bv(t) + D\theta(t+1),$$
$$A_{\mathrm{CL}} = (A - BK). \tag{3}$$

The cost function is a quadratic performance index:

$$\begin{aligned} V(x, \mathbf{v}, \boldsymbol{\theta}) = &\sum_{j=0}^{N-1} x(t+j|t)^{\mathrm{T}} Q x(t+j|t) \\ &+ \sum_{j=0}^{N-1} u(t+j|t)^{\mathrm{T}} R u(t+j|t) \\ &+ x(t+N|t)^{\mathrm{T}} P x(t+N|t), \end{aligned} \tag{4}$$

where $x(t|t) = x$, $x(t+j|t)$ is the prediction of the state for $t+j$ made at t and $u(t+j|t) = -Kx(t+j|t) + v(t+j|t)$. The sequence of future values of $\theta(t)$ over a prediction horizon N is denoted by $\boldsymbol{\theta} = [\theta(t+1)^{\mathrm{T}}, \ldots, \theta(t+N)^{\mathrm{T}}]^{\mathrm{T}}$, and $\boldsymbol{\Theta} = \{\boldsymbol{\theta} \in \mathbb{R}^{N\cdot \dim \theta} : \|\boldsymbol{\theta}\|_\infty \leqslant \varepsilon\}$ is the set of possible uncertainty trajectories. On the other hand, $\mathbf{v} = [v(t|t)^{\mathrm{T}}, \ldots, v(t+N-1|t)^{\mathrm{T}}]^{\mathrm{T}}$ is the control correction sequence. Matrices Q, $P \in \mathbb{R}^{\dim x \times \dim x}$ and $R \in \mathbb{R}^{\dim u \times \dim u}$ are symmetric positive definite matrices used as weighting parameters.

MMMPC minimizes the cost function for the worst possible case of the predicted future evolution of the process state or output signal. This is accomplished through the solution of a min–max problem:

$$\begin{aligned} \mathbf{v}^*(x) = \arg\min_{\mathbf{v}} \max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \quad & V(x, \mathbf{v}, \boldsymbol{\theta}) \\ \text{s.t.} \quad & F_u u(t+j|t) + F_x x(t+j|t) \leqslant g, \\ & j = 0, \ldots, N, \quad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}, \\ & x(t+N|t) \in \Omega, \quad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}. \end{aligned} \tag{5}$$

A terminal region constraint $x(t+N|t) \in \Omega$, where Ω is a polyhedron, is included to assure stability of the control law (Mayne, Rawlings, Rao, & Scokaert, 2000).[1]

[1] In this paper we have used standard assumptions in order to prove stability. There are other frameworks to ensure stability such as in El-Farra, Mhaskar, and Christofides (2004).

The predictions $x(t+j|t)$ and $u(t+j|t)$ depend linearly on x, $\mathbf{v}$ and $\boldsymbol{\theta}$. This means that it is possible to find a vector $d \in \mathbb{R}^p$ and matrices G_x, G_v and G_θ, such that all the robust linear constraints of problem (5) can be rewritten as

$$G_x^i x + G_v^i \mathbf{v} + G_\theta^i \boldsymbol{\theta} \leqslant d_i, \quad i = 1, \ldots, p, \quad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta},$$

where G_x^i, G_v^i, G_θ^i denote the ith rows of G_x, G_v and G_θ, respectively, and d_i is the ith component of $d \in \mathbb{R}^p$. Denote now $\|G_\theta^i\|_1$ the sum of the absolute values of row G_θ^i. Taking into account that $\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} G_\theta^i \boldsymbol{\theta} = \max_{\|\boldsymbol{\theta}\|_\infty \leqslant \varepsilon} G_\theta^i \boldsymbol{\theta} = \varepsilon \|G_\theta^i\|_1$, the robust fulfillment of the constraints is satisfied if and only if $G_x^i x + G_v^i \mathbf{v} + \varepsilon \|G_\theta^i\|_1 \leqslant d_i$, $i = 1, \ldots, p$. Therefore, to guarantee robust constraint satisfaction, the following set of linear constraints must be satisfied:

$$G_x x + G_v \mathbf{v} \leqslant d_\varepsilon,$$

where the ith component of d_ε is equal to $d_i - \varepsilon \|G_\theta^i\|_1$. Note that this is a necessary and sufficient condition.

Taking into account (3), (2) and (4), the cost function can be evaluated as a quadratic function:

$$\begin{aligned} V(x, \mathbf{v}, \boldsymbol{\theta}) = &\mathbf{v}^{\mathrm{T}} M_{vv} \mathbf{v} + \boldsymbol{\theta}^{\mathrm{T}} M_{\theta\theta} \boldsymbol{\theta} + 2\boldsymbol{\theta}^{\mathrm{T}} M_{\theta v} \mathbf{v} + 2x^{\mathrm{T}} M_{vf}^{\mathrm{T}} \mathbf{v} \\ &+ 2x^{\mathrm{T}} M_{\theta f}^{\mathrm{T}} \boldsymbol{\theta} + x^{\mathrm{T}} M_{ff} x, \end{aligned} \tag{6}$$

where the matrices can be obtained from the system and the control parameters (Camacho & Bordóns, 2004).

The terminal region Ω and matrix P are assumed to satisfy the following conditions:

- C1: If $x \in \Omega$ then $A_{\mathrm{CL}} x + D\theta \in \Omega$, for every $\theta \in \{\theta \in \mathbb{R}^{\dim \theta} : \|\theta\|_\infty \leqslant \varepsilon\}$.
- C2: If $x \in \Omega$ then $u(x) = -Kx \in U$, where $U \triangleq \{u : F_u u + F_x x \leqslant g\}$.
- C3: $P - A_{\mathrm{CL}}^{\mathrm{T}} P A_{\mathrm{CL}} > Q + K^{\mathrm{T}} R K$.

The stability of A_{CL} guarantees the existence of a positive definite matrix P satisfying C3.

The maximum cost for a given x and $\mathbf{v}$ is attained at a vertex of $\boldsymbol{\Theta}$ because of the convexity of $V(x, \mathbf{v}, \boldsymbol{\theta})$. The maximum cost can be denoted as

$$\begin{aligned} V^*(x, \mathbf{v}) = &\max_{\boldsymbol{\theta}\in \mathrm{vert}(\boldsymbol{\Theta})} V(x, \mathbf{v}, \boldsymbol{\theta}) = V(x, \mathbf{v}, 0) \\ &+ \max_{\boldsymbol{\theta}\in \mathrm{vert}(\boldsymbol{\Theta})} \boldsymbol{\theta}^{\mathrm{T}} H \boldsymbol{\theta} + 2\boldsymbol{\theta}^{\mathrm{T}} q(x, \mathbf{v}), \end{aligned} \tag{7}$$

where $\mathrm{vert}(\boldsymbol{\Theta})$ is the set of vertices of $\boldsymbol{\Theta}$, $H = M_{\theta\theta}$, $q(x, \mathbf{v}) = M_{\theta v} \mathbf{v} + M_{\theta f} x$ and $V(x, \mathbf{v}, 0) = \mathbf{v}^{\mathrm{T}} M_{vv} \mathbf{v} + 2x^{\mathrm{T}} M_{vf}^{\mathrm{T}} \mathbf{v} + x^{\mathrm{T}} M_{ff} x$ is the part of the cost that does not depend on the uncertainty. With this definition, problem (5) can be rewritten as

$$\begin{aligned} \mathbf{v}^*(x) = \arg\min_{\mathbf{v}} \quad & V^*(x, \mathbf{v}) \\ \text{s.t.} \quad & G_x x + G_v \mathbf{v} \leqslant d_\varepsilon, \end{aligned} \tag{8}$$

and the system is controlled by $K_{\text{MPC}}(x(t)) = -Kx(t) + v^*(t|t)$, where $\mathbf{v}^*(x(t)) = [v^*(t|t)^{\mathrm{T}}, \ldots, v^*(t+N-1|t)^{\mathrm{T}}]^{\mathrm{T}}$.

In order to evaluate $V^*(x, \mathbf{v})$ it is necessary to evaluate the function for all the $2^{N*\dim\theta}$ vertices of $\boldsymbol{\Theta}$. Note that this is a well known NP-hard problem.

3. A QP approach to min–max MPC

In this section it is shown how the min–max problem (8) can be replaced by a tractable QP problem which provides a close approximation of the solution of the original problem. The strategy can be summarized in the following steps:

(1) Obtain an initial guess of the solution of (8), denoted $\tilde{\mathbf{v}}^*$. As seen later, this can be achieved by solving a QP problem.
(2) Using $\tilde{\mathbf{v}}^*$, obtain a quadratic function of $\mathbf{v}$ that bounds the worst case cost.
(3) Compute the control law. This involves the solution of a QP problem.

3.1. Computing $\tilde{\mathbf{v}}^*$

Given H defined as in Eq. (7), denote $T_i = \sum_{j=1}^{N\cdot\dim\theta} |H_{ij}|$, where H_{ij} denotes the (i, j)th component of matrix H. Then, define the diagonal matrix T as

$$T = \mathbf{diag}(T_1, \ldots, T_n). \tag{9}$$

Because of how matrix T is defined, $T - H$ is a symmetric diagonally dominant real matrix with non-negative diagonal entries, thus $T - H \geqslant 0$ which implies that $T \geqslant H$. Let $\tilde{V}(x, \mathbf{v}, \boldsymbol{\theta})$ be:

$$\tilde{V}(x, \mathbf{v}, \boldsymbol{\theta}) = V(x, \mathbf{v}, 0) + \boldsymbol{\theta}^{\mathrm{T}} T \boldsymbol{\theta} + 2q^{\mathrm{T}}(x, \mathbf{v})\boldsymbol{\theta}. \tag{10}$$

From the inequality $T \geqslant H$ it is inferred that $\tilde{V}(x, \mathbf{v}, \boldsymbol{\theta}) \geqslant V(x, \mathbf{v}, \boldsymbol{\theta})$. The maximum of $\tilde{V}(x, \mathbf{v}, \boldsymbol{\theta})$ can be computed as

$$\begin{aligned}\tilde{V}^*(x, \mathbf{v}) &= \max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \tilde{V}(x, \mathbf{v}, \boldsymbol{\theta}) \\ &= V(x, \mathbf{v}, 0) + \operatorname{trace}(T)\varepsilon^2 + 2\varepsilon\|q(x, \mathbf{v})\|_1 \\ &= V(x, \mathbf{v}, 0) + \|H\|_s\varepsilon^2 + 2\varepsilon\|q(x, \mathbf{v})\|_1,\end{aligned} \tag{11}$$

where $\|H\|_s$ denotes the sum of the absolute values of the elements of H. Then an initial guess of the solution of (8) can be obtained as

$$\begin{aligned}\tilde{\mathbf{v}}^*(x) = \arg\min_{\tilde{\mathbf{v}}} \quad & \tilde{V}^*(x, \tilde{\mathbf{v}}) \\ \text{s.t.} \quad & G_x x + G_v \tilde{\mathbf{v}} \leqslant d_\varepsilon.\end{aligned} \tag{12}$$

It is evident that this problem can be casted as a QP problem.

3.2. Obtaining an upper bound of the worst case cost

The upper bound of the maximum will be obtained in two steps. In the first one we compute a set of parameters from $\tilde{\mathbf{v}}^*$ that allows us later, in the second step, to compute the bound as a quadratic function of $\mathbf{v}$.

3.2.1. Computing the parameter vector $\alpha(\mathbf{v})$

Note that

$$\begin{aligned}V^*(x, \mathbf{v}) &= \max_{\boldsymbol{\theta}\in\operatorname{vert}(\boldsymbol{\Theta})} \begin{bmatrix}\boldsymbol{\theta}\\1\end{bmatrix}^{\mathrm{T}} \begin{bmatrix} H & q(x, \mathbf{v}) \\ q^{\mathrm{T}}(x, \mathbf{v}) & V(x, \mathbf{v}, 0)\end{bmatrix} \begin{bmatrix}\boldsymbol{\theta}\\1\end{bmatrix} \\ &= \max_{\|z\|_\infty \leqslant 1} z^{\mathrm{T}} M(\mathbf{v}) z\end{aligned} \tag{13}$$

with $z = \begin{bmatrix}\frac{\boldsymbol{\theta}^{\mathrm{T}}}{\varepsilon} & 1\end{bmatrix}^{\mathrm{T}}$ and $M(\mathbf{v}) = \begin{bmatrix}\varepsilon^2 H & \varepsilon q(x, \mathbf{v}) \\ \varepsilon q^{\mathrm{T}}(x, \mathbf{v}) & V(x, \mathbf{v}, 0)\end{bmatrix} \in \mathbb{R}^{n\times n}$, where $n = N \cdot \dim\theta + 1$.

The following procedure, which is based on that presented in Ramirez et al. (2006), provides an upper bound of the worst case cost for a given $\mathbf{v}$. It computes $\alpha(\mathbf{v}) = [\alpha_1(\mathbf{v}), \ldots, \alpha_{n-1}(\mathbf{v})]^{\mathrm{T}}$ and a diagonal matrix $\Gamma(\mathbf{v}) \geqslant M(\mathbf{v})$ such that its trace is an upper bound of the worst case cost for $\mathbf{v}$ (see Property 1).

Procedure 1. *Computation of* $\alpha(\mathbf{v}) = [\alpha_1(\mathbf{v}), \ldots, \alpha_{n-1}(\mathbf{v})]^{\mathrm{T}}$ *and* $\Gamma(\mathbf{v})$.

(1) *Let* $S^{(0)} = M(\mathbf{v}) \in \mathbb{R}^{n\times n}$.
(2) *For* $k = 1$ *to* $n - 1$
(3) *Let* $M_{\text{sub}}^{(k-1)} = [S_{ij}^{(k-1)}]$ *for* $i, j = k \ldots n$.
(4) *Obtain the partition* $M_{\text{sub}}^{(k-1)} = \begin{bmatrix} a & b^{\mathrm{T}} \\ b & M_r\end{bmatrix}$, *where* $a \in \mathbb{R}$, $b \in \mathbb{R}^{n-k}$ *and* $M_r \in \mathbb{R}^{(n-k)\times(n-k)}$.
(5) *Make* $\alpha_k(\mathbf{v}) = \sqrt{\|b\|_1}$.
(6) *If* $\alpha_k(\mathbf{v}) = 0$ *then* $S^{(k)} = S^{(k-1)}$, *else* $S^{(k)} = S^{(k-1)} + [\mathbf{0}_{k-1,1}^{\mathrm{T}} \ \alpha_k(\mathbf{v}) \ \frac{-b^{\mathrm{T}}}{\alpha_k(\mathbf{v})}]^{\mathrm{T}} [\mathbf{0}_{k-1,1}^{\mathrm{T}} \ \alpha_k(\mathbf{v}) \ \frac{-b^{\mathrm{T}}}{\alpha_k(\mathbf{v})}]$.
(7) *end for*
(8) *Make* $\Gamma(\mathbf{v}) = S^{(n-1)}$.

Note that in the previous procedure, $\mathbf{0}_{m,n}$ denotes a $(m \times n)$ matrix of zeros. The following property shows that the trace of $\Gamma(\mathbf{v})$ constitutes an improved upper bound of $V^*(x, \mathbf{v})$. That is, $V^*(x, \mathbf{v}) \leqslant \operatorname{trace}(\Gamma(\mathbf{v})) \leqslant \tilde{V}^*(x, \mathbf{v})$.

Property 1. *Matrices* $S^{(0)}, S^{(1)}, \ldots, S^{(n-1)}$, *obtained by means of procedure* 1 *satisfy*:

(i) $S^{(k)}$ *is a partially diagonalized matrix. That is, there is a diagonal matrix* $T^{(k)} \in \mathbb{R}^{k\times k}$ *such that* $S^{(k)} = \mathbf{diag}(T^{(k)}, M_{\text{sub}}^{(k)})$.
(ii) $S^{(n-1)} = \Gamma(\mathbf{v})$ *is a diagonal matrix.*
(iii) $V^*(x, \mathbf{v}) \leqslant \operatorname{trace}(\Gamma(\mathbf{v}))$.
(iv) $\|S^{(k)}\|_s \leqslant \|S^{(k-1)}\|_s$.
(v) $\operatorname{trace}(\Gamma(\mathbf{v})) \leqslant \tilde{V}^*(x, \mathbf{v})$, $\forall \mathbf{v}$.

Proof. See Appendix A.

Procedure 1 is the foundation to obtain a QP problem that provides a solution with a worst case cost that is close to the optimal worst case cost but with the advantage of the lower computational burden of a QP problem (see Section 3.2.2).

3.2.2. Obtaining the bound as a quadratic function on $\mathbf{v}$

The diagonalization process shown in 3.2.1 can be used to obtain a matrix denoted by $\hat{\Gamma}(\mathbf{v})$, which allows one to obtain a bound of the maximum that can be computed as a quadratic function of $\mathbf{v}$. This is achieved by means of the following procedure:

Procedure 2. *Obtaining the matrix* $\hat{\Gamma}(\mathbf{v})$.

(1) *Obtain* $\tilde{\mathbf{v}}^*$ *from the QP problem defined in Eq.* (12).
(2) *Compute* $\alpha(\tilde{\mathbf{v}}^*)$ *by Procedure* 1.
(3) *Let* $\hat{S}^{(0)}(\mathbf{v}) = M(\mathbf{v}) \in \mathbb{R}^{n\times n}$.
(4) *For* $k = 1$ *to* $n-1$.
(5) *Let* $\hat{M}_{\text{sub}}(\mathbf{v}) = [\hat{S}_{ij}^{(k-1)}(\mathbf{v})]$ *for* $i, j = k \cdots n$.
(6) *Obtain the partition* $\hat{M}_{\text{sub}}(\mathbf{v}) = \begin{bmatrix} a(\mathbf{v}) & b^{\mathrm{T}}(\mathbf{v}) \\ b(\mathbf{v}) & M_r(\mathbf{v}) \end{bmatrix}$, *where* $a(\mathbf{v}) \in \mathbb{R}$.
(7) *If* $\alpha_k(\tilde{\mathbf{v}}^*) = 0$ *then* $\hat{S}^{(k)}(\mathbf{v}) = \hat{S}^{(k-1)}(\mathbf{v})$, *else* $\hat{S}^{(k)}(\mathbf{v}) = \hat{S}^{(k-1)}(\mathbf{v}) + \left[\mathbf{0}_{k-1,1}^{\mathrm{T}}\, \alpha_k(\tilde{\mathbf{v}}^*)\, \frac{-b(\mathbf{v})^{\mathrm{T}}}{\alpha_k(\tilde{\mathbf{v}}^*)}\right]^{\mathrm{T}} \left[\mathbf{0}_{k-1,1}^{\mathrm{T}}\, \alpha_k(\tilde{\mathbf{v}}^*)\, \frac{-b(\mathbf{v})^{\mathrm{T}}}{\alpha_k(\tilde{\mathbf{v}}^*)}\right]$.
(8) *end for*
(9) *Make* $\hat{\Gamma}(\mathbf{v}) = \hat{S}^{(n-1)}(\mathbf{v})$.

Theorem 1. *Denote* $\hat{V}^*(x, \mathbf{v}) = \|\hat{\Gamma}(\mathbf{v})\|_s$. *Then*

(1) $\hat{\Gamma}(\tilde{\mathbf{v}}^*) = \Gamma(\tilde{\mathbf{v}}^*)$.
(2) $\hat{V}^*(x, \mathbf{v})$ *can be obtained by means of the solution of a QP problem.*
(3) $V^*(x, \mathbf{v}) \leqslant \hat{V}^*(x, \mathbf{v})$.

Proof. See Appendix A.

3.3. Computing the control law

The value of the control signal is obtained by solving the following QP optimization problem:

$$\hat{\mathbf{v}}^*(x) = \arg\min_{\hat{\mathbf{v}}} \quad \hat{V}^*(x, \hat{\mathbf{v}}) \qquad \text{s.t.} \quad G_x x + G_v \hat{\mathbf{v}} \leqslant d_\varepsilon, \tag{14}$$

and the system is controlled by $\hat{K}_{\mathrm{MPC}}(x(t)) = -Kx(t) + \hat{v}^*(t|t)$, where $\hat{v}^*(t|t)$ is the first element of $\hat{\mathbf{v}}^*(x)$.

Remark 1. Note that, in order to reduce the difference between the solution of the exact min–max problem and $\hat{\mathbf{v}}^*(x)$ the whole procedure can be applied twice or more times using at each subsequent step the solution obtained in the previous step as the initial guess used in Procedure 1.

Table 1
Average computational burden (measured in flops) required to compute the control law for different values of the controller horizons and dimension of x

dim, x	Prediction and control horizon			
	6	10	14	18
4	5.97×10^5	2.76×10^6	4.34×10^6	6.64×10^6
12	3.94×10^6	1.65×10^7	4.37×10^7	8.77×10^7
20	1.57×10^7	6.9×10^7	1.69×10^8	3.67×10^8

For each entry, 10 simulations (each of 100 samples) with random systems have been computed.

The computational burden of the proposed strategy is clearly much lower than that of the exact MMMPC. This computational burden is mostly due to the two QP problems that must be solved. Each of these has the same complexity of a standard constrained MPC using a quadratic cost function. As illustrated in Table 1, the complexity is strongly related to the size of $\mathbf{v}$ and x.

4. Stability of the proposed control law

In this section the stability properties of the control $\hat{K}_{\mathrm{MPC}}(x(t))$ are shown. First some properties are presented and then stability is proved. Recall that $\mathbf{v}^*$, $\tilde{\mathbf{v}}^*$ and $\hat{\mathbf{v}}^*$ are the solutions of (8), (12) and (14), respectively. Denote also $J(x) = V^*(x, \mathbf{v}^*)$, $\tilde{J}(x) = \tilde{V}^*(x, \tilde{\mathbf{v}}^*)$ and $\hat{J}(x) = \hat{V}^*(x, \hat{\mathbf{v}}^*)$. Note that problems (8), (12) and (14) have the same feasibility region as the constraints are the same.

Property 2.

$\hat{J}(x) \leqslant \tilde{J}(x)$.

Proof. As $\hat{\mathbf{v}}^*$ is the minimizer of $\hat{V}^*(x, \mathbf{v})$ it follows that

$$\hat{J}(x) = \hat{V}^*(x, \hat{\mathbf{v}}^*) \leqslant \hat{V}^*(x, \tilde{\mathbf{v}}^*). \tag{15}$$

Thus, in order to prove that $\hat{J}(x) \leqslant \tilde{J}(x)$ it suffices to show that $\hat{V}^*(x, \tilde{\mathbf{v}}^*) \leqslant \tilde{V}^*(x, \tilde{\mathbf{v}}^*) = \tilde{J}(x)$. As it was shown in the proof of Theorem 1, $\hat{V}^*(x, \tilde{\mathbf{v}}^*) = \mathrm{trace}\,(\hat{\Gamma}(\tilde{\mathbf{v}}^*)) = \mathrm{trace}\,(\Gamma(\tilde{\mathbf{v}}^*))$. Moreover, from Property 1, $\mathrm{trace}\,(\Gamma(\tilde{\mathbf{v}}^*)) \leqslant \tilde{V}^*(x, \tilde{\mathbf{v}}^*)$. Thus,

$$\hat{V}^*(x, \tilde{\mathbf{v}}^*) = \mathrm{trace}\,(\hat{\Gamma}(\tilde{\mathbf{v}}^*)) = \mathrm{trace}\,(\Gamma(\tilde{\mathbf{v}}^*)) \leqslant \tilde{V}^*(x, \tilde{\mathbf{v}}^*) = \tilde{J}(x).$$

Thus $\hat{V}^*(x, \tilde{\mathbf{v}}^*) \leqslant \tilde{J}(x)$. This completes the proof. □

It is clear that the optimal solution $\hat{\mathbf{v}}^*$ of problem (14) is a suboptimal feasible solution of the original min–max problem (8). As it is claimed in the following property, the difference between the optimal value of the original objective function and the value obtained with $\hat{\mathbf{v}}^*$ is bounded by $\|H\|_s\varepsilon^2$. Note that this result gives an implicit measure of how well $\hat{\mathbf{v}}^*$ approximates the solution of the original min–max problem (8).

Property 3. *It holds that*

$V^*(x, \hat{\mathbf{v}}^*) - \sigma\varepsilon^2 \leqslant J(x)$,

where $\sigma = \|H\|_s$.

Proof. See Appendix A.

The following property, which is proved in Alamo, Muñoz de la Peña, Limón Marruedo, and Camacho (2005) will be used in the proof of the stability of the proposed approach (see Theorem 2).

Property 4. *Assume that* C1–C3 *are satisfied. Let* $\mathbf{v} = [v(t|t)^{\mathrm{T}}, \ldots, v(t+N-1|t)^{\mathrm{T}}]^{\mathrm{T}}$ *and* $\mathbf{v}_s$ *a shifted version of* $\mathbf{v}$ *computed as* $\mathbf{v}_s = [v(t+1|t)^{\mathrm{T}}, \ldots, v(t+N-1|t)^{\mathrm{T}}, 0]^{\mathrm{T}}$.

If $\mathbf{v}$ *is feasible for problem* (8) *at* $x(t)$ *then* $\mathbf{v}_s$ *is also feasible at* $x(t+1)$ *and* $\exists\, \gamma > 0$ *such that*:

$$V^*(x(t+1), \mathbf{v}_s) \leqslant V^*(x(t), \mathbf{v}) - x(t)^{\mathrm{T}} Qx(t) + \gamma\varepsilon^2.$$

The following theorem states the stabilizing properties of the proposed control law:

Theorem 2. *Under assumptions* C1–C3, *the control law* $\mathbf{u}(x(t)) = -Kx(t) + \hat{v}^*(t|t)$ *stabilizes system* (1).

Proof. Let $\hat{\mathbf{v}}_s^*$ be the shifted version (as in Property 4) of $\hat{\mathbf{v}}^*$. Due to the non-optimality of $\hat{\mathbf{v}}_s^*$ for problem (8), it holds that

$$J(x(t+1)) \leqslant V^*(x(t+1), \hat{\mathbf{v}}_s^*). \tag{16}$$

Note that $\hat{\mathbf{v}}_s^*$ is feasible for both (14) and (8), thus by Property 4:

$$V^*(x(t+1), \hat{\mathbf{v}}_s^*) \leqslant V^*(x(t), \hat{\mathbf{v}}^*) - x(t)^{\mathrm{T}} Qx(t) + \gamma\varepsilon^2. \tag{17}$$

Furthermore, by Property (3) $V^*(x(t), \hat{\mathbf{v}}^*) \leqslant J(x(t)) + \sigma\varepsilon^2$. Substituting this inequality in (17) and using (16)

$$J(x(t+1)) \leqslant J(x(t)) - x(t)^{\mathrm{T}} Qx(t) + (\gamma + \sigma)\varepsilon^2$$

which can be rewritten as

$$J(x(t+1)) - J(x(t)) \leqslant -x(t)^{\mathrm{T}} Qx(t) + (\gamma + \sigma)\varepsilon^2. \tag{18}$$

Note that $(\gamma+\sigma)\varepsilon^2 > 0$ and that $-x(t)^{\mathrm{T}} Qx(t) \leqslant 0$, thus it is ensured that the optimal worst case cost will decrease, i.e. $J(x(t+1)) - J(x(t)) < 0$, as long as $x(t)^{\mathrm{T}} Qx(t) > (\gamma+\sigma)\varepsilon^2$. Define

$$\Phi_\varepsilon = \{x \in \mathbb{R}^{\dim x} : \text{(8) is feasible and } x^{\mathrm{T}} Qx \leqslant (\gamma+\sigma)\varepsilon^2\}.$$

It is clear that the system state is steered into set Φ_ε (which contains the origin) from any arbitrary $x(t)$. But, whenever it enters into Φ_ε it may evolve out of it or remain inside, because it is not ensured that the optimal worst case cost decreases.

Taking into account that $-x(t)^{\mathrm{T}} Qx(t) \leqslant 0$ it follows that

$$\begin{aligned} J(x(t+1)) - J(x(t)) &\leqslant -x(t)^{\mathrm{T}} Qx(t) + (\gamma+\sigma)\varepsilon^2 \\ &\leqslant (\gamma+\sigma)\varepsilon^2 \end{aligned}$$

which yields

$$J(x(t+1)) \leqslant J(x(t)) + (\gamma+\sigma)\varepsilon^2. \tag{19}$$

Suppose that $x(t) \in \Phi_\varepsilon$, then $J(x(t)) + (\gamma+\sigma)\varepsilon^2 \leqslant \max_{x\in\Phi_\varepsilon} J(x) + (\gamma+\sigma)\varepsilon^2$, thus, taking into account (19) it follows that $\forall x(t) \in \Phi_\varepsilon$ it holds that $J(x(t+1)) \leqslant \beta$ where $\beta = \max_{x\in\Phi_\varepsilon} J(x) + (\gamma+\sigma)\varepsilon^2$. Thus, whenever the system state enters into Φ_ε it evolves into the set $\Omega_\beta = \{x \in \mathbb{R}^{\dim x} : J(x) \leqslant \beta\}$. Once the state is in Φ_ε it can evolve outside of Φ_ε, but it will remain inside Ω_β. From Ω_β it will be steered again into Φ_ε and so on. The system state is always confined into Ω_β from the moment it enters for the first time in Φ_ε. Thus, the state system is ultimately bounded. This means that the system is stabilized by the control law $\hat{K}_{\mathrm{MPC}}(x(t)) = -Kx(t) + \hat{v}^*(t|t)$. $\square$

Note that the region of ultimate boundedness Ω_β is not necessarily contained in Ω (although this is the most common situation as Ω is the maximal robust positively invariant set for the system). If $\Omega_\beta \nsubseteq \Omega$, the closed loop trajectories of the state under the proposed control law can escape from Ω. Robust stability, however, is guaranteed in spite of this.

5. Example

To illustrate the results presented in this paper, consider the two-tank network example given in Ogunnaike and Ray (1994, Chapter 20). Using the parameters given in Alamo, Ramirez, and Camacho (2005) the following continuous time two inputs, two outputs, state-space model can be obtained:

$$\dot{x} = \begin{bmatrix} -\frac{0.5}{3} & \frac{0.2}{3} \\ \frac{0.5}{2} & -\frac{0.5}{2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x. \tag{20}$$

Constraints are imposed on both states and control actions such that $\|x(k)\|_\infty \leqslant 1.5$ and $\|u(k)\|_\infty \leqslant 0.4$. A discrete time model has been obtained from (20) sampling at 0.2 min using a zero-order holder. Fig. 1 shows the results of the proposed controller applied to the two-tank model. The set-point for the liquid level of each tank was 1 and 0.7 m, respectively. The prediction horizon was $N = 7$. Identity matrices were chosen as Q and R. An uncertainty of ± 0.025 m is considered to affect both liquid levels. In the simulation a random noise of ± 0.01 m has been added to both levels and an unexpected loss of liquid in tank 1 is introduced at sampling time $t = 60$.

The absolute deviation of the solution of (14) from that of (8) (computed as $\hat{\mathbf{v}}^*(x) - \mathbf{v}^*(x)$) is also shown in Fig. 1. It can be seen that it is very small throughout the simulation.

Finally, the lower computational burden of the proposed strategy is illustrated in Table 2. The computational burden is much

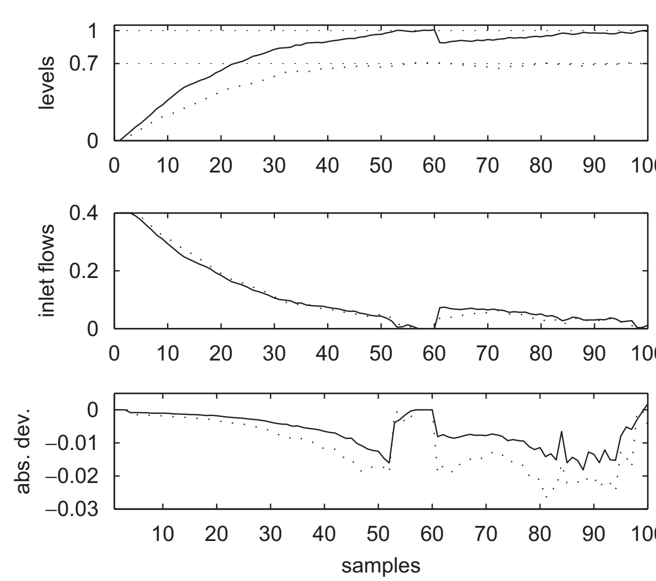

Fig. 1. Liquid levels (system state), inlet flows (control signal values) and absolute deviations (from the exact MMMPC) of the solution obtained by the proposed strategy (tank 1 solid plot, tank 2 dotted plot).

Table 2
Mean flops for the original min–max MPC and the proposed strategy for different values of the prediction and control horizon (N) in the simulation example of Section 5

N	Avg. flops (min–max)	Avg. flops (proposed)
5	3.73×10^7	7.6×10^4
6	3.43×10^8	1.28×10^5
7	1.84×10^9	1.42×10^5

lower in the proposed strategy and the gap broadens exponentially as prediction horizon grows.

6. Conclusions

An MMMPC based on an tractable QP problem has been presented in this paper. The solution of this QP problem is close to that of the min–max problem whereas it has a much lower computational burden. As it is based on a QP problem, it can be implemented in almost any industrial hardware capable to run a constrained MPC controller. Thus it extends broadly the fields of application of MMMPC controllers. The proposed controller is shown to be stable, which together with the explicit consideration of the uncertainty in the computation of the control law guarantees performance.

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Appendix A. Technical or auxiliary proofs

A.1. Proof of Property 1

(i) $S^{(k)}$ is a partially diagonalized matrix: let us suppose that $S^{(k-1)}$ is a partially diagonalized matrix. That is,

$$S^{(k-1)} = \mathbf{diag}(T^{(k-1)}, M_{\text{sub}}^{(k-1)}), \tag{A.1}$$

where $T^{(k-1)} \in \mathbb{R}^{(k-1)\times(k-1)}$ is a diagonal matrix. Two cases must be taken into account: if $\alpha_k(\mathbf{v})$ is equal to zero then $b = \mathbf{0}_{n-k,1}$ and trivially, $S^{(k)} = S^{(k-1)}$ is a partially diagonalized matrix. If $\alpha_k(\mathbf{v}) \neq 0$ then

$$\begin{aligned} S^{(k)} &= S^{(k-1)} + \begin{bmatrix} \mathbf{0}_{k-1,1} \\ \alpha_k(\mathbf{v}) \\ \dfrac{-b}{\alpha_k(\mathbf{v})} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{k-1,1} \\ \alpha_k(\mathbf{v}) \\ \dfrac{-b}{\alpha_k(\mathbf{v})} \end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} T^{(k)} & 0 \\ 0 & M_{\text{sub}}^{(k)} \end{bmatrix}, \end{aligned} \tag{A.2}$$

where $T^{(k)} = \mathbf{diag}(T^{(k-1)}, a + \alpha_k^2(\mathbf{v}))$ and $M_{\text{sub}}^{(k)} = M_r + bb^{\mathrm{T}}/\alpha_k^2(\mathbf{v})$.

(ii) $\Gamma(\mathbf{v})$ is a diagonal matrix: Note that $\Gamma(\mathbf{v}) = S^{(n-1)}$. From the previous claim, $S^{(n-1)} = \mathbf{diag}(T^{(n-1)}, M_{\text{sub}}^{(n-1)}) \in \mathbb{R}^{n\times n}$, where $T^{(n-1)} \in \mathbb{R}^{(n-1)\times(n-1)}$. Thus, $M_{\text{sub}}^{(n-1)} \in \mathbb{R}$. It follows that $\Gamma(\mathbf{v})$ is a diagonal matrix.

(iii) $V^*(x, \mathbf{v}) \leqslant \text{trace}\,(\Gamma(\mathbf{v}))$: By construction, $M(\mathbf{v}) = S^{(0)} \leqslant S^{(1)} \leqslant \cdots \leqslant S^{(n-1)} = \Gamma(\mathbf{v})$. Thus, $\Gamma(\mathbf{v})$ is a diagonal matrix that satisfies $\Gamma(\mathbf{v}) \geqslant M(\mathbf{v})$. From this and (13)

$$\begin{aligned} V^*(x, \mathbf{v}) &= \max_{\|z\|_\infty \leqslant 1} z^{\mathrm{T}} M(\mathbf{v}) z \leqslant \max_{\|z\|_\infty \leqslant 1} z^{\mathrm{T}} \Gamma(\mathbf{v}) z \\ &= \text{trace}\,(\Gamma(\mathbf{v})). \end{aligned}$$

(iv) $\|S^{(k)}\|_s \leqslant \|S^{(k-1)}\|_s$: If $\alpha_k(\mathbf{v}) = \sqrt{\|b\|_1} = 0$ then $S^{(k)} = S^{(k-1)}$ and the claim is satisfied. Suppose now that $\alpha_k(\mathbf{v}) \neq 0$ then, using Eqs. (A.1) and (A.2)

$$\begin{aligned} \|S^{(k)}\|_s &= \|T^{(k-1)}\|_s + a + \alpha_k^2(\mathbf{v}) + \left\| M_r + \frac{bb^{\mathrm{T}}}{\alpha_k^2(\mathbf{v})} \right\|_s \\ &= \|T^{(k-1)}\|_s + a + \|b\|_1 + \left\| M_r + \frac{bb^{\mathrm{T}}}{\|b\|_1} \right\|_s \\ &\leqslant \|T^{(k-1)}\|_s + a + \|b\|_1 + \|M_r\|_s + \left\| \frac{bb^{\mathrm{T}}}{\|b\|_1} \right\|_s \\ &\leqslant \|T^{(k-1)}\|_s + a + \|b\|_1 + \|M_r\|_s + \|b\|_1 \\ &= \|S^{(k-1)}\|_s. \end{aligned}$$

Note that in order to obtain the inequality $\|S^{(k)}\|_s \leqslant \|S^{(k-1)}\|_s$, the equality $\|bb^{\mathrm{T}}\|_s = \|b\|_1^2$ has been used. Also, note that in the previous equation a is non-negative as it is one of the entries of the diagonal of a positive semidefinite matrix.

(v) $\text{trace}\,(\Gamma(\mathbf{v})) \leqslant \tilde{V}^*(x, \mathbf{v}),\ \forall \mathbf{v}$: Note that $V(x, \mathbf{v}, 0) \geqslant 0$, thus, it results that (see Eq. (11)):

$$\begin{aligned} \tilde{V}^*(x, \mathbf{v}) = \|M(\mathbf{v})\|_s &= \left\| \begin{bmatrix} \varepsilon^2 H & \varepsilon q(x, \mathbf{v}) \\ \varepsilon q^{\mathrm{T}}(x, \mathbf{v}) & V(x, \mathbf{v}, 0) \end{bmatrix} \right\|_s \\ &= \|S^{(0)}\|_s. \end{aligned}$$

Moreover, $(\Gamma(\mathbf{v})) = S^{(n-1)}$ is a diagonal matrix positive semidefinite, thus $\text{trace}\,(\Gamma(\mathbf{v})) = \|S^{(n-1)}\|_s$. On the other hand, as it has been shown, $\|S^{(k)}\|_s \leqslant \|S^{(k-1)}\|_s$. This implies that $\|S^{(n-1)}\|_s \leqslant \|S^{(0)}\|_s$, that is, trace $(\Gamma(\mathbf{v})) \leqslant \tilde{V}^*(x, \mathbf{v}),\ \forall \mathbf{v}$. □

A.2. Proof of Theorem 1

(i) $\hat{\Gamma}(\tilde{\mathbf{v}}^*) = \Gamma(\tilde{\mathbf{v}}^*)$. Note that the computation of $\hat{\Gamma}(\tilde{\mathbf{v}}^*)$ relies on $\alpha(\tilde{\mathbf{v}}^*)$. It is clear from Procedures 1 and 2 that if $\mathbf{v} = \tilde{\mathbf{v}}^*$ then $\hat{S}^{(k)}(\tilde{\mathbf{v}}^*) = S^{(k)}(\tilde{\mathbf{v}}^*)$. This implies that $\hat{S}^{(n-1)}(\tilde{\mathbf{v}}^*) = S^{(n-1)}(\tilde{\mathbf{v}}^*)$, thus $\hat{\Gamma}(\tilde{\mathbf{v}}^*) = \Gamma(\tilde{\mathbf{v}}^*)$.

(ii) Consider matrix $\hat{S}^{(0)}(\mathbf{v}) = M(\mathbf{v})$ defined in (13) and partitioned as

$$\hat{S}^{(0)}(\mathbf{v}) = \begin{bmatrix} \varepsilon^2 H_{11} & \varepsilon^2 H_{1r}^{\mathrm{T}} & \varepsilon q_1(x, \mathbf{v}) \\ \varepsilon^2 H_{1r} & \varepsilon^2 H_{rr} & \varepsilon q_r(x, \mathbf{v}) \\ \varepsilon q_1(x, \mathbf{v}) & \varepsilon q_r^{\mathrm{T}}(x, \mathbf{v}) & V(x, \mathbf{v}, 0) \end{bmatrix},$$

where H_{11}, $q_1(x, \mathbf{v})$ and $V(x, \mathbf{v}, 0) \in \mathbb{R}$, $H_{1r}, q_r(x, \mathbf{v}) \in \mathbb{R}^{(N \cdot \dim \theta)-1}$ and $H_{rr} \in \mathbb{R}^{\{(N \cdot \dim \theta)-1\} \times \{(N \cdot \dim \theta)-1\}}$. Note that $q_1(x, \mathbf{v})$ and $q_r(x, \mathbf{v})$ have an affine dependence on $\mathbf{v}$ whereas $V(x, \mathbf{v}, 0)$ is a quadratic function on $\mathbf{v}$. Suppose that $\alpha_1(\tilde{\mathbf{v}}^*) \neq 0$ (the case of $\alpha_1(\tilde{\mathbf{v}}^*) = 0$ is similar). Using $\alpha_1 = \alpha_1(\tilde{\mathbf{v}}^*)$, matrix $\hat{S}^{(0)}(\mathbf{v})$ is partially diagonalized by adding the term $c_1(\mathbf{v})c_1^{\mathrm{T}}(\mathbf{v})$ where

$$c_1(\mathbf{v}) = \begin{bmatrix} \alpha_1 & -\dfrac{\varepsilon^2 H_{1r}^{\mathrm{T}}}{\alpha_1} & -\dfrac{\varepsilon q_1(x, \mathbf{v})}{\alpha_1} \end{bmatrix}^{\mathrm{T}}$$

which yields

$$\hat{S}^{(1)}(\mathbf{v}) = \mathbf{diag}(\varepsilon^2 H_{11} + \alpha_1^2, M_r(x, \mathbf{v})),$$

where the sub-matrix

$$M_r(x, \mathbf{v}) = \begin{bmatrix} \varepsilon^2 H_{rr} + \dfrac{\varepsilon^4 H_{1r} H_{1r}^{\mathrm{T}}}{\alpha_1^2} & \varepsilon q_r(x, \mathbf{v}) + \dfrac{\varepsilon^3 H_{1r} q_1(x, \mathbf{v})}{\alpha_1^2} \\ \varepsilon q_r^{\mathrm{T}}(x, \mathbf{v}) + \dfrac{\varepsilon^3 H_{1r}^{\mathrm{T}} q_1(x, \mathbf{v})}{\alpha_1^2} & V(x, \mathbf{v}, 0) + \dfrac{\varepsilon^2 q_1^2(x, \mathbf{v})}{\alpha_1^2} \end{bmatrix}$$

has the same structure as $M(\mathbf{v})$. That is, the last element is a quadratic function of $\mathbf{v}$, the remaining elements of the last row and column are affine functions of $\mathbf{v}$ and all the other elements are constants. That is, as the structure is preserved, a new iteration of Procedure 2 supposes a further diagonalization in which only the last element has a quadratic dependence on $\mathbf{v}$. At the end of Procedure 2 the diagonal matrix $\hat{S}^{(n-1)}(\mathbf{v}) = \hat{\Gamma}(\mathbf{v})$ is obtained with all its elements constant (i.e., they do not depend on $\mathbf{v}$) except the last one which has the form

$$\hat{\Gamma}_{nn}(\mathbf{v}) = V(x, \mathbf{v}, 0) + \frac{\varepsilon^2 q_1^2(x, \mathbf{v})}{\alpha_1^2} + \cdots . \tag{A.3}$$

Once $\hat{\Gamma}(\mathbf{v})$ has been obtained, the bound of the maximum can be computed as $\hat{V}^*(x, \mathbf{v}) = \|\hat{\Gamma}(\mathbf{v})\|_s$ which is a quadratic function of $\mathbf{v}$.

(iii) $V^*(x, \mathbf{v}) \leqslant \hat{V}^*(x, \mathbf{v})$. This follows from the fact that by construction $\hat{\Gamma}(\mathbf{v}) = M(\mathbf{v}) + c_1(\mathbf{v})c_1^{\mathrm{T}}(\mathbf{v}) + \cdots + c_{n-1}(\mathbf{v})c_{n-1}^{\mathrm{T}}(\mathbf{v})$. Thus, $M(\mathbf{v}) \leqslant \hat{\Gamma}(\mathbf{v})$ and, together with the fact that by construction $\hat{\Gamma}(\mathbf{v})$ is diagonal, this implies that

$$V^*(x, \mathbf{v}) = \max_{\|z\|_\infty \leqslant 1} z^{\mathrm{T}} M(\mathbf{v}) z \leqslant \max_{\|z\|_\infty \leqslant 1} z^{\mathrm{T}} \hat{\Gamma}(\mathbf{v}) z = \|\hat{\Gamma}(\mathbf{v})\|_s .$$

As $\|\hat{\Gamma}(\mathbf{v})\|_s = \hat{V}^*(x, \mathbf{v})$, then $V^*(x, \mathbf{v}) \leqslant \hat{V}^*(x, \mathbf{v})$. □

A.3. Proof of Property 3

Note that $J(x) = V^*(x, \mathbf{v}^*)$. From Eqs. (7) and (10) it results that

$$\tilde{V}(x, \mathbf{v}, \boldsymbol{\theta}) = V(x, \mathbf{v}, \boldsymbol{\theta}) + \boldsymbol{\theta}^{\mathrm{T}}(T - H)\boldsymbol{\theta},$$

where T is a diagonal matrix defined as in (9). Taking into account that $T \geqslant H \geqslant 0$, $\|\boldsymbol{\theta}\|_\infty \leqslant \varepsilon$ and that T is diagonal

$$\tilde{V}(x, \mathbf{v}, \boldsymbol{\theta}) \leqslant V(x, \mathbf{v}, \boldsymbol{\theta}) + \boldsymbol{\theta}^{\mathrm{T}} T \boldsymbol{\theta} \leqslant V(x, \mathbf{v}, \boldsymbol{\theta}) + \mathrm{trace}(T)\varepsilon^2.$$

As $\mathrm{trace}(T) = \|H\|_s$ it can be inferred that $V^*(x, \mathbf{v}^*) \geqslant \tilde{V}^*(x, \mathbf{v}^*) - \sigma\varepsilon^2$ with $\sigma = \|H\|_s$. As $\tilde{\mathbf{v}}^*$ is the minimizer of $\tilde{V}^*(x, \tilde{\mathbf{v}})$, $V^*(x, \mathbf{v}^*) \geqslant \tilde{V}^*(x, \tilde{\mathbf{v}}^*) - \sigma\varepsilon^2$ which in turn can be rewritten as $J(x) \geqslant \tilde{J}(x) - \sigma\varepsilon^2$. Recall that from Property 2 $\hat{J}(x) \leqslant \tilde{J}(x)$; thus $J(x) \geqslant \hat{J}(x) - \sigma\varepsilon^2 = \hat{V}^*(x, \hat{\mathbf{v}}^*) - \sigma\varepsilon^2$. From Theorem 1 $\hat{V}^*(x, \mathbf{v}) \geqslant V^*(x, \mathbf{v})$ thus

$$J(x) \geqslant V^*(x, \hat{\mathbf{v}}^*) - \sigma\varepsilon^2.$$

This completes the proof. □

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