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Automatica 42 (2006) 1745–1751
www.elsevier.com/locate/automatica

Brief paper

Robust MPC of constrained discrete-time nonlinear systems based on approximated reachable sets☆

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Received 18 April 2005; received in revised form 10 February 2006; accepted 8 May 2006
Available online 7 July 2006

Abstract

A robust MPC for constrained nonlinear systems with uncertainties is presented. Outer bounds of the reachable sets of the system are used to predict the evolution of the system under uncertainty. A method that uses zonotopes to represent the approximated reachable sets is proposed. The closed-loop system is ultimately bounded thanks to a contractive constraint that drives the system to a robust invariant set.

Keywords: Predictive control; Nonlinear systems; Constrained control; Robust stability

1. Introduction

Model predictive control is a control strategy that has been widely adopted in industry (Qin & Badwell, 2003) and academia (Camacho & Bordons, 1999). The reason for this success is the ability to deal with constraints and multivariable systems. A survey about model predictive control can be found in Mayne, Rawlings, Rao, and Scokaert (2000) where sufficient conditions to guarantee asymptotic stability are given.

The problem of robust nonlinear model predictive control is addressed in this paper. When uncertainties are present, they should be taken into account in the computation of the control law in order to guarantee robust stability. Some authors have formulated this problem as in Michalska and Mayne (1993) where a dual-mode receding horizon controller is proposed and robustness under decaying additive uncertainties is achieved by a proper choice of the terminal region. In Magni, Nijmeijer, and van der Shaft (2001) a robust MPC strategy based on an H_∞ cost function is presented. In Limon, Bravo, Alamo, and Camacho (2005) a robust nonlinear predictive controller based on reachable sets is presented. Natural interval extension is used to bound the uncertain evolution of the system.

A linear difference inclusion of the original nonlinear system is used by some authors to implement a robust control (Angeli, Casavola, & Mosca, 2002; Cannon, Deshmukh, & Kouvaritakis, 2002; Casavola, Famularo, & Franze, 2002; Kothare, Balakrishnan, & Morari, 1996). In Langson, Chryssochoos, Rakovic, and Mayne (2004) a tube predictive controller is proposed for linear constrained systems with additive uncertainty.

In this paper, a new robust MPC for nonlinear systems with parametric uncertainty is proposed. To improve the results obtained in Limon et al. (2005), the new approach relies on a prediction method that uses zonotopes (Alamo, Bravo, & Camacho, 2005; Kühn, 1998) to bound the reachable sets.

The paper is organized as follows: In Section 2, the class of nonlinear uncertain systems under consideration is introduced. In Section 3, an outer bound of the uncertain trajectory of the system is presented. The proposed robust nonlinear MPC controller is introduced in Section 4. The stability of the closed-loop system is analyzed in Section 5. An illustrative example is given in Section 6. The paper draws to a close with a section of conclusions.

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Tor Arne Johansen under the direction of Editor Frank Allgöwer.

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doi:10.1016/j.automatica.2006.05.003

2. Problem statement

Consider an uncertain nonlinear discrete-time system of the form:

$$x_{k+1} = f(x_k, u_k, w_k), \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state of the system and $u_k \in \mathbb{R}^m$ is the control vector at sample time k. The vector $w_k \in \mathbb{R}^{n_w}$ represents the uncertainty. It is assumed that the uncertainty is bounded in a compact set that contains the origin: $w_k \in W$. The system is subject to constraints on the state and on the control action: $x_k \in X$ and $u_k \in U$ where X is a closed set and U a compact set, both containing the origin.

In order to provide some amount of feedback to the predictions (Rossiter, Kouvaritakis, & Rice, 1998), the control action is given by a stabilizing control law plus a correction term: $u_k = Kx_k + v_k$. It will be assumed that $u_k = Kx_k$ stabilizes the system around the origin. With this notation, the dynamics of the system can be rewritten as

$$x_{k+1} = f(x_k, Kx_k + v_k, w_k) = f_K(x_k, v_k, w_k). \tag{2}$$

At sample time k, the objective of the robust predictive controller is the computation of a sequence of correction control inputs $v(k) = \{v(k|k), v(k+1|k), \ldots, v(k+N-1|k)\}$ in such a way that the uncertain evolution of system (2) satisfies the constraints of the problem and a given cost function is minimized.

The next section presents the concept of reachable set and bounding operator. A bounding operator is used to build a guaranteed prediction of the system trajectory.

3. Computation of outer bounds of the reachable sets

In what follows, some preliminary notations are introduced. An interval $X = [a, b]$ is the set $\{x : a \leqslant x \leqslant b\}$. The unitary interval is $\mathbf{B} = [-1, 1]$. A box is an interval vector. A unitary box, denoted as $\mathbf{B}^m$, is a box composed by m unitary intervals. Given a box $Q = ([a_1, b_1], \ldots, [a_n, b_n])^\top$: mid (Q) denotes its center and diam $(Q) = (b_1 - a_1, \ldots, b_n - a_n)^\top$. The Minkowski sum of two sets X and Y is defined by $X \oplus Y = \{x + y : x \in X, y \in Y\}$. Given a set Ω and a scalar $\delta > 0$, $\delta\Omega$ represents the set $\{x : x \cdot 1/\delta \in \Omega\}$. Given a vector $p \in \mathbb{R}^n$ and a matrix $H \in \mathbb{R}^{n \times m}$, the set: $p \oplus H\mathbf{B}^m = \{p + Hz : z \in \mathbf{B}^m\}$ is called a zonotope of order m. Note that this is the Minkowski sum of the segments defined by the columns of matrix H. Given a function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and a set $Y \subset \mathbb{R}^n$, $g(Y)$ denotes the set $\{g(y) : y \in Y\}$. With this notation, it is possible to present a definition of reachable set of a system.

Definition 1 (*Reachable set*). Consider a system given by (2), consider also that the state at sample time k is x_k and that a sequence of correction control inputs $v(k)$ is given, then the reachable sets $\{X(k|k), X(k+1|k), \ldots, X(k+N|k)\}$ are obtained from the recursion: $X(k+j|k) = f_K(X(k+j-1|k), v(k+j-1|k), W)$ where $X(k|k) = x_k$.

Note that $X(k+j|k)$ is the set of all states that can be reached by the evolution of the uncertain system at sample time $k+j$ applying the sequence of control correction inputs $v(k)$. The exact computation of these sets is a difficult task. In order to reduce the complexity of the computation of these sets, approximated approaches can be used. The computation of the approximated reachable sets are based on the concept of bounding operator:

Definition 2. $\Psi(\cdot,\cdot,\cdot)$ is a bounding operator of $f(\cdot,\cdot,\cdot)$ if $f(X, u, W) \subseteq \Psi(X, u, W)$, for all (X, u, W).

A bounding operator, based on natural interval extension, was used to calculate the approximated reachable sets in Limon et al. (2005). Although it is an efficient solution, direct natural interval extension can produce a large overestimation of the exact reachable set.

An alternative way of obtaining approximated reachable sets was presented in Kühn (1998). Kühn's method is a procedure to bound the orbits of discrete dynamic systems without uncertainty by means of zonotopes. The method exhibits subexponential overestimation (Kühn, 1998).

In this paper, a generalization of Kühn's method for uncertain nonlinear systems is presented. Based on this generalization, a bounded operator with improved approximation properties is obtained. The following theorem (see, Alamo et al., 2005, for a proof), introduces the zonotope inclusion operator which is required to present the proposed bounded operator.

Theorem 1 (*Zonotope inclusion*). *Consider a family of zonotopes represented by $Z = p \oplus \mathbf{M}\mathbf{B}^m$ where $p \in \mathbb{R}^n$ is a real vector and $\mathbf{M} \in \mathbb{I}^{n \times m}$ is an interval matrix. A zonotope inclusion, denoted by $\diamond(Z)$, is defined by*

$$\diamond(Z) = p \oplus [\text{mid}\,(\mathbf{M})\ \ G] \begin{bmatrix} \mathbf{B}^m \\ \mathbf{B}^n \end{bmatrix} = p \oplus J\mathbf{B}^{m+n},$$

where $G \in \mathbb{R}^{n \times n}$ is a diagonal matrix that satisfies:

$$G_{ii} = \sum_{j=1}^{m} \frac{\text{diam}\,(\mathbf{M}_{ij})}{2}, \quad i = 1, \ldots, n.$$

Under these definitions it results that: $Z \subseteq \diamond(Z)$.

The following theorem is a generalization of Kühn's result (Kühn, 1998) as it considers uncertainty and non-autonomous systems. An analogous result for autonomous systems can also be found in (Alamo et al., 2005).

Theorem 2. *Given a function $f(x, u, w)$ where $x \in \mathbb{R}^n$ and $w \in \mathbb{R}^{n_w}$, a zonotope $X = p \oplus H\mathbf{B}^m$, and a zonotope $W = c_w \oplus C_w \mathbf{B}^{s_w}$, consider the following:*

- *A zonotope $q \oplus S\mathbf{B}^d$ such that $f(p, u, W) \subseteq q \oplus S\mathbf{B}^d$.*
- *An interval matrix $\mathbf{M} \supseteq \nabla_x f(X, u, W)H$.*
- *A zonotope $\Psi(X, u, W) = q \oplus S\mathbf{B}^d \oplus \diamond(\mathbf{M}\mathbf{B}^m) = q \oplus H_q\mathbf{B}^l$ with $l = d + n + m$*

Under the previous assumptions it results that $f(X,u,W) \subseteq \Psi(X,u,W)$.

Proof. Given $w \in W$, the application of the mean value extension yields: $f(X,u,w) \subseteq f(p,u,w) \oplus (\nabla_x f(X,u,w))H\mathbf{B}^m$. Thus: $f(X,u,W) \subseteq f(p,u,W) \oplus (\nabla_x f(X,u,W))H\mathbf{B}^m \subseteq q \oplus S\mathbf{B}^d \oplus \mathbf{M}\mathbf{B}^m \subseteq q \oplus S\mathbf{B}^d \oplus \diamond(\mathbf{M}\mathbf{B}^m) = q \oplus H_q \mathbf{B}^l$. □

In order to obtain zonotope $q \oplus S\mathbf{B}^d$ of Theorem 2 interval arithmetic (Moore, 1966; Kühn, 1998) can be used. If $f(x,u,w)$ is an affine function of w (that is, $f(x,u,w) = g(x,u) + G(x,u)w$) then $f(p,u,W) = (g(p,u) + G(p,u)c_w) \oplus (G(p,u)C_w)\mathbf{B}^{s_w}$. In this case, choosing $q = g(p,u) + G(p,u)c_w$, $S = G(p,u)C_w$ and $d = s_w$, an exact bound is obtained. Interval matrix $\mathbf{M}$ can also be computed by means of interval arithmetics. Note that the proposed operator $\Psi(\cdot,\cdot,\cdot)$ relies on the use of the mean value extension (Krawczyk & Nickel, 1982), which constitutes an interval approximation of second order.

Remark 1. If a linear system with bounded uncertainty is considered, zonotopes provide an exact description of the reachable sets. Consider the linear system $f(x,u,w) = Ax + Bu + Ew$, as well as the zonotopes $X = p \oplus H\mathbf{B}^m$ and $W = c_w \oplus C_w \mathbf{B}^{s_w}$ then $f(X,u,W) = (Ap + Bu + Ec_w) \oplus [AH \;\; EC_w]\mathbf{B}^{m+s_w}$.

4. Robust MPC

This section presents a dual predictive controller (Michalska & Mayne, 1993). These controllers split the space into two parts. One of them is a control invariant set around the origin that constitutes the terminal region Ω. In the terminal region, a local control law is used $u = Kx$. In the rest of the state space, a predictive controller is applied. In what follows, the details of the proposed robust predictive controller are given. In order to prove that the closed-loop system is ultimately bounded, the following assumption will be required:

Assumption 1. Consider system (1). There is a convex region $\Omega \subseteq X$, containing the origin, with an associated local control law $u = Kx \in U$ such that $f(x,Kx,w) = f_K(x,0,w) \in \Omega$ for all $x \in \Omega$, $w \in W$.

Assumption 1 is not too restrictive as it only requires that the local controller stabilizes the system in a neighborhood of the steady state. A linear difference inclusion (Boyd, Ghaoui, Feron, & Balakrishnan, 1994) can be used to approximate the nonlinear system and to compute a robust linear controller and set Ω (Blanchini, 1999).

In order to check if the evolution of the uncertain system verifies the constraints of the optimization problem, approximated reachable sets are considered. These are based on the existence of a bounding operator $\Psi(\cdot,\cdot,\cdot)$. This bounding operator can be obtained, for example, by means of Theorem 2. Other bounding operators could be used.

Assumption 2. Consider System (2). The operator $\Psi(\cdot,\cdot,\cdot)$ is a bounding operator of function $f_K(\cdot,\cdot,\cdot)$. That is, $f_K(X,v,W) = f(X,Kx+v,W) \subseteq \Psi(X,v,W)$, for all (X,v,W).

The following definition introduces the concept of approximated reachable sets:

Definition 3. Given an initial condition x_k, a sequence of correction control inputs $v(k) = \{v(k|k), v(k+1|k), \ldots, v(k+N-1|k)\}$ and the bounding operator $\Psi(\cdot,\cdot,\cdot)$, a sequence of approximated reachable sets $\{\hat{X}(k|k), \hat{X}(k+1|k), \ldots, \hat{X}(k+N|k)\}$ is obtained from the following recursion:

$$\hat{X}(k+j|k) = \begin{cases} \Psi(\hat{X}(k+j-1|k), \quad v(k+j-1|k), W) \\ \qquad\qquad\qquad\qquad\qquad \text{if } \hat{X}(k+j-1|k) \not\subseteq \Omega, \\ 0 \qquad\qquad\qquad\qquad\quad \text{otherwise}, \end{cases}$$

where $j = 1, \ldots, N$ and $\hat{X}(k|k) = x_k$.

Remark 2. The computational burden required to compute the approximated reachable sets depends on the chosen bounding operator: bounding operators based on a global Lipschitz constant or on natural interval arithmetics require the same order of operations as the evaluation of the function $f(x,u,w)$. The computational burden corresponding to the bounding operator proposed in Theorem 2 is bounded by the number of floating point evaluations required to evaluate $(\nabla_x f(x,u,w))H$ multiplied by a small constant. Although this bounding operator has an increased computational burden when compared with mere natural interval evaluation, it has much better convergence properties.

A cost function $J_E(\cdot)$ is introduced in order to assure convergence to the terminal region. This function penalizes the uncertain trajectory that is not included in the terminal region. This penalization is made by means of an operator that serves as a measure of the distance to the terminal region:

Definition 4. Given sets Y and $\Phi = 1/(1+\alpha)\Omega$ (where α is a scalar greater than zero), $\|Y\|_\Phi$ denotes the maximum between zero and the smallest scalar $\beta \geqslant 0$ that verifies $Y \subseteq (1+\beta)\Phi$.

Note that $\|Y\|_\Phi$ is equal to zero if and only if $Y \subseteq \Phi$. If Φ is a polyhedron defined by a set of linear constraints $\Phi = \{x : d_i^\top x \leqslant e_i,\ i = 1, \ldots, p\}$ that contains the origin ($e_i > 0$, $i = 1, \ldots, p$) and $Y = p \oplus H\mathbf{B}^m$ is a zonotope, then $\|Y\|_\Phi$ can be obtained from the equality:

$$\|Y\|_\Phi = \max\left\{0, \max_{i=1,\ldots,p} \frac{d_i^\top p - e_i + \|H^\top d_i\|_1}{e_i}\right\},$$

where $\|H^\top d_i\|_1$ denotes the sum of the absolute values of vector $H^\top d_i$. Set Φ is equal to $1/(1+\alpha)\Omega$ where $\alpha > 0$. It is not difficult to see that $x \notin \Omega$ implies that $\|x\|_\Phi > \alpha$.

The cost function $J_E(k)$ is defined by $J_E(-1) = \infty$ and $J_E(k) = \sum_{i=0}^{N-1} \|\check{X}(k+i|k)\|_\Phi$, where $\Phi = 1/(1+\alpha)\Omega$ and

$$\check{X}(k+l|k) = \begin{cases} \hat{X}(k+l|k) & \text{if } P(x_k) \text{ is feasible,} \\ \check{X}(k+l|k-1) & \text{otherwise,} \end{cases}$$

where $l = 0, \ldots, N-1$, $\check{X}(k+N|k) = 0$ and $P(x_k)$ denotes the proposed optimization problem which is detailed below.

Given an initial condition x_k and a sequence of correction control inputs $v(k) = \{v(k|k), v(k+1|k), \ldots, v(k+N-1|k)\}$, $\hat{x}(i|k)$ denotes the nominal evolution of the system. That is, $\hat{x}(k|k) = x_k$ and $\hat{x}(k+j+1|k) = f_K(\hat{x}(k+j|k), v(k+j|k), 0)$ with $j = 0, \ldots, N-1$.

The proposed predictive controller solves at sample time k the following optimization problem $P(x_k)$:

$$P(x_k) = \min_{v(k)} J(x_k, v(k)) = \sum_{j=0}^{N-1} L(\hat{x}(k+j|k), v(k+j|k)) + V(\hat{x}(k+N)|k)$$

subject to:

$$\hat{x}(k|k) = x_k, \tag{3}$$

$$\hat{x}(k+j+1|k) = f_K(\hat{x}(k+j|k), v(k+j|k), 0), \tag{4}$$

$$v(k+j|k) \oplus K\hat{X}(k+j|k) \subseteq U, \tag{5}$$

$$\hat{X}(k+j|k) \subseteq X, \quad j = 0, 1, \ldots, N-1, \tag{6}$$

$$\hat{X}(k+N|k) \subseteq \Omega, \tag{7}$$

$$\sum_{j=0}^{N-1} \|\hat{X}(k+j|k)\|_\Phi - J_E(k-1) < -\alpha, \tag{8}$$

where $L(\cdot,\cdot)$ is a positive definite state cost and $V(\cdot)$ is a positive definite terminal cost. The optimization problem is subject to the system dynamics, that is, given $v(k)$, the sequence $\{\hat{X}(k|k), \hat{X}(k+1|k), \ldots, \hat{X}(k+N|k)\}$ is obtained as in Definition 3.

Note that the robust satisfaction of the input constraints are considered in (5). The predictive controller has finite horizon N. As it will be shown in the following section, if $P(x_0)$ is feasible then there is a sequence of control signals that drives the system to the terminal region in a finite number of sample instants. The contractive constraint (8) assures convergence to the terminal region. It will be shown that the closed-loop system reaches the terminal region in a finite number of sample instants. The constraints are applied to the approximated reachable sets so, given an initial state and a sequence of control signals, it is assured that the state satisfies the constraints for every possible realization of the uncertainty. The dual controller applies the control signal $u_k = K_{\mathrm{MPC}}(x_k)$ at time k, which is obtained by means of the following algorithm:

Algorithm[*Controller Algorithm* (x_k)]
  If $x_k \in \Omega$ then $u_k = Kx_k$
  Else
    Compute $J_E(k-1)$
    If $P(x_k)$ is feasible then $\bar{v}(k) = v^*(k)$,
      where $v^*(k)$ is an optimal (or suboptimal)
      feasible solution of the optimization
      problem $P(x_k)$
    Else $\bar{v}(k) = \{\bar{v}(k|k-1), \bar{v}(k+1|k-1), \ldots, \bar{v}(k+N-2|k-1), 0\}$
    Endif
    $u_k = \bar{v}(k|k) + Kx_k$
  Endif
End Algorithm

5. Robust properties of the proposed controller

The origin is not a steady state of the uncertain system because the uncertainties may not be decaying. Hence, the aim of a stabilizing controller is to steer the state to a neighborhood of the origin and keep the state evolution in it. This section proves that for any feasible initial state, the proposed predictive controller steers the uncertain system to the terminal region where it remains for all the time.

Denote $\bar{v}(k) = \{\bar{v}(k|k), \ldots, \bar{v}(k+N-1|k)\}$ the sequence of control correction inputs obtained at sample time k by means of the controller algorithm. Denoting $X(k|k) = x_k$, the sequence of exact reachable sets $\{X(k|k), \ldots, X(k+N|k)\}$ is obtained from the following recursion: $X(k+j|k) = f_K(X(k+j-1|k), \bar{v}(k+j-1|k), W)$, $j = 1, \ldots, N$. The following theorem states that the proposed controller satisfies robustly both the state and the control constraints.

Theorem 3. *If $P(x_0)$ is feasible then the controller $u_k = K_{\mathrm{MPC}}(x_k)$ guarantees that the closed loop trajectories of the system satisfy robustly the constraints of the problem. That is, $x_k \in X$ and $u_k \in U$ $\forall k \geqslant 0$.*

Proof. Suppose that $x_k \notin X$. As $\Omega \subseteq X$, it is inferred that $x_k = X(k|k) \not\subseteq \Omega$. Taking now into account Property 1 (see Appendix A): $X(k|k) \subseteq \check{X}(k|k) \subseteq X$. This is a contradiction.

In order to prove that $u_k \in U$, two cases must be taken into account. If $x_k \in \Omega$ then $u_k = Kx_k$. In this case, Assumption 1 guarantees that $u_k = Kx_k \in U$. Suppose now that $x_k = X(k|k) \notin \Omega$. It is inferred from Property 1 (see Appendix A) that $u_k = \bar{v}(k|k) + Kx_k = \bar{v}(k|k) \oplus KX(k|k) \subseteq U$. □

Remark 3. Denote l the smallest integer such that $P(x_{k-l})$ is feasible. In case of unfeasibility at x_k, the proposed controller algorithm makes $\bar{v}(k|k)$ equal to $\bar{v}(k|k-l)$. As the control correction sequence $\{\bar{v}(k-l|k-l), \bar{v}(k-l+1|k-l), \ldots, \bar{v}(k-l+N-1|k-l)\}$ if feasible at x_{k-l} the application of the sequence of control actions

$$u_{k-l+j} = \bar{v}(k-l+j|k-l) + Kx_{k-l+j}, \quad j = 0, \ldots, N-1$$

drives the initial condition x_{k-l} to Ω for every possible realization of the uncertainty. As this is precisely the control input that the algorithm applies in case of unfeasibility the admissible robust convergence to set Ω is guaranteed.

Next, a lemma used to prove that the state of the system is ultimately bounded is enunciated.

Lemma 1. *If $P(x_0)$ is feasible and $x_k \notin \Omega$ then $J_E(k) - J_E(k-1) < -\alpha$.*

Proof. Suppose first that $P(x_k)$ is feasible. In this case, $\check{X}(k+j|k) = \hat{X}(k+j|k)$, $j = 0, \dots, N-1$. Therefore:

$$\begin{aligned} J_E(k) - J_E(k-1) &= \sum_{i=0}^{N-1} \|\check{X}(k+i|k)\|_\Phi - J_E(k-1) \\ &= \sum_{i=0}^{N-1} \|\hat{X}(k+i|k)\|_\Phi - J_E(k-1) < -\alpha. \end{aligned}$$

Note that the last inequality is due to the feasibility of $P(x_k)$ (see constraint 8).

Suppose now that $P(x_k)$ is unfeasible. In this case, $\check{X}(k+j|k) = \check{X}(k+j|k-1)$, $j = 0, \dots, N-1$. Thus:

$$\begin{aligned} &J_E(k) - J_E(k-1) \\ &= \sum_{i=0}^{N-1} \|\check{X}(k+i|k)\|_\Phi - J_E(k-1) \\ &= \sum_{i=0}^{N-1} \|\check{X}(k+i|k-1)\|_\Phi - \sum_{i=0}^{N-1} \|\check{X}(k-1+i|k-1)\|_\Phi \\ &= \|\check{X}(k+N-1|k-1)\|_\Phi - \|\check{X}(k-1|k-1)\|_\Phi. \end{aligned}$$

By definition, $\check{X}(k-1+N|k-1) = 0$. It is then concluded that $J_E(k) - J_E(k-1) = -\|\check{X}(k-1|k-1)\|_\Phi$. From $x_k \notin \Omega$ it is inferred that $x_{k-1} \in \check{X}(k-1|k-1) \not\subseteq \Omega$ which implies that $\|\check{X}(k-1|k-1)\|_\Phi > \alpha$. Therefore, $J_E(k) - J_E(k-1) < -\alpha$. □

The controller proposed in this paper steers the uncertain system to the terminal region, which is a robust invariant set. The next theorem proves that the closed loop system is ultimately bounded for all x_0 such that the optimization problem $P(x_0)$ is feasible.

Theorem 4. *The system controlled by $u_k = K_{\mathrm{MPC}}(x_k)$ is ultimately bounded in Ω for all x_0 such that the optimization problem $P(x_0)$ is feasible.*

Proof. Lemma 1 assures that if the optimization problem $P(x_0)$ is feasible and $x_k \notin \Omega$, then the proposed predictive controller satisfies the contractive constraint $J_E(k) - J_E(k-1) < -\alpha$. If $P(x_0)$ is feasible and the evolution of the system does not reach Ω then $J_E(k) < J_E(0) - k\alpha, \forall k > 0$. Therefore, there exists $\hat{k}$ such that $J_E(\hat{k}) < J_E(0) - \hat{k}\alpha < 0$. This is a contradiction because $J_E(k)$, $k \geqslant 0$, is a positive definite function, so it is inferred that $x_{\hat{k}} \in \Omega$. The system reaches the terminal region in a finite number of sample times and the close loop system is ultimately bounded. Note that Assumption 1 guarantees that once the state vector reaches Ω it remains in it. □

Remark 4. If the bounding operator Ψ is monotonic, that is, $X_a \subseteq X_b$ implies $\Psi(X_a, v, W) \subseteq \Psi(X_b, v, W)$, $\forall v, \forall W$, then it is possible to assure that feasibility for the initial state implies feasibility of the subsequent optimization problems (this can be proved following the ideas presented in Limon et al. (2005), where a monotonic operator based on interval arithmetic was used to bound the evolution of the system). However, the monotonic assumption can overrestrict the class of bounding operator that can be used (for example, the bounding operator proposed in Theorem 2 is not monotonic). Therefore, no assumption on the fulfilment of the monotonic property is considered in this paper.

6. Example

The proposed MPC controller is applied to a highly nonlinear system: a continuous stirred tank reactor (CSTR) simulation model. The continuous time model of a CSTR for an exothermic, irreversible reaction $A \longrightarrow B$ with constant liquid volume is given by Magni, De Nicolao, Magnani, and Scattolini (2001):

$$\begin{aligned} \frac{\mathrm{d}C_A}{\mathrm{d}t} &= \frac{q}{V}(C_{Af} - C_A) - k_0 \exp\left(-\frac{E}{RT}\right) \cdot C_A, \\ \frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{q}{V}(T_f - T) - \frac{\Delta H \cdot k_0}{\rho C_p} \exp\left(-\frac{E}{RT}\right) C_A \\ &\quad + \frac{U \cdot A}{V \cdot \rho \cdot C_p}(T_c - T), \end{aligned}$$

where C_A is the concentration of A in the reactor, T is the reactor temperature and T_c is the temperature of the coolant stream. The parameters of the model are: $\rho = 1000\,\mathrm{g/l}$, $C_p = 0.239\,\mathrm{J/g\,K}$, $\Delta H = -5 \times 10^4\,\mathrm{J/mol}$, $E/R = 8750\,\mathrm{K}$, $k_0 = 7.2 \times 10^{10}\,\mathrm{min}^{-1}$, $U \cdot A = 5 \times 10^4\,\mathrm{J/\,min\,K}$. The nominal operating conditions are given by: $q = 100\,\mathrm{l/\,min}$, $T_f = 350\,\mathrm{K}$, $V = 100\,\mathrm{l}$, $C_{Af} = 1.0\,\mathrm{mol/l}$. The steady state is $C_A^o = 0.5\,\mathrm{mol/l}$, $T^o = 350\,\mathrm{K}$, $T_c^o = 300\,\mathrm{K}$. The temperature of the coolant is constrained to $280K \leqslant T_c \leqslant 370$. The state of the system is defined as $x = [(C_A - C_A^o), (T - T^o)]^T$, and the input as $u = T_c - T_c^o$.

The model is discretized with a sampling period $T_s = 0.03$ min. An additive uncertainty is considered in the continuous-time model of the system. The uncertainty is bounded by $w_1 \in 0.1\mathbf{B}^1$ and $w_2 \in 2\mathbf{B}^1$, where $w = [w_1\ w_2]^T$.

In Fig. 1 the solution of $P(x_0)$ with $x_0 = [-0.15, -45]^\top$ is shown. In the figure, and in order to visualize the exact reachable sets, a sufficient dense cloud of trajectories corresponding to random realizations of the uncertainty has been shown. The sequence of approximated reachable sets obtained using the bounding operator proposed in Theorem 2 are represented with thin solid lines. The terminal set Ω corresponding to the local control law $u = Kx = [-3, -6.9]x$ has been obtained using a linear difference inclusion (Boyd et al., 1994) obtained by means of the application of the interval extension of the mean value theorem. The resulting optimization prob-

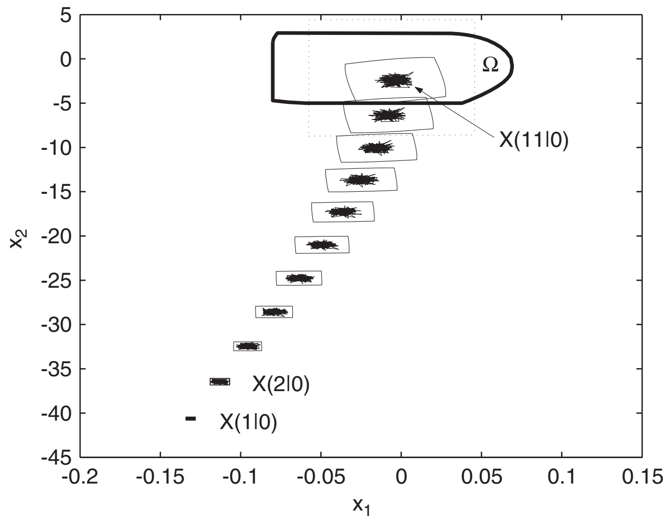

Fig. 1. Approximated reachable sets.

lem $P(x_0)$ is feasible when this bounding operator is used. However, if a bounding operator based only on natural interval arithmetic is used, the feasibility of $P(x_0)$ is lost (in this case, the obtained approximated set $\hat{X}(11|0)$ represented as a dotted line box in the figure is too large to be included in region Ω). As was to be expected, the use of zonotopes provides tighter outer bounds that enlarge the domain of attraction of the controller.

7. Conclusion

A robust dual-mode MPC controller for constrained discrete-time nonlinear systems with uncertainties has been presented. Approximated reachable sets have been added to the MPC optimization problem. These sets are computed using a technique based on zonotopes. The closed-loop system is shown to be robustly ultimately bounded.

Acknowledgments

The authors are indebted to Daniel Limon for the constructive and valuable comments and stimulating discussions. The authors would like to acknowledge MCYT-Spain (contract DPI2005-04568, DPI2003-07146-C02-01 and DPI2004-07444-C04-01). Work partially done in the framework of the HYCON Network of Excellence, contract number FP6-IST-511368. We are also grateful to the anonymous reviewers for their helpful comments.

Appendix A

Property 1. *If $P(x_0)$ is feasible and $X(k+j|k) \not\subseteq \Omega$, $j = 0, 1, \ldots, i$ then*:

$$X(k+j|k) \subseteq \check{X}(k+j|k) \subseteq X, \quad j = 0, 1, \ldots, i,$$

$$\bar{v}(k+j|k) \oplus KX(k+j|k) \subseteq U, \quad j = 0, 1, \ldots, i.$$

Proof. Given a sample time $k > 0$, two cases must be considered:

(1) $P(x_k)$ *is feasible*: It will be shown first that $X(k+j|k) \subseteq \hat{X}(k+j|k)$, $j = 0, \ldots, i$. This claim is proved by induction. By definition, $X(k|k) = \hat{X}(k|k) = x_k$. Assume now that $X(k+j-1|k) \subseteq \hat{X}(k+j-1|k)$ with $1 \leqslant j \leqslant i$. As $X(k+j-1|k) \not\subseteq \Omega$ it is inferred that $\hat{X}(k+j-1|k) \not\subseteq \Omega$. From this and the fact that Ψ is a bounding operator: $X(k+j|k) = f_K(X(k+j-1|k), \bar{v}(k+j-1|k), W) \subseteq f_K(\hat{X}(k+j-1|k), \bar{v}(k+j-1|k), W) \subseteq \Psi(\hat{X}(k+j-1|k), \bar{v}(k+j-1|k), W) = \hat{X}(k+j|k)$.

Therefore, it has been proved by induction that:

$$X(k+j|k) \subseteq \hat{X}(k+j|k), \quad j = 0, \ldots, i. \tag{A.1}$$

Taking into account that $P(x_k)$ is feasible and the definition of $\check{X}(k+j|k)$: $\check{X}(k+j|k) = \hat{X}(k+j|k) \subseteq X$. Therefore, it is concluded from Eq. (A.1) that: $X(k+j|k) \subseteq \hat{X}(k+j|k) = \check{X}(k+j|k) \subseteq X$. From this inclusion and the feasibility of $P(x_k)$: $\bar{v}(k+j|k) \oplus KX(k+j|k) \subseteq \bar{v}(k+j|k) \oplus K\hat{X}(k+j|k) \subseteq U$, $j = 0, 1, \ldots, i$.

(2) $P(x_k)$ *is not feasible*: Denote l the smallest integer such that $P(x_{k-l})$ is feasible. It is easy to see from the proposed controller algorithm that:

$$\begin{aligned}\bar{v}(k) &= \{\bar{v}(k|k-1), \ldots, \bar{v}(k+N-2|k-1), 0\} \\ &= \{\bar{v}(k|k-2), \ldots, \bar{v}(k+N-3|k-2), 0, 0\} \\ &\ \ \vdots \\ &= \{\bar{v}(k|k-l), \ldots, \bar{v}(k+N-l-1|k-l), 0, \ldots, 0\}.\end{aligned}$$

As $\bar{v}(k)$ is composed of the control signals used to compute the reachable sets at instant $k-l$: $X(k+j|k) \subseteq X(k+j|k-l)$, $j = 0, \ldots, i$.

From the last equation and the assumption $X(k+j|k) \not\subseteq \Omega$, $j = 0, 1, \ldots, i$, it is inferred that $X(k+j|k-l) \not\subseteq \Omega$, $j = 0, \ldots, i$. Note also that $X(k+j|k-l) \not\subseteq \Omega$ for $j = -l, \ldots, -1$. This affirmation can be proved by contradiction: if there is $j^* \in [-l, \ldots, -1]$ such that $X(k+j^*|k-l) \subseteq \Omega$ then $x_{k+j^*} \in \Omega$ implies (because of the dual controller) that $x_{k+j} \in \Omega, \forall j \geqslant j^*$. This is a contradiction as $x_k \notin \Omega$. From the above considerations it is concluded that: $X(k-l+j|k-l) \not\subseteq \Omega$, $j = 0, \ldots, i+l$.

Taking into account the feasibility of $P(x_{k-l})$ and considering the previous item of the proof, it is inferred that $X(k+j|k-l) \subseteq \check{X}(k+j|k-l)$ for $j = 0, \ldots, i$. Applying the definition of $\check{X}(k+j|k)$ and from the feasibility of $P(x_{k-l})$: $\check{X}(k+j|k) = \check{X}(k+j|k-1) = \cdots = \check{X}(k+j|k-l) = \hat{X}(k+j|k-l) \subseteq X$.

It is therefore concluded that: $X(k+j|k) \subseteq X(k+j|k-l) \subseteq \check{X}(k+j|k-l) = \check{X}(k+j|k) \subseteq X$, $j = 0, \ldots, i$. From the already proved inclusion: $X(k+j|k) \subseteq \check{X}(k+j|k-l) = \hat{X}(k+j|k-l)$, $j = 0, \ldots, i$. This and the feasibility of $P(x_{k-l})$ yield: $\bar{v}(k+j|k) \oplus KX(k+j|k) \subseteq \bar{v}(k+j|k) \oplus K\hat{X}(k+j|k-l) \subseteq U$, $j = 0, 1, \ldots, i$. □

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