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Brief paper

Input to state stability of min–max MPC controllers for nonlinear systems with bounded uncertainties ☆

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Abstract

Min–max model predictive control (MPC) is one of the control techniques capable of robustly stabilize uncertain nonlinear systems subject to constraints. In this paper we extend existing results on robust stability of min–max MPC to the case of systems with uncertainties which depend on the state and the input and not necessarily decaying, i.e. state and input dependent bounded uncertainties. This allows us to consider both plant uncertainties and external disturbances in a less conservative way.

It is shown that the input-to-state practical stability (ISpS) notion is suitable to analyze the stability of worst-case based controllers. Thus, we provide Lyapunov-like sufficient conditions for ISpS. Based on this, it is proved that if the terminal cost is an ISpS-Lyapunov function then the optimal cost is also an ISpS-Lyapunov function for the system controlled by the min–max MPC and hence, the controlled system is ISpS. Moreover, we show that if the system controlled by the terminal control law locally admits certain stability margin, then the system controlled by the min–max MPC retains the stability margin in the feasibility region.

Keywords: Nonlinear systems; Input-to-state stability; Model predictive control; Constraints

1. Introduction

Model predictive control (MPC) is an optimal control technique which deals with constraints on the states and the inputs. This strategy is based on the solution of a finite horizon optimization problem, which can be posed as a mathematical programming problem. The control law is obtained by means of the receding horizon strategy that requires the solution of the optimization problem at each sample time (Camacho & Bordons, 2004). It is well known that considering a terminal cost and a terminal constraint in the optimization problem, the MPC stabilizes asymptotically the constrained system (Mayne, Rawlings, Rao, & Scokaert, 2000).

If there are uncertainties in the process model, then the stabilizing properties of the MPC may provide a certain degree of robustness under some assumptions (Limon, Alamo, & Camacho, 2002b; Grimm, Messina, Tuna, & Teel, 2004). One of the approaches to the design of MPC controllers incorporating the uncertainty is the so-called open loop formulations (Michalska & Mayne, 1993; Limon, Alamo, & Camacho, 2002a; Alamo, Muñoz de la Peña, Limon, & Camacho, 2005). These controllers guarantee robust stability and constraint satisfaction but, they may end up being very conservative since they are likely to have a very small feasible region or a poor performance.

This conservativeness can be overcome thanks to the closed-loop formulations for nonlinear systems (Mayne, 2001; Magni, Nijmeijer, & van der Shaft, 2001; Fontes & Magni, 2003; Magni, De Nicolao, Scattolini, & Allgöwer, 2003), where a sequence of control laws is computed instead of a sequence of control actions. In (Mayne, 2001) sufficient conditions to design an asymptotically stabilizing min–max MPC in case of uncertainties that decay with the state are given. In the case of persistent disturbances, the standard min–max MPC problem is modified using a *dual* stage cost, which yields to an

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optimization problem difficult to solve and provides a discontinuous control law. An H_∞ approach to the robust stabilization of nonlinear systems is presented in (Magni et al., 2001). In (Fontes & Magni, 2003), min–max MPC for continuous-time systems with persistent disturbances is analyzed. Under some robust stability conditions and sufficiently small sampling time, asymptotic convergence to a robust invariant set of the uncontrolled (zero-input) system is proved. This result has been applied to design a stabilizing bang-bang controller.

The goal of this paper is to obtain sufficient conditions to design robust stable MPC for constrained systems with bounded uncertainties which may depend on both input and state of the system as well as on a persistent term. This considered class of uncertainty allows us to represent model mismatches and persistent disturbances, which may reduce the conservativeness in the uncertainties modelling (Raković, Kerrigan, & Mayne, 2004). Input to state practical stability (ISpS) (Sontag & Wang, 1996) provides an appropriate framework to analyze the stability of this considered class of closed loop systems. Sufficient conditions to ensure ISpS are obtained by means of a Lyapunov-like function. Based on this result, the main contribution of this paper is presented: sufficient conditions for robust stability (ISpS) of min–max MPC for systems with bounded uncertainties.

The paper is organized as follows: in Section 2, the system to be controlled is presented. In Section 3 sufficient conditions for ISpS are given. The closed-loop min–max MPC is presented in Section 4 together with the proposed sufficient conditions of robust stability of min–max MPC under bounded uncertainties, which is the main contribution of the paper. The paper finishes with some conclusions.

Notation. $\|x\|$ denotes a norm of a given vector x; for a given sequence of N bounded signals $\mathbf{w}=\{w_0, w_1, \ldots, w_{N-1}\}$, we denote the norm $\|\mathbf{w}\|_\infty = sup\{\|w_k\|, k=0, 1, \ldots, N-1\}$, where the cardinality of the sequence is inferred from the context. A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{K}$-function if $\alpha(0)=0$, $\alpha(s)>0$ for all $s>0$ and it is strictly increasing. A function $\alpha(\cdot)$ is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and $\alpha(s)\to\infty$ when $s\to\infty$. A continuous function $\beta : \mathbb{R}_+ \times Z_+ \to \mathbb{R}_+$ is a $\mathcal{KL}$-function if $\beta(s,t)$ is a $\mathcal{K}$-function in s for every $t \geqslant 0$, it is strictly decreasing in t for every $s>0$ and $\beta(s,t)\to 0$ when $t\to\infty$. For a given pair of functions $\theta_1(\cdot)$ and $\theta_2(\cdot)$, $\theta_1 \circ \theta_2(s)$ denotes the function $\theta_1(\theta_2(s))$ and $\theta_1^k(s)=\theta_1 \circ \theta_1^{k-1}(s)$, with $\theta_1^0(s)=s$ whereas $[\theta_1(s)]^k$ denotes $\theta_1(s)$ to the power of k.

2. System description

Consider a system described by an uncertain nonlinear time-invariant discrete time model

$$x^+ = f(x, u, w), \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector, the disturbance input $w \in \mathbb{R}^q$ models the uncertainty and x^+ is the successor state (Mayne et al., 2000). The system is subject to constraints on both the state and the control action given by

$$u \in U \quad \text{and} \quad x \in X, \tag{2}$$

where X and U are compact sets containing the origin.

It is assumed that the uncertainty is contained in set $W(x,u)$ that may depend on the state and input of the system and it is compact for every (x,u). However, for the design of robust controllers, the uncertainties are modelled as confined in a given compact set

$$W_m(x,u) = \{w \in \mathbb{R}^q : \|w\| \leqslant \gamma(\|(x,u)\|) + \rho(\mu)\}, \tag{3}$$

where $\gamma(\cdot)$ and $\rho(\cdot)$ are $\mathcal{K}$-functions and for all $x \in X$ and $u \in U$, $W(x,u) \subseteq W_m(x,u)$.

Constant $\mu \geqslant 0$ describes the fact that $W(x,u)$ at $x=0$ and $u=0$ may be not zero, which usually happens for instance when persistent disturbances are present. Therefore, this description is suitable for modelling both plant uncertainties and persistent external disturbances. For notational convenience, the dependence on (x,u) of set $W_m(x,u)$ and $W(x,u)$ may be omitted, denoting them as W_m and W, respectively.

Effective control in the presence of uncertainties requires a feedback structure. So, a sequence of control laws $\pi(x)$ to be applied to the system at current state x must be considered. This control policy for a prediction horizon N is given by $\pi(x) = \{\pi_0(x), \pi_1(\cdot), \ldots, \pi_{N-1}(\cdot)\}$, where $\pi_i(x) : \mathbb{R}^n \to \mathbb{R}^m$. Note that, for a given state x, the first term is a control action, so it may be denoted as $u(0)$. The solution to (1) at time j when the initial state is x at time 0, the uncertainty realization is $\mathbf{w}=\{w_0, \ldots, w_{j-1}\}$ and the control policy π is applied will be denoted as $x(j)=\phi(j; x, \pi, \mathbf{w})$. Thus, sequence $\mathbf{w}$ is said to be possible if $w_i \in W_m(x(i), \pi_i(x(i)))$. This will be shortly denoted as $\mathbf{w} \in W_m^j$. In the sequel, x_k and u_k will denote the state and the control action applied to the system at sampling time k.

In what follows, some well established definitions and results on invariant sets used in the paper are recalled (Blanchini, 1999).

Definition 1. Consider the uncertain system $x^+ = \mathscr{F}(x,w)$, where $w \in \mathscr{W}_m \subset \mathbb{R}^q$ models the uncertainty. Then the set $\Omega \subset \mathbb{R}^n$ is a robust positively invariant set if $\mathscr{F}(x,w) \in \Omega$, for all $x \in \Omega$ and for all $w \in \mathscr{W}_m$.

Definition 2. A set $\Omega \subset \mathbb{R}^n$ is a robust control invariant set for the system (1) subject to constraint (2) if for all $x \in \Omega$, there exists an admissible input $u=u(x) \in U$ such that $f(x,u,w) \in \Omega$ for all $w \in W_m$.

Definition 3. Let $\Omega \subset \mathbb{R}^n$ be a robust positively (or control) invariant set for system (1) subject to constraints (2), then the i-step robust stabilizable set $X_i(\Omega)$ is the set of admissible states which can be steered to the target set Ω in i steps or less by a sequence of admissible control laws $\pi(x)$ for all $\mathbf{w} \in W_m^i$.

This set satisfies that $X_i(\Omega) \supseteq X_{i-1}(\Omega)$ and hence $X_i(\Omega)$ is a robust control invariant set, for $i \geqslant 0$.

3. A sufficient condition for ISpS

In this section a suitable framework to analyze stability of system (1) controlled by a min–max MPC is presented. Since the calculation of this control law is based on the worst possible case of the modelled uncertainties, the obtained control law implicitly depends on the considered model of the uncertainties. To emphasize this point, the closed-loop system is expressed by

$$x^+ = \mathscr{F}_\mu(x, w), \tag{4}$$

where subscript μ denotes this dependence.

In this case, the uncertainties of the closed loop system are such that $w \in \mathscr{W}(x)$, while the modelled uncertainties are

$$w \in \mathscr{W}_m(x) = \{w \in \mathbb{R}^q : \|w\| \leqslant \gamma(\|x\|) + \rho(\mu)\}, \tag{5}$$

where $\gamma(\cdot)$ and $\rho(\cdot)$ are $\mathscr{K}$-functions and $\mathscr{W}(x) \subseteq \mathscr{W}_m(x)$ for all $x \in X$.

In order to analyze the stability of this class of systems, ISpS provides an adequate framework. This concept is stated by means of comparison functions, in the following definition.

Definition 4 (*Sontag & Wang, 1996*). The system (4) is input-to-state practically stable if there is a $\mathscr{KL}$-function $\beta(\cdot, \cdot)$ and a pair of $\mathscr{K}$-functions $\delta_1(\cdot)$, $\delta_2(\cdot)$ such that

$$\|x_k\| \leqslant \beta(\|x_0\|, k) + \delta_1(\|\mathbf{w}\|_\infty) + \delta_2(\mu),$$

where $x_j = \mathscr{F}_\mu(x_{j-1}, w_{j-1})$, for all $w_{j-1} \in \mathscr{W}(x_{j-1})$, for all $j = 1, \ldots, k$ and $\mathbf{w} = \{w_0, \ldots, w_{k-1}\}$.

Notice that $\delta_1(\|\mathbf{w}\|_\infty)$ depends on the actual realization of the uncertainties whereas $\delta_2(\mu)$ depends on the modelled uncertainty $\mathscr{W}_m(x)$.

The definition of ISpS is closely related to the notion of Input to state stability (ISS) (Jiang & Wang, 2001). Indeed, ISS implies ISpS, since taking $\mu = 0$, the given definition corresponds to the standard ISS notion; however the converse is not true, since an ISS system satisfies that 0-input, i.e. $w_k = 0$, for all $k \geqslant 0$ (that means no model mismatches), implies asymptotic stability to the origin, while for an ISpS system, 0-input implies asymptotic stability to a compact set containing the origin. In (Sontag & Wang, 1996), it has been proved that ISpS is equivalent to ISS extended to point-to-set distance from the state to a proper compact invariant set for the system; however Definition 4 is used in this paper because it is more adequate for the analysis of min–max MPC.

In the ISS framework, when the uncertainty is state dependent, the notion of stability margin is introduced to establish an upper bound of the uncertainties that provides asymptotic stability to the origin. In the case of ISpS for systems with state dependent plus a persistent term uncertainties, this notion cannot be used since the system is not asymptotically stable even when the uncertainty is null; however this can be extended considering an upper bound on the state-dependent part of the uncertainty that ensures asymptotic stability when no persistent term is present. This is rigourously stated in the following definition:

Definition 5. Consider system (4) where uncertainties are modelled by set $\mathscr{W}_m(x)$ given by (5). We say that $\gamma(\cdot)$ is a stability margin of system (4) if there exists a $\mathscr{KL}$-function $\beta(\cdot, \cdot)$ and a $\mathscr{K}$-function $\delta(\cdot)$ such that

$$\|x_k\| \leqslant \beta(\|x_0\|, k) + \delta(\mu),$$

where $x_j = \mathscr{F}_\mu(x_{j-1}, w_{j-1})$ and $w_{j-1} \in \mathscr{W}_m(x_{j-1})$ $j = 1, \ldots, k$.

In the following definition, an ISpS-Lyapunov function, which provides a sufficient condition for Input-to-State practical Stability, is presented.

Definition 6. Consider system (4) and suppose that the uncertainty vector is bounded by $w \in \mathscr{W}(x)$. A function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$ is called an ISpS-Lyapunov function if there are some $\mathscr{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ and $\sigma(\cdot)$, and some $\mathscr{K}$-functions $\rho_1(\cdot)$ and $\rho_2(\cdot)$ such that

$$\alpha_1(\|x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|) + \sigma(\mu)$$

$$V(\mathscr{F}_\mu(x, w)) - V(x) \leqslant -\alpha_3(\|x\|) + \rho_1(\|w\|) + \rho_2(\mu). \tag{6}$$

Note that the ISpS-Lyapunov function may not be bounded above by a $\mathscr{K}_\infty$-function of the state. This allows us to consider candidate Lyapunov functions such that the value of $V(x)$ at the origin $x=0$ depends on the modelled bound of the uncertainties.

Based on the previous definition, a sufficient condition for ISpS is presented in the following theorem.

Theorem 1. *If system* (4) *admits an ISpS-Lyapunov function* (6), *then it is input-to-state practically stable.*

Proof. Consider the $\mathscr{K}_\infty$-function $\bar{\alpha}_2(s) = \alpha_2(s) + \sigma(s)$. With this choice of $\bar{\alpha}_2(\cdot)$ it results that[1] $\alpha_2(\|x\|) + \sigma(\mu) \leqslant \bar{\alpha}_2(\|x\| + \mu)$. Therefore, $V(x) \leqslant \bar{\alpha}_2(\|x\| + \mu)$ and hence, $\|x\| + \mu \geqslant \bar{\alpha}_2^{-1}(V(x))$.

Let $\varepsilon(\cdot)$ be a given $\mathscr{K}_\infty$-function, and consider the $\mathscr{K}_\infty$-function given by $\underline{\alpha}_3(s) = \min(\alpha_3(s/2), \varepsilon(s/2))$ then

$$\alpha_3(\|x\|) + \varepsilon(\mu) \geqslant \underline{\alpha}_3(\|x\| + \mu) \geqslant \underline{\alpha}_3 \circ \bar{\alpha}_2^{-1}(V(x)),$$

where $\alpha_4(s) = \underline{\alpha}_3 \circ \bar{\alpha}_2^{-1}(s)$ is a $\mathscr{K}_\infty$-function. Then, from this

$$\begin{aligned} V(\mathscr{F}_\mu(x, w)) &\leqslant V(x) - \alpha_3(\|x\|) + \rho_1(\|w\|) + \rho_2(\mu) \\ &\leqslant V(x) - \alpha_4(V(x)) + \rho_1(\|w\|) + \hat{\rho}_2(\mu) \end{aligned}$$

with $\hat{\rho}_2(\mu) = \varepsilon(\mu) + \rho_2(\mu)$. In virtue of Property 1, there exists a $\mathscr{K}_\infty$-function $\alpha_5(s)$ such that $\alpha_5(s) \leqslant \alpha_4(s)$ for all $s \geqslant 0$ and $\psi(s) = s - \alpha_5(s)$ is a $\mathscr{K}$-function and then

$$V(\mathscr{F}_\mu(x, w)) \leqslant \psi(V(x)) + \rho_1(\|w\|) + \hat{\rho}_2(\mu). \tag{7}$$

Consider the $\mathscr{K}$-function given by $\phi(s) = s - 1/2 \cdot \alpha_5(s)$, and consider that the initial state of the system is x_0, then we are going to prove that

$$V(x_{k+1}) \leqslant \phi^{k+1}(V(x_0)) + \theta(a), \tag{8}$$

[1] A collection of properties of the comparison functions used in the proof can be found in Property 1 in the appendix.

where a is defined as $a=\rho_1(\|\mathbf{w}\|_\infty)+\hat{\rho}_2(\mu)$, $\mathbf{w}=\{w_0,\ldots,w_k\}$ and $\theta(a)=\alpha_5^{-1}(2\cdot a)$. This is proved by induction: since $\theta(a)>2a$, we derive that

$$V(x_1)\leqslant\psi(V(x_0))+a\leqslant\phi(V(x_0))+\theta(a).$$

Assume that $V(x_k)\leqslant\phi^k(V(x_0))+\theta(a)$. Taking into account Lemma 2 and the property

$$\phi\circ\theta(a)+a=\theta(a)-0.5\alpha_5\circ\theta(a)+a=\theta(a)$$

it is concluded

$$\begin{aligned}V(x_{k+1})&\leqslant\psi(V(x_k))+a\leqslant\psi(\phi^k(V(x_0))+\theta(a))+a\\&\leqslant\phi(\phi^k(V(x_0)))+\phi(\theta(a))+a\\&=\phi^{k+1}(V(x_0))+\theta(a).\end{aligned}$$

Properties of the $\mathscr{K}$-functions and Lemma 3 yields

$$\begin{aligned}\alpha_1(\|x_k\|)&\leqslant\phi^k(\alpha_2(\|x_0\|)+\sigma(\mu))+\theta(a)\\&\leqslant\phi^k(2\cdot\alpha_2(\|x_0\|))+\phi^k(2\cdot\sigma(\mu))+\theta(a)\\&\leqslant\phi^k(2\cdot\alpha_2(\|x_0\|))+2\cdot\sigma(\mu)+\theta(a).\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_k\|&\leqslant\alpha_1^{-1}(\phi^k(2\cdot\alpha_2(\|x_0\|))+2\cdot\sigma(\mu)\\&\quad+\theta(\rho_1(\|\mathbf{w}\|_\infty)+\hat{\rho}_2(\mu)))\\&\leqslant\alpha_1^{-1}(\phi^k(2\cdot\alpha_2(\|x_0\|))+2\cdot\sigma(\mu)\\&\quad+\theta(2\cdot\rho_1(\|\mathbf{w}\|_\infty))+\theta(2\cdot\hat{\rho}_2(\mu)))\\&\leqslant\alpha_1^{-1}(2\cdot\phi^k(2\cdot\alpha_2(\|x_0\|)))\\&\quad+\alpha_1^{-1}(4\cdot\theta(2\cdot\rho_1(\|\mathbf{w}\|_\infty)))\\&\quad+\alpha_1^{-1}(8\cdot\sigma(\mu)+4\cdot\theta(2\cdot\hat{\rho}_2(\mu)))\\&=\beta(\|x_0\|,k)+\varphi_1(\|\mathbf{w}\|_\infty)+\varphi_2(\mu).\end{aligned}\tag{9}$$

From the properties of the $\mathscr{K}$-functions, it is easy to see that $\beta(\cdot,\cdot)$ is a $\mathscr{KL}$-function and $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$ are $\mathscr{K}$-functions. □

Corollary 1. *Assume that system* (4) *admits an ISpS-Lyapunov function* (6), *then*

(i) *If any of the functions $\sigma(\cdot)$, $\rho_1(\cdot)$ or $\rho_2(\cdot)$ is zero, then the system is ISpS.*
(ii) *If there exists a $\mathscr{K}_\infty$-function $\tilde{\alpha}_3(\cdot)$ and a $\mathscr{K}$-function $\tilde{\rho}_2(\cdot)$ such that*

$$V(\mathscr{F}_\mu(x,w))-V(x)\leqslant-\tilde{\alpha}_3(\|x\|)+\tilde{\rho}_2(\mu)$$

for every $w\in\mathscr{W}_m(x)$ given in (5), *then $\gamma(\cdot)$ is a stability margin for the system.*

Proof. (i) It is immediate from the proof of Theorem 1.

(ii) In this case, $V(x)$ can be posed as an ISpS-Lyapunov function with $\tilde{\rho}_1(\cdot)$ equals to zero. From (9) in the proof of Theorem 1, we infer that $\varphi_1(s)=\alpha_1^{-1}(4\cdot\theta(2\cdot\tilde{\rho}_1(s)))$ is also zero, and hence $\|x_k\|\leqslant\beta(\|x_0\|,k)+\varphi_2(\mu)$, proving the claim. □

Remark 1. Notice that the obtained results do not rely on the continuity of system (4) or continuity of ISpS-Lyapunov function (6); therefore these can be used for discontinuous systems such as system (1) controlled by a discontinuous feedback controller.

Based on these results, we derive sufficient conditions to guarantee ISpS of a system with bounded uncertainties controlled by a closed-loop min–max MPC controller.

4. Stability analysis of min–max MPC

MPC control law is obtained by minimizing a cost function of the predicted evolution of the system. In the case of min–max MPC, the cost associated to the future evolution of the system depends on the control policy and the future realization of the uncertainties

$$J_N(x,\pi,\mathbf{w})=\sum_{i=0}^{N-1}\ell(x(i),\pi_i(x(i)))+F(x(N)),$$

where $x(i)=\phi(i;x,\pi,\mathbf{w})$, $\pi=\{\pi_i(\cdot)\}$, and the stage cost $\ell(\cdot,\cdot)$ is a positive definite function. The control policy is derived from the solution of the following optimization problem $P_N(x)$

$$J_N^*(x)=\min_{\pi}\max_{\mathbf{w}\in W_m}J_N(x,\pi,\mathbf{w})$$

s.t.

$$\pi_i(x(i))\in U,x(i)\in X,\quad i=0,\ldots,N-1,\ \ \forall\mathbf{w}\in W_m^N$$

$$x(N)\in\Omega,\ \forall\mathbf{w}\in W_m^N.$$

Due to the receding horizon policy, the min–max MPC controller is given by $K_N(x)=\pi_0(x)$. This problem is feasible in the region of initial states that can be robustly steered to the terminal set Ω in N steps. The proposed min–max MPC controller satisfies the following assumptions:

Assumption 1. *Consider system* (1) *and assume that the uncertainties vector w is contained in the compact set $W(x,u)$. Assume that the modelled set of uncertainties $W_m(x,u)$ given by* (3) *is used to design the local control law $u=h(x)$. Assume that Ω is an admissible robust invariant set for the system controlled by the control law $u=h(x)$ such that the origin is in its interior. Consider that $F(x)$ is an associated ISpS-Lyapunov function such that for all $x\in\Omega$ and for all $w\in W_m(x,h(x))$ we have that*

$$\alpha_1(\|x\|)\leqslant F(x)\leqslant\alpha_2(\|x\|)+\sigma(\mu)$$

$$F(f(x,h(x),w))-F(x)\leqslant-\ell(x,h(x))+\varrho(\mu),$$

where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\sigma(\cdot)$ and $\varrho(\cdot)$ are $\mathscr{K}_\infty$-functions and the stage cost satisfies $\ell(x,u)\geqslant\alpha_3(\|(x,u)\|)$, being $\alpha_3(\cdot)$ a $\mathscr{K}_\infty$-function.

Notice that this assumption states that the control law $u=h(x)$ is designed in such a way that $\gamma(\cdot)$ is a stability margin of the closed loop system in Ω (see Corollary 1). Suitable terminal cost and set that satisfy this assumption can be computed using the method proposed in (Magni et al., 2003) in the context of the H_∞ paradigm for a class of nonlinear systems.

Before presenting the main result, an upper bound of the optimal cost is obtained in the following lemma.

Lemma 1. *Consider system* (1) *and suppose that the uncertainty vector w is modelled by* (3). *Let Ω and $F(x)$ satisfy Assumption* 1, *then there exists a couple of $\mathscr{K}_\infty$-functions $\alpha_s^J(\cdot)$ and $\sigma^J(\cdot)$ such that*

$$J_N^*(x) \leqslant \alpha_2^J(\|x\|) + \sigma^J(\mu)$$

for all $x \in X_N(\Omega)$ and for all $w \in W_m(x, K_N(x))$.

Proof. The compactness of X and U implies that the predicted evolution of the system, $x(k)$, and the feasible control action, $u(k)$, are bounded. This fact and Assumption 1 guarantee that the optimal cost is upper bounded, that is, there exists a finite real number $\bar{J}_N$ such that $J_N^*(x) \leqslant \bar{J}_N$ for all $x \in X_N(\Omega)$.[2]

Let $B_r \subset \mathbb{R}^n$ be a ball $B_r = \{x \in \mathbb{R}^n : \|x\| \leqslant r\}$ such that $B_r \subseteq \Omega$. Note that this ball exists since the origin is in the interior of Ω. Let ε be a positive constant $\varepsilon = max(1, \bar{J}_N/\alpha_2(r))$. Consider the $\mathscr{K}_\infty$-functions given by $\alpha_2^J(s) = \varepsilon \cdot \alpha_2(s)$ and $\sigma^J(s) = \sigma(s) + N \cdot \varrho(s)$. Two cases must be taken into account:

- $x \in \Omega$: from Assumption 1 we have that $F(x(k+1)) - F(x(k)) \leqslant -\ell(x(k), h(x(k))) + \varrho(\mu)$, where $x(k) = \phi(k; x, \pi_h, \mathbf{w})$ and $\pi_{h_i}(x) = h(x)$. Summing this inequality from $k = 0$ to $N - 1$ we have

  $$F(x) \geqslant \sum_{i=0}^{N-1} \ell(x(k), h(x(k))) + F(x(N)) - N \cdot \varrho(\mu).$$

  In virtue of Assumption 1, the control policy π_h is feasible. By optimality, it is derived that $F(x) \geqslant J_N^*(x) - N \cdot \varrho(\mu)$. Hence we have that

  $$J_N^*(x) \leqslant F(x) + N \cdot \varrho(\mu) \leqslant \alpha_2(\|x\|) + \sigma(\mu) + N \cdot \varrho(\mu) \leqslant \alpha_2^J(\|x\|) + \sigma^J(\mu).$$

- If $x \notin \Omega$, then $x \notin B_r$ and hence $\alpha_2(\|x\|) > \alpha_2(r)$. Hence

  $$J_N^*(x) \leqslant \bar{J}_N \leqslant \bar{J}_N \cdot \frac{\alpha_2(\|x\|)}{\alpha_2(r)} \leqslant \alpha_2^J(\|x\|) + \sigma^J(\mu). \qquad \square$$

In the following theorem, we present the main result of the paper.

Theorem 2. *Consider system* (1) *and suppose that the uncertainty vector w is contained in the compact set $W(x,u)$ and that the modelled bound of the uncertainties is $W_m(x,u)$ defined in* (3). *Let terminal set Ω and terminal cost $F(x)$ satisfy Assumption* 1 *and let $u = K_N(x)$ the obtained min–max MPC controller. Then*

(i) *The optimal cost is an ISpS-Lyapunov function.*
(ii) *The closed-loop system is ISpS for all initial state $x_0 \in X_N(\Omega)$ and for every uncertainty $w_k \in W(x_k, K_N(x_k))$.*
(iii) *Function $\gamma(\cdot)$ is a stability margin of the closed-loop system for all $x_0 \in X_N(\Omega)$.*

Proof. (i) Thanks to the invariance of the terminal set, the feasible region of the controller $X_N(\Omega)$ is a robust invariant set for the closed loop system and the controller is well defined all the time (Mayne, 2001).

In virtue of the previous lemmas we have that

$$\alpha_3(\|x\|) \leqslant \ell(x, K_N(x)) \leqslant J_N^*(x) \leqslant \alpha_2^J(\|x\|) + \sigma^J(\mu),$$

which is the first requirement to be an ISpS-Lyapunov function. The decreasing property of the optimal cost is proved in what follows by means of the dynamic programming approach to the min–max problem (in an analogous way to the proof presented in Mayne, 2001). Define the optimal cost in i-steps:

$$J_i^*(x) = \min_{u \in U} \left\{ \max_{w \in W_m} \{\ell(x,u) + J_{i-1}^*(f(x,u,w))\} \right. \\ \left. \text{such that } f(x,u,w) \in X_{i-1}(\Omega),\ \forall w \in W_m \right\},$$

where $J_0^*(x) = F(x)$ defined in $X_0(\Omega) = \Omega$. Define $u = K_i(x)$ as the argument of the optimal solution to $P_i(x)$. Denote $J_i^*(x) - J_{i-1}^*(x)$ as $\Delta J_i^*(x)$.

For all $x \in \Omega$, $u = h(x)$ is feasible and hence

$$\Delta J_1^*(x) \leqslant \max_{w \in W_m} \{\ell(x, h(x)) + F(f(x, h(x), w))\} - F(x) \leqslant \varrho(\mu).$$

Assume that $\Delta J_i^*(x) \leqslant \varrho(\mu)$ for all $x \in X_{i-1}(\Omega)$. Consider any $x \in X_i(\Omega)$, then the control action $u = K_i(x)$ is well defined and it is feasible for the optimization problem since $x_i^+ = f(x, K_i(x), w) \in X_{i-1}(\Omega) \subseteq X_i(\Omega)$. Then it follows that

$$J_{i+1}^*(x) \leqslant \max_{w \in W_m} \{\ell(x, K_i(x)) + J_i^*(x_i^+)\}$$

and we have that

$$\begin{aligned} \Delta J_{i+1}^*(x) &\leqslant \max_{w \in W_m} \{\ell(x, K_i(x)) + J_i^*(x_i^+)\} \\ &\quad - \max_{w \in W_m} \{\ell(x, K_i(x)) + J_{i-1}^*(x_i^+)\} \\ &\leqslant \max_{w \in W_m} \{[\ell(x, K_i(x)) + J_i^*(x_i^+)] \\ &\quad - [\ell(x, K_i(x)) + J_{i-1}^*(x_i^+)]\} \\ &= \max_{w \in W_m} \{\Delta J_i^*(x_i^+)\} \leqslant \varrho(\mu). \end{aligned}$$

Hence, by induction it is inferred that $J_{i+1}^*(x) - J_i^*(x) \leqslant \varrho(\mu)$ for all $i \geqslant 0$ and $x \in X_i(\Omega)$.

[2] In virtue of Theorem 2, the compactness assumption on X and U can be relaxed assuming a bounded set of initial states.

Consider that the state of the system is x_k and that the min–max MPC control law $u_k = K_N(x_k)$ is applied, then the system evolves to $x_{k+1} = f(x_k, K_N(x_k), w_k)$. Since $x_k \in X_N(\Omega)$ and $w_k \in W(x_k, K_N(x_k)) \subseteq W_m(x_k, K_N(x_k))$, it is clear that $x_{k+1} \in X_N(\Omega)$.

Based on the monotonicity result, it follows that

$$\begin{aligned} J_N^*(x_{k+1}) - J_N^*(x_k) &= J_N^*(x_{k+1}) - \max_{w \in W_m} \{\ell(x_k, K_N(x_k)) \\ &\quad + J_{N-1}^*(f(x_k, K_N(x_k), w))\} \\ &\leqslant J_N^*(x_{k+1}) - \ell(x_k, K_N(x_k)) \\ &\quad - J_{N-1}^*(f(x_k, K_N(x_k), w_k)) \\ &= \Delta J_N^*(x_{k+1}) - \ell(x_k, K_N(x_k)) \\ &\leqslant -\ell(x_k, K_N(x_k)) + \varrho(\mu). \end{aligned} \tag{10}$$

(ii) Since the optimal cost is an ISpS-Lyapunov function, in virtue of Theorem 1 the closed loop system is ISpS.

(iii) It is inferred from (10) and Corollary 1. □

Remark 2. Notice that the obtained sufficient conditions generalize the ones presented in (Mayne, 2001). In fact, if we consider the uncertainty model analyzed in (Mayne, 2001), that is $w \in W(x)$ with $w \to 0$ when $x \to 0$, this model is included in (3) making $\mu = 0$. In this case, the conditions proposed in this paper coincides with the ones presented in (Mayne, 2001).

Remark 3. An interesting consequence of this result is that the robustness of the design of the terminal controller is translated to the MPC; that is, if the local controller $u = h(x)$ is robust enough to ensure Assumption 1 and to guarantee that the bound of the uncertainty is a stability margin locally in Ω, then the min–max MPC controller $K_N(x)$ inherits these properties extended to $X_N(\Omega)$. Moreover, the effect of the persistent part of the uncertainties on the decreasing of the terminal cost remains in the optimal cost.

5. Conclusions

Sufficient conditions for stability of uncertain systems controlled by min–max MPC for a general class of bounded uncertainties are presented in this paper. Since this controller is a worst-case based controller, it is shown that the ISpS notion is suitable for this aim; an ISpS-Lyapunov function which provides sufficient condition for ISpS is presented. Based on this result, it is proved that if the terminal region is a robust invariant set and the terminal cost is chosen as an ISpS-Lyapunov function such that $\gamma(\cdot)$ is a stability margin, then the system controlled by the derived min–max MPC controller is ISpS. Moreover the stability margin holds for the obtained predictive controller.

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Appendix A. Some properties of comparison functions

In what follows, a collection of some well-known properties of comparison functions used in this paper are presented (see for instance, Khalil, 1996; Jiang & Wang, 2001).

Property 1. *Let* $\theta_1 : [0, a_1] \to \mathbb{R}_+$ *and* $\theta_2 : [0, a_2] \to \mathbb{R}_+$ *be* $\mathcal{K}$*-functions, let* $\theta_3(\cdot)$ *and* $\theta_4(\cdot)$ *be* $\mathcal{K}_\infty$*-functions and let* $\beta(\cdot,\cdot)$ *be a* $\mathcal{KL}$-function, *then*:

(1) $\theta_1^{-1}(\cdot)$ *is a* $\mathcal{K}$*-function defined in* $[0, \theta_1(a_1)]$.
(2) $\theta_1 \circ \theta_2(\cdot)$ *is a* $\mathcal{K}$*-function defined in* $[0, b]$, *with* $b = \min(a_2, \theta_2^{-1}(a_1))$.
(3) $\theta_1 \circ \beta(\cdot)$ *is a* $\mathcal{KL}$*-function.*
(4) $\max(\theta_1(s), \theta_2(s))$ *is a* $\mathcal{K}$*-function defined in* $[0, b]$ *with* $b = \min(a_1, a_2)$.
(5) $\min(\theta_1(s), \theta_2(s))$ *is a* $\mathcal{K}$*-function defined in* $[0, b]$ *with* $b = \min(a_1, a_2)$.
(6) $\theta_1(s_1 + s_2) \leqslant \theta_1(2 \cdot s_1) + \theta_1(2 \cdot s_2)$ *for all* $s_1, s_2 \in [0, a_1/2]$
(7) $\theta_1(s_1) + \theta_2(s_2) \leqslant \theta_5(s_1 + s_2)$, *where* $\theta_5(s) = \theta_1(s) + \theta_2(s)$, *for all* $s_1 + s_2 \leqslant \min(a_1, a_2)$.
(8) $\theta_1(s_1) + \theta_2(s_2) \geqslant \theta_6(s_1 + s_2)$, *where* $\theta_6(s) = \min(\theta_1(s/2), \theta_2(s/2))$, *for all* $s_1 \in [0, a_1]$ *and* $s_2 \in [0, a_2]$ *such that* $s_1 + s_2 \leqslant 2 \cdot \min(a_1, a_2)$.
(9) *There exists a* $\mathcal{K}_\infty$*-function* $\theta_7(s)$ *such that* $\theta_7(s) \leqslant \theta_3(s)$ *for all* $s \geqslant 0$ *and* $\theta_8(s) = s - \theta_7(s)$ *is a* $\mathcal{K}$*-function.*

Notice that $\mathcal{K}_\infty$-functions are a class of $\mathcal{K}$-functions; hence all the properties of $\mathcal{K}$-functions can be extended to $\mathcal{K}_\infty$-functions.

Lemma 2. *Consider a* $\mathcal{K}$*-function* $\psi(s) = s - \theta(s)$ *where* $\theta(\cdot)$ *is a* $\mathcal{K}_\infty$*-function. Consider the* $\mathcal{K}$*-function given by* $\phi(s) = s - 1/2 \cdot \theta(s)$, *then* $\psi(s_1 + s_2) \leqslant \phi(s_1) + \phi(s_2)$.

Proof. First, we have that $\theta(s_1 + s_2) = 1/2 \cdot \theta(s_1 + s_2) + 1/2 \cdot \theta(s_1 + s_2) \geqslant 1/2 \cdot \theta(s_1) + 1/2 \cdot \theta(s_2)$. Based on this result, we derive that $\psi(s_1 + s_2) = s_1 + s_2 - \theta(s_1 + s_2) \leqslant s_1 + s_2 - 1/2 \cdot \theta(s_1) - 1/2 \cdot \theta(s_2) = \phi(s_1) + \phi(s_2)$. □

Lemma 3. *Let* $\phi(\cdot)$ *be a* $\mathcal{K}$*-function such that* $\phi(s) < s$ *for all* $s > 0$, *then the function* $\delta(s, k) = \phi^k(s)$ *is a* $\mathcal{KL}$*-function.*

Proof. It is immediate that $\phi^k(s)$ is a $\mathcal{K}$-function in s. The fact that $\phi^k(s)$ is decreasing in k for all $s > 0$ is proved by induction: by assumption, $\phi^1(s) = \phi(s) < s = \phi^0(s)$. Assume that $\phi^i(s) < \phi^{i-1}(s)$, then $\phi^{i+1}(s) = \phi \circ \phi^i(s) < \phi \circ \phi^{i-1}(s) = \phi^i(s)$.

To finish it suffices to prove that $\phi^k(s) \to 0$ when $k \to \infty$. To this aim, define $\lambda(s) = \max\{\phi(z)/z, \ \forall z \in (0, s]\}$. It is clear that $\lambda(s) \in (0, 1)$ for all s and for every $s_1 \leqslant s_2$, $\lambda(s_1) \leqslant \lambda(s_2)$. It is going to be proved by induction that $\phi^k(s) \leqslant [\lambda(s)]^k \cdot s$. For $k = 0$ is immediate; assume that $\phi^k(s) \leqslant [\lambda(s)]^k \cdot s$, then

$\phi^{k+1}(s)=\phi(\phi^k(s))\leqslant\phi([\lambda(s)]^k\cdot s)$. Given that $[\lambda(s)]^k\cdot s<s$ we have that $\lambda([\lambda(s)]^k\cdot s)\leqslant\lambda(s)$ and hence $\phi^{k+1}(s)\leqslant\lambda([\lambda(s)]^k\cdot s)\cdot[\lambda(s)]^k\cdot s\leqslant\lambda(s)\cdot[\lambda(s)]^k\cdot s=[\lambda(s)]^{k+1}\cdot s$. Since for every s $\lambda(s)\in(0,1)$, we derive that $[\lambda(s)]^k\cdot s\to 0$ when $k\to\infty$ and hence $\phi^k(s)\to 0$ when $k\to\infty$,which completes the proof. □

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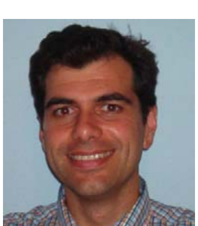

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