[12] K. S. Narendra and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting A-Matrices," *IEEE Trans. Autom. Control*, vol. 39, no. 12, pp. 2469–2471, Dec. 1994.
[13] P. Peleties and R. DeCarlo, "Asymptotic stability of m-switched systems using Lyapunov-like functions," in *Proc. Amer. Control Conf.*, Boston, MA, 1991, pp. 1679–1684.
[14] S. Sastry and M. Bosdon, *Adaptive Control: Stability, Convergence and Robustness*. Englewood Cliffs, NJ: Prentice-Hall, 1989.
[15] G. Tao, *Adaptive Control Design and Analysis*. New York: Wiley, 2003.
[16] K. S. Tsakalis, "Model reference adaptive control of linear time-varying plants: The case of 'jump' parameter variations," *Int. J. Control*, vol. 56, no. 6, pp. 1299–1345, 1992.

Iterative Distributed Model Predictive Control of Nonlinear Systems: Handling Asynchronous, Delayed Measurements

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***Abstract*—In this work, we focus on iterative distributed model predictive control (DMPC) of large-scale nonlinear systems subject to asynchronous, delayed state feedback. The motivation for studying this control problem is the presence of asynchronous, delayed measurement samplings in chemical processes and the potential use of networked sensors and actuators in industrial process control applications to improve closed-loop performance. Under the assumption that there exist upper bounds on the time interval between two successive state measurements and on the maximum measurement delay, we design an iterative DMPC scheme for nonlinear systems via Lyapunov-based control techniques. Sufficient conditions under which the proposed distributed MPC design guarantees that the state of the closed-loop system is ultimately bounded in a region that contains the origin are provided. The theoretical results are illustrated through a catalytic alkylation of benzene process example.**

***Index Terms*—Asynchronous measurements, distributed model predictive control (DMPC), measurement delays, nonlinear systems, process control.**

I. Introduction

Model predictive control (MPC) is a popular control strategy for the design of high performance process control systems and is typically studied within the centralized control paradigm in which all the manipulated inputs are optimized in a single optimization problem [1]. While the centralized paradigm to MPC has been successful, in recent years, there is a trend for the development of decentralized and distributed MPC due to the significantly increased computational complexity, organization and maintenance difficulties as well as reduced fault tolerance of centralized MPC (e.g.,[2], [3]).

In the literature, several approaches for the design of decentralized and distributed MPC have been reported; please see [4], [2], [3] for reviews of results in this area. Specifically, in [5], a distributed MPC (DMPC) scheme for coupled nonlinear systems subject to decoupled constraints was designed. In [6], a robust DMPC design was developed for linear systems with coupling between subsystems modeled as bounded disturbances. In [7], a decentralized MPC was proposed for nonlinear systems with no information exchange between the local controllers and the stability of the decentralized control system was ensured by a set of contractive constraints. In [8], a cooperative DMPC scheme was developed for linear systems with guaranteed stability of the closed-loop system and convergence of the cost to its optimal value, and in [9], a game theory based DMPC scheme for constrained linear systems was proposed. In our previous work [10], [11], a sequential DMPC architecture and an iterative DMPC architecture were designed for nonlinear systems via Lyapunov-based control techniques. Specifically, in the sequential DMPC architecture, the distributed controllers communicate via one-directional communication, are evaluated in sequence and once in each sampling time; and in the iterative DMPC architecture, the distributed controllers communicate via bi-directional communication, are evaluated in parallel and iterate to achieve convergence in each sampling time. However, all of the above results are based on the assumption of continuous sampling of the entire plant state vector and assuming no delays and perfect communication between the sensors/actuators and the controllers.

In many chemical process applications, the assumption of continuous, undelayed process state sampling and perfect communication between the sensors/actuators and the controllers may not hold because of measuring difficulties of some process states (e.g., species concentrations) and communication network malfunctions introducing data losses and time-varying delays [12]. Previous work on MPC design for systems subject to delayed feedback has primarily focused on centralized MPC designs [13]–[15] and little attention has been given to the design of DMPC for systems subject to delayed measurements. In [16], the issue of delays in the communication between distributed controllers was addressed. In our previous work [17], we developed sequential DMPC schemes for nonlinear systems subject to asynchronous and delayed state feedback. The approach used in [17] can be extended to handle asynchronous measurements in an iterative DMPC, however, it can not be used to handle measurement delays in iterative DMPC.

Motivated by the above considerations, in this work, we focus on iterative DMPC of large-scale nonlinear systems subject to asynchronous, delayed state feedback. Under the assumption that there exist upper bounds on the time interval between two successive state measurements and on the maximum measurement delay, we design an iterative DMPC scheme for nonlinear systems via Lyapunov-based control techniques. Sufficient conditions under which the proposed distributed MPC design guarantees that the state of the closed-loop system is ultimately bounded in a region that contains the origin are provided. The theoretical results are illustrated through a catalytic alkylation of benzene process example.

II. Preliminaries

The operator $|\cdot|$ is used to denote Euclidean norm of a vector while $|\cdot|_Q$ refers to the weighted Euclidean norm, defined by $|x|_Q = x^T Qx$. A continuous function $\alpha : [0, a) \rightarrow [0, b)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and satisfies $\alpha(0) = 0$. The symbol Ω_r is used to denote the set $\Omega_r := \{x \in R^{n_x} : V(x) \leq r\}$ where V is a scalar positive definite, continuous differentiable function and $V(0) = 0$, and the operator '/' denotes set subtraction, that is, $A/B := \{x \in R^{n_x} : x \in A, x \notin B\}$. The symbol x^e denotes an estimate of x. The symbol

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$\mathrm{diag}(v)$ denotes a square diagonal matrix whose diagonal elements are the elements of vector v.

We consider nonlinear systems of the form

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t) + k(x(t))w(t) \quad (1)$$

where $x(t) \in R^{n_x}$ denotes the vector of state variables, $u_i(t) \in R^{m_{u_i}}$, $i = 1, \ldots, m$, are m sets of control (manipulated) inputs and $w(t) \in R^{n_w}$ denotes the vector of disturbance variables. The m sets of inputs are restricted to be in m nonempty convex sets $U_i \subseteq R^{m_{u_i}}$, $i = 1, \ldots, m$, which are defined as $U_i := \{u_i \in R^{m_{u_i}} : |u_i| \leq u_i^{\max}\}$, $i = 1, \ldots, m$ where $u_i^{\max}$, $i = 1, \ldots, m$, are the magnitudes of the input constraints. The disturbance vector is bounded, i.e., $w(t) \in W$ where $W := \{w \in R^{n_w} : |w| \leq \theta, \theta > 0\}$.

We assume that $f(x)$, $g_i(x)$, $i = 1, \ldots, m$, and $k(x)$ are locally Lipschitz vector functions and that the origin is an equilibrium point of the unforced nominal system (i.e., system of (1) with $u_i(t) = 0$, $i = 1, \ldots, m$, $w(t) = 0$ for all t) which implies that $f(0) = 0$.

We further assume that there exists an explicit control law $h(x) = [h_1(x)^T \; \ldots \; h_m(x)^T]^T$ with $u_i = h_i(x)$, $i = 1, \ldots, m$, which renders (under continuous state feedback) the origin of the nominal closed-loop system asymptotically stable while satisfying the input constraints for all x inside a given stability region; please see [19] for results on the explicit construction of $h(x)$. This assumption implies that there exist functions $\alpha_i(\cdot)$, $i = 1, 2, 3, 4$ of class $\mathcal{K}$ and a continuously differentiable Lyapunov function $V(x)$ for the nominal closed-loop system, that satisfy the following inequalities [18], [19]:

$$\begin{aligned} \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|),\ & \left|\frac{\partial V(x)}{\partial x}\right| \leq \alpha_4(|x|) \\ \frac{\partial V(x)}{\partial x}\left(f(x) + \sum_{i=1}^{m} g_i(x)h_i(x)\right) & \leq -\alpha_3(|x|) \\ h_i(x) \in U_i,\ & i = 1, \ldots, m \end{aligned} \quad (2)$$

with $i = 1, \ldots, m$ for all $x \in \Omega_\rho \subseteq R^{n_x}$ where Ω_ρ (typically taken to be a level set of $V(x)$) denotes the stability region of the closed-loop system under $h(x)$. We denote the region $\Omega_\rho \subseteq O$ as the stability region of the closed-loop system under $h(x)$. The construction of $V(x)$ can be carried out in a number of ways using systematic techniques like, for example, sum-of-squares methods. Because of the local Lipschitz property assumed for the vector fields $f(x)$, $g_i(x)$, $i = 1, \ldots, m$, and $k(x)$ and of the boundedness of the manipulated inputs u_i, $i = 1, \ldots, m$, and the disturbance w, there exists a positive constant M such that:

$$\left|f(x) + \sum_{i=1}^{m} g_i(x)u_i + k(x)w\right| \leq M \quad (3)$$

for all $x \in \Omega_\rho$. Moreover, if we take into account the continuous differentiable property of the Lyapunov function $V(x)$, there exist positive constants L_x, L_w and L_{u_i}, $i = 1, \ldots, m$ such that:

$$\begin{aligned} \left|\frac{\partial V}{\partial x} f(x) - \frac{\partial V}{\partial x} f(x')\right| \leq L_x \left|x - x'\right|, \ \left|\frac{\partial V}{\partial x} k(x)\right| \leq L_w \\ \left|\frac{\partial V}{\partial x} g_i(x) - \frac{\partial V}{\partial x} g_i(x')\right| \leq L_{u_i} |x - x'|, \ i = 1, \ldots, m \end{aligned} \quad (4)$$

for all $x, x' \in \Omega_\rho$, $u_i \in U_i$, $i = 1, \ldots, m$, and $w \in W$.

III. Iterative DMPC With Asynchronous, Delayed Measurements

In this section, we design an iterative DMPC scheme which takes into account asynchronous, delayed measurements explicitly and provides deterministic closed-loop stability properties. In the proposed design, we will design m Lyapunov-based MPC (LMPC) controllers to compute u_i, $i = 1, \ldots, m$, and refer to the LMPC computing the input trajectories of u_i as LMPC i. A schematic of the proposed iterative DMPC for systems subject to asynchronous, delayed measurements is shown in Fig. 1.

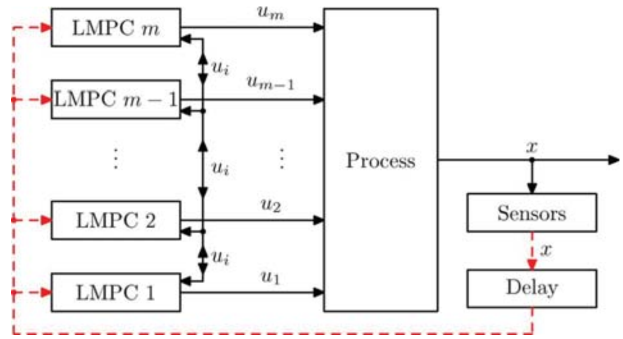

Fig. 1. Iterative DMPC with asynchronous, delayed measurements.

We assume that the full state of the system (1) is received by the controllers at asynchronous time instants t_k where $\{t_{k\geq 0}\}$ is a random increasing sequence of times and that there are delays in the measurements received by the controllers. In order to model the delays in measurements, an auxiliary variable d_k is introduced to indicate the delay corresponding to the measurement received at time t_k, that is, at time t_k, the measurement $x(t_k - d_k)$ is received. In order to study the stability properties in a deterministic framework, we assume that there exists an upper bound T_m on the interval between two successive measurements and the delays associated with the measurements are smaller than an upper bound D, which is, in general, related to measurement sensors and data transmission networks. We note that for chemical processes, the delay in the measurements received by a controller are mainly caused in the measurement sampling process. We also assume that the time instant in which a measurement is sampled is recorded and transmitted together with the measurement. This assumption is practical for many process control applications and implies that the delay in a measurement received by the controllers can be assumed to be known. Note that because the delays are time-varying, it is possible that at a time instant t_k, the controllers may receive a measurement $x(t_k - d_k)$ which does not provide new information (i.e., $t_k - d_k < t_{k-1} - d_{k-1}$); that is, the controller has already received a measurement of the state after time $t_k - d_k$. In this case, the controllers only use measurements that provide new information. Based on the above modeling of the measurements, we can calculate that the maximum amount of time the system might operate in open-loop following t_k is $D + T_m - d_k$ [17]. This upper bound will be used in the formulation of the iterative DMPC design below.

We propose to take advantage of the system model both to estimate the current system state from a delayed measurement and to control the system in open-loop when new information is not available. To this end, when a delayed measurement is received, the distributed controllers use the system model and the input trajectories that have been applied to the system to get an estimate of the current state and then based on the estimate, MPC optimization problems are solved to compute the optimal future input trajectory that will be applied until new measurements are received. The proposed implementation strategy for the iterative DMPC design is as follows:

1) When $x(t_k - d_k)$ is available at t_k, all the distributed controllers receive it and check whether it provides new information. If it does, go to step 2. Else, go to step 5.

2) The controllers estimate the current system state $x^e(t_k)$ and then evaluate their future input trajectories in an iterative fashion with initial input guesses generated by $h(\cdot)$.
3) At iteration c ($c \geq 1$):
 3.1. Each controller evaluates its future input trajectory based on $x^e(t_k)$ and the latest received input trajectories of all the other controllers (when $c = 1$, initial input guesses generated by $h(\cdot)$ are used).
 3.2. The controllers exchange their future input trajectories. Based on all the input trajectories, each controller calculates and stores the value of the cost function.
4) If a termination condition is satisfied, each controller sends its entire future input trajectory corresponding to the smallest value of the cost function to its actuators; if the termination condition is not satisfied, go to step 3 ($c \leftarrow c + 1$).
5) When a new measurement is received, go to step 1 ($k \leftarrow k + 1$).

In order to estimate the current system state $x^e(t_k)$ based on $x(t_k - d_k)$, the distributed controllers take advantage of the input trajectories that have been applied to the system from $t_k - d_k$ to t_k and the system model of (1) with $w(t) = 0$. Note that since the controllers exchange their input trajectories at the end of each iteration, they are able to determine the inputs the other controllers implement which correspond to the smallest cost value in each sampling time. Let us denote the input trajectories that have been applied to the system as $u_i^*(t)$, $i = 1, \ldots, m$. Therefore, $x^e(t_k)$ is evaluated by integrating the following differential equation:

$$\dot{x}^e(t) = f(x^e(t)) + \sum_{i=1}^{m} g_i(x^e(t)) u_i^*(t),\ \forall t \in [t_k - d_k, t_k) \tag{5}$$

with $x^e(t_k - d_k) = x(t_k - d_k)$.

In order to proceed, we define $x_n(\tau|t_k)$ for $\tau \in [0, N\Delta]$ as the nominal sampled trajectory of the system of (1) associated with the feedback control law $h(x)$ and sampling time Δ starting from $x^e(t_k)$. Note that N is the prediction horizon of the DMPC. This nominal sampled trajectory is obtained by integrating the following differential equation recursively:

$$\dot{x}_n(\tau|t_k) = f(x_n(\tau|t_k)) + \sum_{i=1}^{m} g_i(x_n(\tau|t_k)) h_i(x_n(l\Delta|t_k)),\quad \forall \tau \in [l\Delta, (l+1)\Delta) \tag{6}$$

where $l = 0, \ldots, N-1$. Note that in (6), the control laws h_i, $i = 1, \ldots, m$, are implemented in a sample-and-hold fashion. Based on $x_n(\tau|t_k)$, we define:

$$u_{n,j}(\tau|t_k) = h_j(x_n(l\Delta|t_k)),\ \forall \tau \in [l\Delta, (l+1)\Delta) \tag{7}$$

where $j = 1, \ldots, m$ and $l = 0, \ldots, N-1$. The sampled trajectory $x_n(\tau|t_k)$ and the input trajectory $u_{n,j}(\tau|t_k)$ will be used in the design of the LMPC to construct the stability constraint and used as the initial input guess for iteration 1 (i.e., $u_i^{*,0} = u_{n,i}$ for $i = 1, \ldots, m$). Specifically, the design of LMPC j, $j = 1, \ldots, m$, at iteration c is based on the following optimization problem:

$$\min_{u_j \in S(\Delta)} \int_0^{N\Delta} \left[|\tilde{x}(\tau)|_{Q_c} + \sum_{i=1}^{m} |u_i(\tau)|_{R_{ci}} \right] d\tau \tag{8a}$$

$$\text{s.t. } \dot{\tilde{x}}(\tau) = f(\tilde{x}(\tau)) + \sum_{i=1}^{m} g_i(\tilde{x}(\tau)) u_i(\tau),\ \tilde{x}(0) = x^e(t_k) \tag{8b}$$

$$u_i(\tau) = u_i^{*,c-1}(\tau|t_k),\ \forall i \neq j \tag{8c}$$

$$\left| u_j(\tau) - u_j^{*,c-1}(\tau|t_k) \right| \leq \Delta u_j,\ \forall \tau \in [0, N_{Dk}\Delta] \tag{8d}$$

$$u_j(\tau) \in U_j \tag{8e}$$

$$\begin{aligned} &\frac{\partial V(\tilde{x}(\tau))}{\partial \tilde{x}} \left(\frac{1}{m} f(\tilde{x}(\tau)) + g_j(\tilde{x}(\tau)) u_j(\tau) \right) \\ &\quad \leq \frac{\partial V(x_n(\tau|t_k))}{\partial x_n} \cdot \left(\frac{1}{m} f(x_n(\tau|t_k)) + g_j(x_n(\tau|t_k)) u_{n,j}(\tau|t_k) \right), \\ &\quad \forall \tau \in [0, N_{Dk}\Delta] \end{aligned} \tag{8f}$$

where $S(\Delta)$ is the family of piece-wise constant functions, Q_c and R_{ci}, $i = 1, \ldots, m$, are positive definite weighting matrices, $\tilde{x}$ is the predicted state trajectory of the nominal system, and N_{Dk} is the smallest integer satisfying $N_{Dk}\Delta \geq T_m + D - d_k$. The optimal solution to the optimization problem of (8) is denoted $u_j^{*,c}(\tau|t_k)$ for $\tau \in [0, N\Delta)$. Accordingly, we define the final optimal input trajectory of LMPC j as $u_j^*(\tau|t_k)$. Note that the value of N_{Dk} depends on d_k, so it may have different values at different time instants and has to be updated before solving the optimization problems. The constraint of (8d) imposes a limit on the input change in two consecutive iterations, i.e., for LMPC j, the magnitude of input change in two consecutive iterations is restricted to be smaller than a positive constant Δu_j. Given that $h_j(x)$ provides a feasible, stabilizing initial solution to the optimization problem of LMPC j (8), the constraint of (8d) allows LMPC j to gradually (depending on the value of Δu_j) optimize its input trajectory and ensures that the iterations can be terminated at any number without loss of closed-loop stability. The constraint of (8f) is used to guarantee the closed-loop stability.

In the design of (8), the number of iterations c may be restricted to be smaller than a maximum iteration number $c_{\max}$ (i.e., $c \leq c_{\max}$) and/or the iterations may be terminated when a maximum computational time is reached.

The manipulated inputs of the closed-loop system under the above iterative DMPC with delayed measurements are defined as follows:

$$u_i(t) = u_i^*(t - t_k|t_k),\ i = 1, \ldots, m, \forall t \in [t_k, t_{k+q}) \tag{9}$$

for all t_k such that $t_k - d_k > \max_{l<k} t_l - d_l$ and the variable q denotes the smallest integer that satisfies $t_{k+q} - d_{k+q} > t_k - d_k$.

Remark 1: For general nonlinear systems, there is no guaranteed convergence of the optimal cost of the distributed optimization of (8) to any value. Note also that the implementation strategy of the DMPC guarantees that the optimal cost of the distributed optimization of (8) is upper bounded by the cost of the controller $h(\cdot)$ at each sampling time. We further note that in the case of linear systems, the constraint of (8f) can be written in a quadratic form with respect to u_j and it can be verified that the optimization problem of (8) is convex. If the input given by LMPC j of (8) at each iteration is defined as a convex combination of the current optimal input solution and the previous one (e.g., $u_j^c(\tau|t_k) = \sum_{i=1}^{m, i\neq j} w_i u_j^{c-1}(\tau|t_k) + w_j u_j^{*,c}(\tau|t_k)$ where $\sum_{i=1}^{m} w_i = 1$ with $0 < w_i < 1$, $u_j^{*,c}$ is the current solution given by the optimization problem of (8) and u_j^{c-1} is the convex combination of the solutions obtained at iteration $c - 1$), then it can be proved that the optimal cost of the distributed LMPC of (8) converges to the one of the corresponding centralized control system [20], [8]. These considerations imply that there is a balance between controller evaluation time and closed-loop performance that should be struck in the control system architecture (i.e., iterative or centralized) and/or the determination of the maximum iteration number, $c_{\max}$.

A. Stability Analysis

The stability properties of the iterative DMPC of (8)–(9) are stated in Theorem 1 below. To state Theorem 1, we need the following propositions.

Proposition 1 (cf. [21], [11]): Consider the nominal sampled trajectory x_n of the system of (1) in closed-loop with the controller $h(x)$ applied in a sample-and-hold fashion and with open-loop state estimation. Let $\Delta, \epsilon_s > 0$ and $\rho > \rho_s > 0$ satisfy:

$$-\alpha_3(\alpha_2^{-1}(\rho_s)) + L^* M \leq -\frac{\epsilon_s}{\Delta} \tag{10}$$

with $L^* = L_x + \sum_{i=1}^m L_{u_i} u_i^{\max}$. If $\rho_{\min} < \rho$ where $\rho_{\min} = \max\{V(x_n(t+\Delta)) : V(x_n(t)) \leq \rho_s\}$, $x_n(0) \in \Omega_\rho$ and $d_k = 0$ for all k, then $V(x_n(k\Delta)) \leq \max\{V(x_n(0)) - k\epsilon_s, \rho_{\min}\}$.

Proposition 1 ensures that if there is no measurement delay and the nominal system under the control of $h(\cdot)$ implemented in a sample-and-hold fashion starts in Ω_ρ, then it is ultimately bounded in $\Omega_{\rho_{\min}}$. Proposition 2 below provides an upper bound on the deviation of the nominal state trajectory from the actual state trajectory when the same control actions are applied.

Proposition 2 (cf. [11]): Consider the systems:

$$\begin{aligned}\dot{x}_a(t) &= f(x_a(t)) + \sum_{i=1}^m g_i(x_a(t))u_i(t) + k(x_a(t))w(t)\\ \dot{x}_b(t) &= f(x_b(t)) + \sum_{i=1}^m g_i(x_b(t))u_i(t)\end{aligned}$$

with initial states $x_a(t_0) = x_b(t_0) \in \Omega_\rho$. There exists a class $\mathcal{K}$ function $f_W(\cdot)$ such that $|x_a(t) - x_b(t)| \leq f_W(t - t_0)$ for all $x_a(t)$, $x_b(t) \in \Omega_\rho$ and all $w(t) \in W$ where $f_W(\tau) = R_w \theta (e^{R_x \tau} - 1)/R_x$ with θ being the upper bound of the disturbance $w(t)$ and R_w, R_x being positive real numbers.

Proposition 3 bounds the difference between the magnitudes of the Lyapunov function of two states in Ω_ρ.

Proposition 3 (cf. [21]): Consider the Lyapunov function $V(\cdot)$ of the system of (1). There exists a quadratic function $f_V(\cdot)$ such that $V(x_a) \leq V(x_b) + f_V(|x_a - x_b|)$ for all $x_a, x_b \in \Omega_\rho$ with $f_V(s) = \alpha_4(\alpha_1^{-1}(\rho))s + M_v s^2$ and $M_v > 0$.

Proposition 4 bounds the difference between the nominal state trajectory (i.e., $x_a(t)$ in Proposition 4) under the optimized control inputs at the current iteration (i.e., $u_i^c(t)$, $i = 1, \ldots, m$, in Proposition 4) and the predicted nominal state trajectory (i.e., $x_b(t)$ in Proposition 4) generated in the optimization problem of LMPC j with u_i, $i \neq j$, determined at a previous iteration (i.e., $u_i = u_i^{c-1}$, $\forall\, i \neq j$) and u_j calculated at the current iteration (i.e., $u_j = u_j^c$).

Proposition 4: Consider the systems:

$$\begin{aligned}\dot{x}_a(t) &= f(x_a(t)) + \sum_{i=1}^m g_i(x_a(t))u_i^c(t)\\ \dot{x}_b(t) &= f(x_b(t)) + \sum_{i=1}^{m,\, i\neq j} g_i(x_b(t))u_i^{c-1}(t) + g_j(x_b(t))u_j^c(t)\end{aligned}$$

with initial states $x_a(t_0) = x_b(t_0) \in \Omega_\rho$. There exists a class $\mathcal{K}$ function $f_{X,j}(\cdot)$ such that $|x_a(t) - x_b(t)| \leq f_{X,j}(t - t_0)$ for all $x_a(t)$, $x_b(t) \in \Omega_\rho$, and $u_i^c(t)$, $u_i^{c-1} \in U_i$ and $|u_i^c(t) - u_i^{c-1}(t)| \leq \Delta u_i$ with $i = 1, \ldots, m$.

Proof: Define $e(t) = x_a(t) - x_b(t)$. The time derivative $\dot{e}(t)$ can be calculated as $\dot{e}(t) = \dot{x}_a(t) - \dot{x}_b(t)$. Adding and subtracting $\sum_{i=1}^{m,\, i\neq j} g_i(x_b(t))u_i^c(t)$ to/from the expression of $\dot{e}(t)$ and taking into account the local Lipschitz properties assumed for the vector fields $f(\cdot)$ and $g_i(\cdot)$, $i = 1, \ldots, m$, the boundedness of the manipulated inputs, and the boundedness of the difference between $u_i^c(t)$ and $u_i^{c-1}(t)$, we obtain the following inequality:

$$|\dot{e}(t)| \leq C_x |e(t)| + \sum_{i=1}^{m,\, i\neq j} C_{g,i} u_i^{\max} |e(t)| + \sum_{i=1}^{m,\, i\neq j} M_{g,i} \Delta u_i.$$

where C_x, $C_{g,i}$ and $M_{g,i}$ $(i = 1, \ldots, m)$ are positive constants. Denoting $C_{1,j} = C_x + \sum_{i=1}^{m,\, i\neq j} C_{g,i} u_i^{\max}$ and $C_{2,j} = \sum_{i=1}^{m,\, i\neq j} M_{g,i}\Delta u_i$, and integrating $|\dot{e}(t)|$ with initial condition $e(t_0) = 0$, we can obtain that $|e(t)| \leq (C_{2,j}/C_{1,j})\left(e^{C_{1,j}(t-t_0)} - 1\right)$. This proves Proposition 4 with $f_{X,j}(\tau) = (C_{2,j}/C_{1,j})\left(e^{C_{1,j}\tau} - 1\right)$. ■

To simplify the proof of Theorem 1, we define a new function $f_X(\tau)$ based on $f_{X,i}$, $i = 1, \ldots, m$, as follows:

$$f_X(\tau) = \sum_{i=1}^m \left(\frac{1}{m} L_x + L_{u_i} u_i^{\max}\right)\left(\frac{1}{C_{1,i}} f_{X,i}(\tau) - \frac{C_{2,i}}{C_{1,i}}\tau\right). \tag{11}$$

It is easy to verify that $f_X(\tau)$ is a strictly increasing and convex function of its argument. In Theorem 1 below, we provide sufficient conditions under which the iterative DMPC guarantees that the state of the closed-loop system is ultimately bounded in a region that contains the origin.

Theorem 1: Consider the system of (1) in closed-loop with the DMPC design of (8)–(9) based on the controller $h(x)$ that satisfies the conditions of (2) with class $\mathcal{K}$ functions $\alpha_i(\cdot)$, $i = 1, 2, 3, 4$. Let $\Delta, \epsilon_s > 0$, $\rho > \rho_{\min} > 0$, $\rho > \rho_s > 0$, $N \geq 1$ and $D \geq 0$ satisfy the condition of (10) and the following inequality:

$$-N_R\epsilon_s + f_X(N_D\Delta) + f_V(f_W(N_D\Delta)) + f_V(f_W(D)) < 0 \tag{12}$$

with N_D being the smallest integer satisfying $N_D\Delta \geq T_m + D$ and N_R being the smallest integer satisfying $N_R\Delta \geq T_m$. If $x(t_0) \in \Omega_\rho$, $N \geq N_D$ and $d_0 = 0$, then $x(t)$ is ultimately bounded in $\Omega_{\rho_d} \subseteq \Omega_\rho$ where $\rho_d = \rho_{\min} + f_X(N_D\Delta) + f_V(f_W(N_D\Delta)) + f_V(f_W(D))$.

Proof: We assume that at t_k, a delayed measurement $x(t_k - d_k)$ containing new information is received, and that the next measurement with new state information is not received until t_{k+i}. This implies that $t_{k+i} - d_{k+i} > t_k - d_k$ and that the iterative DMPC of (8)–(9) is solved at t_k and the optimal input trajectories $u_i^*(\tau|t_k)$, $i = 1, \ldots, m$, are applied from t_k to t_{k+i}. In this proof, we will refer to $\tilde{x}(t)$ for $t \in [t_k, t_{k+i}]$ as the state trajectory of the nominal system of (1) under the control of the iterative DMPC with $\tilde{x}(t_k) = x^e(t_k)$.

Part I: In this part, we prove that the stability results stated in Theorem 1 hold for $t_{k+i} - t_k = N_{Dk}\Delta$ (recall that N_{Dk} is the smallest integer satisfying $N_{Dk}\Delta \geq T_m + D - d_k$) and all $d_k \leq D$. By Proposition 1 and taking into account that $x_n(t_k) = x^e(t_k)$, the following inequality can be obtained:

$$V(x_n(t_{k+i} - t_k|t_k)) \leq \max\{V(x^e(t_k)) - N_{Dk}\epsilon_s, \rho_{\min}\}. \tag{13}$$

By Proposition 2 and taking into account that $x^e(t_k - d_k) = x(t_k - d_k)$, $\tilde{x}(t_k) = x^e(t_k)$ and $N_D\Delta \geq N_{Dk}\Delta + d_k$, the following inequalities can be obtained:

$$\begin{aligned}|x^e(t_k) - x(t_k)| &\leq f_W(d_k)\\ |\tilde{x}(t_{k+i}) - x(t_{k+i})| &\leq f_W(N_D\Delta).\end{aligned} \tag{14}$$

When $x(t) \in \Omega_\rho$ for all times (this point will be proved below), we can apply Proposition 3 to obtain the following inequalities:

$$\begin{aligned}V(x^e(t_k)) &\leq V(x(t_k)) + f_V(f_W(d_k))\\ V(x(t_{k+i})) &\leq V(\tilde{x}(t_{k+i})) + f_V(f_W(N_D\Delta)).\end{aligned} \tag{15}$$

From (13) and (15), the following inequality is obtained:

$$\begin{aligned}V(x_n(t_{k+i} - t_k|t_k)) \leq{}& \max\{V(x(t_k)) - N_{Dk}\epsilon_s, \rho_{\min}\}\\ &+ f_V(f_W(d_k)).\end{aligned} \tag{16}$$

The derivative of the Lyapunov function of the nominal system of (1) under the control of the iterative DMPC from t_k to t_{k+i} is expressed as follows for $\tau \in [0, N_{Dk}\Delta]$:

$$\dot{V}(\tilde{x}(\tau)) = \frac{\partial V(\tilde{x}(\tau))}{\partial x}\left(f(\tilde{x}(\tau)) + \sum_{i=1}^m g_i(\tilde{x}(\tau))u_i^*(\tau|t_k)\right).$$

Adding the above equation and the constraints of (8f) in each LMPC together, and reworking the resulting inequality, we can obtain the following inequality for $\tau \in [0, N_{Dk}\Delta]$ by accounting for (3) and Proposition 4:

$$\dot{V}(\tilde{x}(\tau)) \leq \dot{V}(x_n(\tau|t_k)) + \left(\frac{1}{m}L_x + L_{u_1}u_1^{\max}\right) f_{X,1}(\tau) + \cdots + \left(\frac{1}{m}L_x + L_{u_m}u_m^{\max}\right) f_{X,m}(\tau).$$

Integrating the above inequality from $\tau = 0$ to $\tau = N_{Dk}\Delta$ and taking into account that $\tilde{x}(t_k) = x_n(t_k), t_{k+i} - t_k = N_{Dk}\Delta$ and the definition of $f_X(\cdot)$, the following inequality can be obtained:

$$V(\tilde{x}(t_{k+i})) \leq V(x_n(t_{k+i} - t_k|t_k)) + f_X(N_{Dk}\Delta). \quad (17)$$

From (15), (16) and (17), the following inequality is obtained:

$$V(x(t_{k+i})) \leq \max\{V(x(t_k)) - N_{Dk}\epsilon_s, \rho_{\min}\} + f_V(f_W(d_k)) + f_V(f_W(N_D\Delta)) + f_X(N_{Dk}\Delta). \quad (18)$$

In order to prove that the Lyapunov function is decreasing between two consecutive measurements, the following inequality must hold:

$$N_{Dk}\epsilon_s > f_V(f_W(d_k)) + f_V(f_W(N_D\Delta)) + f_X(N_{Dk}\Delta) \quad (19)$$

for all possible $0 \leq d_k \leq D$. Taking into account that f_W, f_V and f_X are strictly increasing functions of their arguments, N_{Dk} is a decreasing function of the delay d_k and that if $d_k = D$ then $N_{D_k} = N_R$, then if the condition of (12) is satisfied, the condition of (19) holds for all possible d_k and there exists $\epsilon_w > 0$ such that the following inequality holds:

$$V(x(t_{k+i})) \leq \max\{V(x(t_k)) - \epsilon_w, \rho_d\} \quad (20)$$

which implies that if $x(t_k) \in \Omega_\rho/\Omega_{\rho_d}$, then $V(x(t_{k+i})) < V(x(t_k))$, and if $x(t_k) \in \Omega_{\rho_d}$, then $V(x(t_{k+i})) \leq \rho_d$. Because the upper bound on the difference between the Lyapunov function of the actual trajectory x and the nominal trajectory $\tilde{x}$ is a strictly increasing function of time, the inequality of (20) also implies that:

$$V(x(t)) \leq \max\{V(x(t_k)), \rho_d\}, \ \forall t \in [t_k, t_{k+i}]. \quad (21)$$

Using the inequality of (21) recursively, it can be proved that if $x(t_0) \in \Omega_\rho$, then the closed-loop trajectories of the system of (1) under the proposed iterative DMPC stay in Ω_ρ for all times (i.e., $x(t) \in \Omega_\rho, \ \forall t$). Moreover, it can be proved that if $x(t_0) \in \Omega_\rho, \limsup_{t\to\infty} V(x(t)) \leq \rho_d$. This proves that $x(t) \in \Omega_\rho$ for all times and $x(t)$ is ultimately bounded in Ω_{ρ_d} when $t_{k+i} - t_k = N_{Dk}\Delta$.

Part 2: In this part, we extend the results of Part 1 to the general case, that is, $t_{k+i} - t_k \leq N_{Dk}\Delta$. Taking into account that f_V, f_W and f_X are strictly increasing functions of their arguments and following similar steps as in Part 1, it can be readily proved that the inequality of (19) holds for all possible $d_k \leq D$ and $t_{k+i} - t_k \leq N_{Dk}\Delta$. Using this inequality and following a similar line argument as in Part 1, the stability results stated in Theorem 1 can be proved. ■

Remark 2: Note that in the case that the open-loop operation time is larger than $D + T_m - d_k$, we may still apply the proposed DMPC design but the closed-loop stability cannot be guaranteed, depending on the open-loop process dynamic behavior.

IV. Application to an Alkylation of Benzene Process

We consider an alkylation of benzene with ethylene process which consists of four continuously stirred tank reactors (CSTRs) and a flash tank separator and is modeled by 25 nonlinear ordinary differential equations. Please see [11] for the detailed modeling of the process. Each of the tanks has an external heat/coolant input. The manipulated inputs to the process are the heat injected to or removed from the five vessels, Q_1, Q_2, Q_3, Q_4 and Q_5, and the feed stream flow rates to CSTR-2 and CSTR-3, F_4 and F_6. The states of the process consist of the concentrations of benzene (A), ethylene (B), ethylbenzene (C), and 1,3-diethylbenzene (D) in each of the five vessels and the temperatures of the vessels. We consider a steady state (operating point), x_s, of the process which is defined by the steady-state inputs Q_{1s}, Q_{2s}, Q_{3s}, Q_{4s}, Q_{5s}, F_{4s} and F_{6s}, shown in Table I. The steady-state temperatures in the five vessels are the following: $T_{1s} = 477.24$ K, $T_{2s} = 476.97$ K, $T_{3s} = 473.47$ K, $T_{4s} = 470.60$ K, $T_{5s} = 478.28$ K. The control objective is to regulate the system from an initial state to the steady state. The initial temperatures of the five vessels are: $T_{1o} = 443.02$ K, $T_{2o} = 437.12$ K, $T_{3o} = 428.37$ K, $T_{4o} = 433.15$ K, $T_{5o} = 457.55$ K. The first distributed controller (LMPC 1) will be designed to compute the values of Q_1, Q_2 and Q_3, the second distributed controller (LMPC 2) will be designed to compute the values of Q_4 and Q_5, and the third distributed controller (LMPC 3) will be designed to compute the values of F_4 and F_6. Taking this into account, the process model belongs to the class of nonlinear systems: $\dot{x}(t) = f(x) + g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3$ where the state x is the deviation of the state of the process from the steady state, $u_1^T = [u_{11} \ u_{12} \ u_{13}] = [Q_1 - Q_{1s} \ Q_2 - Q_{2s} \ Q_3 - Q_{3s}]$, $u_2^T = [u_{21} \ u_{22}] = [Q_4 - Q_{4s} \ Q_5 - Q_{5s}]$ and $u_3^T = [u_{31} \ u_{32}] = [F_4 - F_{4s} \ \ F_6 - F_{6s}]$ are the manipulated inputs which are subject to the constraints shown in Table II. We use the same design of $h(x)$ as in [11] based on a quadratic Lyapunov function $V(x) = x^T P x$ with P being the following weighting matrix: $P = \mathrm{diag}([1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1])$. Based on $h(x)$, we design the iterative DMPC with the weighting matrices being $Q_c = \mathrm{diag}([1\ 1\ 1\ 1\ 10^3\ 1\ 1\ 1\ 1\ 10^3\ 10\ 10\ 10\ 10\ 3000\ 1\ 1\ 1\ 10^3\ 1\ 1\ 1\ 1\ 10^3])$, $R_{c1} = \mathrm{diag}([1 \times 10^{-8}\ 1 \times 10^{-8}\ 1 \times 10^{-8}\ ])$, $R_{c2} = \mathrm{diag}([1 \times 10^{-8}\ 1 \times 10^{-8}\ ])$ and $R_{c3} = \mathrm{diag}([10\ 10])$. The sampling time of the LMPCs is chosen to be $\Delta = 30$ s. Δu_i is chosen to be $0.25u_i^{\max}$ for the distributed LMPCs and maximum iteration number (i.e., $c \leq c_{\max}$) is used as the termination condition. In the simulations, bounded process noise is considered.

We consider that the state of the process is sampled at asynchronous time instants $\{t_{k\geq 0}\}$ with $T_m = 50$ s. Moreover, we consider that there are delays involved in the measurements with $D = 40$ s. Measurement delays can naturally arise in the context of species concentration measurements. We will compare the proposed iterative DMPC with a centralized LMPC which takes into account delayed measurements explicitly [14]. The centralized LMPC uses the same weighting matrices, sampling time and prediction horizon as used in the DMPC. In order to model the sampling time instants, a bounded Poisson process (see [17]) is used to generate $\{t_{k\geq 0}\}$ and another bounded random process is used to generate the associated delay sequence $\{d_{k\geq 0}\}$. We choose the horizon of all the LMPCs to be $N = 3$ so that the horizon covers the maximum possible open-loop operation interval (i.e., $T_m + D = 90$ s). Note that the maximum possible open-loop operation interval only depends on the frequency of measurement sampling and the delays in

TABLE I
Steady-State Input Values for x_s

Q_{1s}	-4.4×10^6 $[J/s]$	Q_{2s}	-4.6×10^6 $[J/s]$
Q_{3s}	-4.7×10^6 $[J/s]$	Q_{4s}	9.2×10^6 $[J/s]$
Q_{5s}	5.9×10^6 $[J/s]$	F_{4s}, F_{6s}	8.70×10^{-4} $[m^3/s]$

TABLE II
Manipulated Input Constraints

$\lvert u_{12}\rvert \leq 1 \times 10^5\ [J/s]$	$\lvert u_{1i}\rvert \leq 0.5 \times 10^5\ [J/s], (i = 1, 3)$
$\lvert u_{21}\rvert \leq 1 \times 10^5\ [J/s]$	$\lvert u_{31}\rvert \leq 4.93 \times 10^{-5}\ [m^3/s]$
$\lvert u_{22}\rvert \leq 1 \times 10^5\ [J/s]$	$\lvert u_{32}\rvert \leq 4.93 \times 10^{-5}\ [m^3/s]$

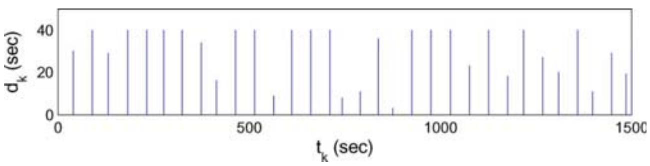

Fig. 2. Asynchronous time sequence $\{t_{k\geq 0}\}$ and corresponding delay sequence $\{d_{k\geq 0}\}$ with $T_m = 50$ s and $D = 40$ s: the x–axis indicates $\{t_{k\geq 0}\}$ and the y–axis indicates the size of d_k.

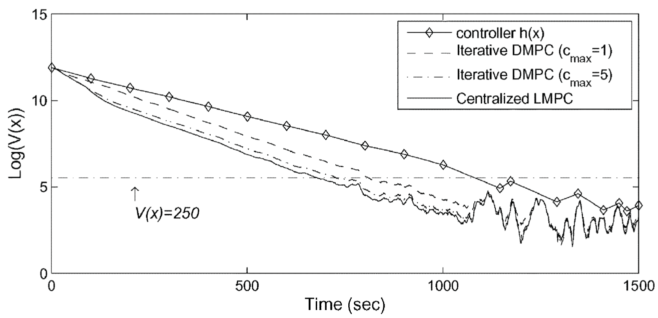

Fig. 3. Trajectories of $V(x)$ under $h(x)$ implemented in a sample-and-hold fashion and with open-loop state estimation, the iterative DMPC with $c_{\max} = 1$, 5 and the centralized LMPC.

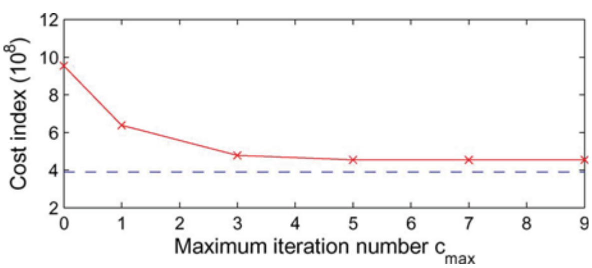

Fig. 4. Total performance cost along the closed-loop system trajectories of centralized LMPC (dashed line) and iterative DMPC (solid line).

the measurements and is not related to the dynamics of the chemical plant. Note also that, in terms of practical considerations, it is possible, particularly in the context of species concentration measurements, for the measurement delays to exceed 30 s and the use of a 40 s delay upper bound for species concentration measurements is realistic from a practical standpoint. Fig. 2 shows the time instants when new state measurements are received and the associated delay sizes

Fig. 3 shows the trajectory of $V(x)$ under different control designs. From Fig. 3, we see that both the proposed iterative DMPC and the centralized LMPC are able to drive the system state to a region very close to the desired steady state ($V(x) \leq 250$); the trajectories of $V(x)$ generated by the iterative DMPC design are bounded by the corresponding trajectory of $V(x)$ under the controller $h(x)$ implemented in a sample-and-hold fashion and with open-loop state estimation. From Fig. 3, we can also see that the centralized LMPC and the iterative DMPC with $c_{\max} = 5$ give very similar $V(x)$ trajectories.

Next, we compare the centralized LMPC and the iterative DMPC from a performance index point of view. To carry out this comparison, the same initial condition and parameters were used for the different control schemes and the total cost under each control scheme was computed as follows: $J = \int_0^{t_f} \left(|x(t)|_{Q_c} + \sum_{i=1}^{3} |u_i(t)|_{R_{ci}} \right) dt$ where $t_f = 1500$ s is the final simulation time. Fig. 4 shows the total cost along the closed-loop system trajectories under the iterative DMPC and the centralized LMPC. For the iterative DMPC design, different maximum numbers of iterations, $c_{\max}$, are used. From Fig. 4, we can see that as the iteration number c increases, the performance cost given by the iterative DMPC design decreases and converges to a value which is very close to the cost of the one corresponding to the centralized LMPC. However, we note that there is no guaranteed convergence of the cost of iterative DMPC to the cost of a centralized MPC because of the non-convexity of the LMPC optimization problems, and the different stability constraints imposed in the centralized LMPC and the iterative DMPC (Remark 1).

Finally, we compare the evaluation times of the various control designs. The simulations are carried out by Java programming language in a *Pentium 3.20 GHz* computer. The optimization problems are solved by the interior point optimizer Ipopt. We evaluate the LMPC optimization problems for 100 runs. The mean evaluation time of the centralized LMPC is about 23.7 s. The mean evaluation time of the iterative DMPC with $c_{\max} = 1$ is 6.3 s which is the largest time among the three LMPC evaluation times (1.6 s, 6.3 s and 4.3 s). The mean evaluation time of the iterative DMPC with $c_{\max} = 4$ is 18.7 s with the evaluation times of the three LMPCs being 6.9 s, 18.7 s and 14.0 s, respectively. From the results, we see that the proposed DMPC leads to a reduction in the evaluation time compared to the centralized LMPC though both provide a similar closed-loop performance. The results also imply that the iterative DMPC may be applicable to processes which require smaller sampling times to maintain closed-loop stability and for which centralized MPC is not a feasible option due to larger evaluation time.

References

[1] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[2] J. B. Rawlings and B. T. Stewart, "Coordinating multiple optimization-based controllers: New opportunities and challenges," *J. Process Control*, vol. 18, pp. 839–845, 2008.

[3] R. Scattolini, "Architectures for distributed and hierarchical model predictive control – A review," *J. Process Control*, vol. 19, pp. 723–731, 2009.

[4] E. Camponogara, D. Jia, B. H. Krogh, and S. Talukdar, "Distributed model predictive control," *IEEE Control Syst. Mag.*, vol. 22, pp. 44–52, 2002.

[5] W. B. Dunbar, "Distributed receding horizon control of dynamically coupled nonlinear systems," *IEEE Trans. Autom. Control*, vol. 52, no. 7, pp. 1249–1263, Jul. 2007.

[6] A. Richards and J. P. How, "Robust distributed model predictive control," *Int. J. Control*, vol. 80, pp. 1517–1531, 2007.

[7] L. Magni and R. Scattolini, "Stabilizing decentralized model predictive control of nonlinear systems," *Automatica*, vol. 42, pp. 1231–1236, 2006.

[8] B. T. Stewart, A. N. Venkat, J. B. Rawlings, S. J. Wright, and G. Pannocchia, "Coorperative distributed model predictive control," *Syst. Control Lett.*, vol. 59, pp. 460–469, 2010.

[9] J. M. Maestre, D. M. de la Peña, and E. F. Camacho, "Distributed model predictive control based on a cooperative game," *Optim. Control Appl. Methods*, vol. 32, pp. 153–176, 2011.

[10] J. Liu, D. M. de la Peña, and P. D. Christofides, "Distributed model predictive control of nonlinear process systems," *AIChE J.*, vol. 55, pp. 1171–1184, 2009.

[11] J. Liu, X. Chen, D. M. de la Peña, and P. D. Christofides, "Sequential and iterative architectures for distributed model predictive control of nonlinear process systems," *AIChE J.*, vol. 56, pp. 2137–2149, 2010.

[12] Y. Tipsuwan and M. Chow, "Control methodologies in networked control systems," *Control Eng. Pract.*, vol. 11, pp. 1099–1111, 2003.

[13] G.-P. Liu, Y. Xia, J. Chen, D. Rees, and W. Hu, "Networked predictive control of systems with random networked delays in both forward and feedback channels," *IEEE Trans. Ind. Electron.*, vol. 54, no. 3, pp. 1282–1297, Jun. 2007.

[14] J. Liu, D. M. de la Peña, P. D. Christofides, and J. F. Davis, "Lyapunov-based model predictive control of nonlinear systems subject to time-varying measurement delays," *Int. J. Adaptive Control Signal Processing*, vol. 23, pp. 788–807, 2009.

[15] L. Grüne, J. Pannek, and K. Worthmann, "A networked unconstrained nonlinear MPC scheme," in *Proc. ECC'09*, Budapest, Hungary, 2009, pp. 371–376.

[16] E. Franco, L. Magni, T. Parisini, M. M. Polycarpou, and D. M. Raimondo, "Cooperative constrained control of distributed agents with nonlinear dynamics and delayed information exchange: A stabilizing receding-horizon approach," *IEEE Trans. Autom. Control*, vol. 53, no. 1, pp. 324–338, Feb. 2008.

[17] J. Liu, D. M. de la Peña, and P. D. Christofides, "Distributed model predictive control of nonlinear systems subject to asynchronous and delayed measurements," *Automatica*, vol. 46, pp. 52–61, 2010.

[18] Y. Lin, E. D. Sontag, and Y. Wang, "A smooth converse Lyapunov theorem for robust stability," *SIAM J. Control Optim.*, vol. 34, pp. 124–160, 1996.

[19] P. D. Christofides and N. H. El-Farra, *Control of Nonlinear and Hybrid Process Systems: Designs for Uncertainty, Constraints and Time-Delays*. Berlin, Germany: Springer-Verlag, 2005.

[20] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation*. Belmont, MA: Athena Scinetific, 1997.

[21] D. M. de la Peña and P. D. Christofides, "Lyapunov-based model predictive control of nonlinear systems subject to data losses," *IEEE Trans. Autom. Control*, vol. 53, no. 9, pp. 2076–2089, Oct. 2008.

Integrated Design of Symbolic Controllers for Nonlinear Systems

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***Abstract*—Symbolic models of continuous and hybrid systems have been studied for a long time, because they provide a formal approach to solve control problems where software and hardware interact with the physical world. While being powerful, this approach often encounters some limitations in concrete applications, because of the large size of the symbolic models needed to be constructed. Inspired by on–the–fly techniques for verification and control of finite state machines, in this note we propose an algorithm that integrates the construction of the symbolic models with the design of the symbolic controllers. Computational complexity of the proposed algorithm is discussed and an illustrative example is included.**

***Index Terms*—Approximate bisimulation, digital control systems, nonlinear systems, on–the–fly design, symbolic models.**

I. Introduction

Symbolic models of continuous and hybrid systems have been studied for a long time, because they provide a formal approach to solve control problems where software and hardware interact with the physical world. Symbolic models are abstract descriptions of control systems in which a symbolic state corresponds to an aggregate of states. Several classes of dynamical and control systems that admit symbolic models were identified during the last few years, see, e.g., [1], [12] and the references therein. In particular, incrementally stable [2] nonlinear control systems were shown in [7], [10] to admit symbolic models. This last result has been further generalized to incrementally stable nonlinear switched systems in [6], incrementally stable nonlinear time–delay systems in [8], [9] and incrementally forward complete nonlinear control systems in [15]. The use of symbolic models for the control design of continuous and hybrid systems has been investigated in [11], [14]. As discussed in [12], this approach provides the designer with a systematic method to address a wide spectrum of novel specifications, that are difficult to enforce by means of conventional control design paradigms. Examples of such specifications include logic specifications expressed in terms of linear temporal logic formulae or automata on infinite strings. The use of these specifications has been shown to be relevant in the control design of important domains of application, including robot motion planning and systems biology (see, e.g., [14] and the references therein). While being powerful, this approach often encounters some limitations in concrete applications, because of the large size of the symbolic models needed to be constructed. In this note we propose one approach to cope with this drawback. We consider a symbolic control design problem for nonlinear control systems. Given a nonlinear control plant and a specification expressed in terms of a finite automaton on infinite strings, we face the problem of designing a symbolic controller that implements the specification with arbitrarily good accuracy. The symbolic controller is furthermore requested to avoid blocking behaviors, when interacting with the plant. This problem can be viewed as an approximate version of similarity games, as discussed in [12]. Related control design problems have been studied in [11] and [14]. The first contribution of this note lies in the derivation of an explicit solution to the control problem under study. *The symbolic controller is proven to be the non–blocking part [3] of the approximate parallel composition [12] between the specification automaton and the symbolic model of the plant*. The synthesis of such a controller requires the preliminary construction of the symbolic model of the plant, which is generally demanding from the computational complexity point of view. Inspired by the research line on on–the–fly verification and control of finite state machines (see e.g., [4], [13]), we give the second contribution of this note consisting in *an efficient algorithm that integrates the construction of the symbolic model of the plant with the design of the symbolic controller*. Computational complexity of the proposed algorithm is discussed and an illustrative example is included.

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II. Preliminary Definitions

Notation

The symbol $|A|$ denotes the cardinality of a finite set A. The identity map on a set A is denoted by 1_A. Given a relation $\mathcal{R} \subseteq A \times B$, the symbol $\mathcal{R}^{-1}$ denotes the inverse relation of $\mathcal{R}$, i.e., $\mathcal{R}^{-1} = \{(b,a) \in B \times A : (a,b) \in A \times B\}$. The symbols $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of integer, real, positive real, and nonnegative real numbers, respectively. The symbol $\|x\|$ denotes the infinity norm of $x \in \mathbb{R}^n$. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum of f is denoted by $\|f\|_\infty$. Given $x \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}^+$, the symbols $\mathcal{B}_\varepsilon(x)$ and $\mathcal{B}_{[\varepsilon[}(x)$ denote the set $\{x \in \mathbb{R}^n \,|\, \|x\| \leq \varepsilon\}$ and the set $[-\varepsilon+x_1, x_1+\varepsilon[\times[-\varepsilon+x_2, x_2+\varepsilon[\times\cdots\times[-\varepsilon+x_n, x_n+\varepsilon[$, respectively. Given $\mu \in \mathbb{R}^+$ and $A \subseteq \mathbb{R}^n$, we denote by μA the set $\{b \in \mathbb{R}^n \,|\, \exists a \in A \text{s.t. } b = \mu a\}$. For any $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^+$ the symbol $[x]_\mu$ denotes the unique vector in $\mu\,\mathbb{Z}^n$ such that $x \in \mathcal{B}_{[\mu/2[}([x]_\mu)$.

A. Control Systems

In this note we consider the nonlinear control system

$$\Sigma : \begin{cases} \dot{x}(t) = f(x(t), u(t)), t \in \mathbb{R}_0^+, \\ x(0) \in X_0, \end{cases} \tag{1}$$