# Dynamic Output Feedback for Discrete-Time Systems Under Amplitude and Rate Actuator Constraints

J. M. Gomes da Silva, D. Limon, T. Alamo, and E. F. Camacho

*Abstract*—**This work proposes a technique for the design of stabilizing dynamic output feedback controllers for discrete-time linear systems with rate and amplitude saturating actuators. The nonlinear effects introduced by the saturations in the closed-loop system are taken into account by using a generalized sector condition, which leads to theoretical conditions for solving the problem directly in the form of linear matrix inequalities.**

*Index Terms*—**Constrained control, control saturation, discrete-time systems, output feedback, stabilization.**

## I. Introduction

The physical impossibility of applying unlimited control signals makes actuator saturation a ubiquitous problem in control systems. In particular, it is well known that input saturation is a source of performance degeneration, limit cycles, different equilibrium points, and even instability. Hence, there has been a great interest in studying these negative effects and also in proposing control design procedures, in global, semiglobal and local contexts of stability, which take directly into account the control bounds (see for instance [1]–[3] and references therein). It should be pointed out that most of these works only consider amplitude saturation and state feedback control strategies.

Works formally addressing stabilization in the presence of both amplitude and rate saturation started to appear in the late 90s. Semi-global stabilization results have been proposed in [4]. Using a low and high gain approach, solutions to the global and semi-global stabilization problems, via both state and output feedback, are given in [5]. In [6], the global stabilization problem is addressed by means of a scheduled low-gain state feedback. As global and semi-global stabilization are concerned, it should be pointed out that these approaches cannot be applied to exponentially unstable open-loop systems. In the local (regional) stabilizing context, we can cite the results presented in [7], [8] and [9], where the synthesis of state feedback control laws is proposed. The synthesis of dynamic output feedback controllers ensuring local stability is considered in [10] and [11]. In [10], a method for designing dynamic output controllers using the Positive Real Lemma is proposed; the main objective is the minimization of an LQG criterion. It should be pointed out that the size and the shape of the stability region are not taken into account in the design procedure, and this may lead to very conservative domains of stability. Moreover, the controller is computed from the solution of strong coupled Riccati equations which, in general, are not simple to solve. A time-varying dynamic controller is proposed in [11]. The stabilizing conditions are given, in this case, in the form of nonlinear matrix inequalities, which implies the use of iterative linear matrix inequality (LMI) relaxation schemes for computing the controller. Furthermore, no explicit consideration is made about the region of attraction associated to the controller. On the other hand, it should be pointed out that all the works cited above are only concerned with continuous-time systems and the rate limitation is considered in the modelling of the actuator. The actuator is considered to present first order dynamics and, in fact, rate saturation is modelled as a saturation of the actuator state. This is the so-called *position-feedback-type* model [10] for rate saturation. In this case, when the time constant of the actuator dynamics tends to zero, the behavior of the position-feedback-type model tends to the "ideal" rate limiter, or, equivalently to the notion of a rate saturation operator as introduced in [5]. However, as pointed out in [12] and [13], if, in fact, the actuator dynamics is not represented by a first order model, the closed-loop stability cannot be ensured by the proposed methods. Furthermore, the position-feedback-type model seems to be unsuitable or imprecise for dealing with the rate saturation phenomenon in a discrete-time framework, representing a digital control system. Note that in this case, due to the saturating state, the actuator cannot be seen as a linear system. Thus, standard discretization methods for linear systems driven by zero-order-holders cannot be directly applied. In fact, the effective signal sent to the plant during a sampling period will depend on continuous-time nonlinear actuator dynamics, which renders a more involved formal analysis.

An alternative approach for dealing with the rate saturation problem without resorting to the position-feedback-type model, has been proposed in [12], [13] (considering continuous-time systems) and in [14] (considering discrete-time systems). The basic idea consists of introducing a rate limiter inside the controller in order to prevent the control signal (to be sent to the actuator) violating the rate bounds. This is accomplished by introducing a nonlinear integrator in the controller structure. In particular, in [12] and [14], stabilizing synthesis conditions for this kind of controller structure have been proposed in the form of nonlinear matrix inequalities, both for state and dynamic output feedback cases. The saturation nonlinear effects are taken into account by the application of classical sector bound conditions. The nonlinearities in the matrix inequalities are due, in this case, to the product of some variables and the multipliers associated to the classical sector conditions. Following the same ideas, but using a Riccati equation approach, in [13] a method for computing fixed-order dynamic output feedback controllers is proposed. The multipliers associated to the sector conditions are seen as tuning parameters and are supposed to be a priori fixed. As in [10], in [12] and [14] the size and the shape of the domain of ensured stability is not taken directly into account in the design procedure. This domain is computed a posteriori, which can lead to small regions of stability for the closed-loop system.

This note proposes a technique for designing stabilizing dynamic output feedback controllers for discrete-time linear systems with rate and amplitude constrained actuators. In order to deal with the rate limitation, similar ideas to the ones proposed in [14] are used. We propose the synthesis of a nonlinear dynamic controller which is composed of a classical linear dynamic controller in cascade with input saturating integrators and two static anti-windup loops. It should be pointed out that, contrary to the anti-windup approaches (see for instance [15], [16], [17], [18], [19], [20], and references therein), where the controller is considered to be given, the idea here consists of computing the controller and the anti-windup gains simultaneously. The anti-windup gains appear, therefore, as extra degrees of freedom in the synthesis problem. In particular, they are also useful to obtain conditions in LMI form. The theoretical conditions for solving the synthesis problem are based on the application of a generalized sector condition proposed in [20], which is applicable for deadzone nonlinearities. This condition encompasses the classical sector condition used, for instance,

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J. M. Gomes da Silva, Jr.is with the Department of Electrical Engineering, UFRGS, Porto Alegre-RS 90035-190, Brazil (e-mail: jmgomes@ece.ufrgs.br).

D. Limon, T. Alamo, and E.F. Camacho are with the Department Ingeniería de Sistemas y Automática, Universidad de Sevilla, Sevilla 41092, Spain (e-mail: limon,alamo,eduardo@cartuja.us.es).

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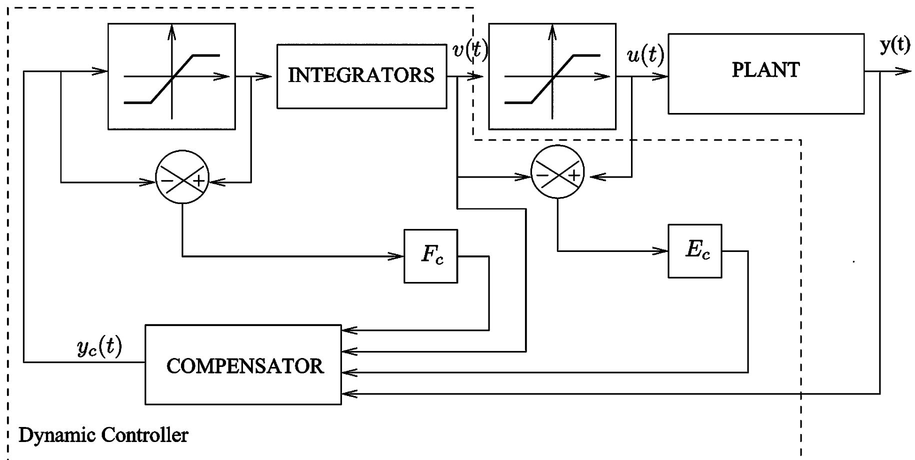



Fig. 1. closed-loop system.

in [10], [13], [21], [14] and allows (contrary to the classical one) the formulation of regional stabilization conditions directly in LMI form. Then, using classical variable transformations, as proposed in [22], it is possible to formulate LMI conditions that allow dynamic nonlinear controllers to be computed in order to ensure the regional asymptotic stability of the closed-loop system. Convex optimization problems for determining the controller in order to enlarge the basin of attraction of the closed-loop as well as enhance the time-domain performance of the closed-loop system are, therefore, proposed. A numerical example is provided to illustrate the application of the proposed method.

*Notations:* The elements of a matrix $A \in \Re^{m\times n}$ are denoted by $A_{(i,l)}, i = 1,\ldots,m, l = 1,\ldots,n$ and $A_{(i)}$ denotes the $i$th row of matrix $A$. For two symmetric matrices, $A$ and $B$, $A > B$ means that $A-B$ is positive definite. $A'$ denotes the transpose of $A$. $blockdiag(A,B)$ is the block-diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. $I_m$ denotes the $m$-order identity matrix. $sat_\rho$ is a componentwise saturation map $\Re^m \to \Re^m$ defined as follows: $(sat_\rho(v))_{(i)} = sat_{\rho_{(i)}}(v_{(i)}) = sign(v_{(i)})\min(\rho_{(i)}, |v_{(i)}|)$, $\forall\ i = 1,\ldots,m$, where $\rho_{(i)}$, denotes the $i$th bound of the saturation function.

## II. Problem Statement

Consider the discrete-time linear system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, the input and the measured output vectors, respectively, and $t \in \mathcal{N}$. Matrices $A$, $B$ and $C$ are real constant matrices of appropriate dimensions. Pairs $(A,B)$ and $(C,A)$ are assumed to be controllable and observable respectively.

The input vector $u$ is subject to amplitude and rate limitations defined as follows.

- Amplitude constraints: $|u_{(i)}(t)| \leq \rho_{a(i)}$, $i = 1,\ldots,m$, where $\rho_{a(i)} > 0$ denote the $i$th control amplitude bound.
- Rate constraints: $|\Delta u_{(i)}(t)| = |u_{(i)}(t) - u_{(i)}(t-1)| \leq \rho_{r(i)}$, $i = 1,\ldots,m$, where $\rho_{r(i)} > 0$ denote the $i$th rate control bound.

We suppose that only the output $y(t)$ is available for measurement. Hence, our aim is to compute a stabilizing dynamic output feedback controller. In particular, we consider a controller composed by an $n+m$ order dynamic compensator in cascade with $m$ input saturating integrators and two anti-windup loops, described by the following equations:

$$v(t+1) = I_m v(t) + sat_{\rho_r}(y_c(t)) \tag{2}$$

$$\begin{aligned} x_c(t+1) &= A_c x_c(t) + B_c[y(t)'v(t)']' \\ &\quad + E_c(sat_{\rho_a}(v(t)) - v(t)) \\ &\quad + F_c(sat_{\rho_r}(y_c(t)) - y_c(t)) \end{aligned} \tag{3}$$

$$y_c(t) = C_c x_c(t) + D_c[y(t)'v(t)']' \tag{4}$$

where $x_c(t) \in \Re^{n+m}$ and $y_c(t) \in \Re^m$ are, respectively, the state and the output of the dynamic compensator and $v(t) \in \Re^m$ corresponds to the state of a sub-system composed by $m$ decoupled integrators. Matrices $A_c$, $B_c$, $C_c$, $D_c$, $E_c$ and $F_c$ have appropriate dimensions. $E_c$ and $F_c$ are anti-windup gains. As a consequence of the amplitude control bounds, the effective control signal applied to system (1) is a saturated one:

$$u(t) = sat_{\rho_a}(v(t)). \tag{5}$$

Under this connection the whole closed-loop system is depicted in Fig. 1.

The set of $m$ input saturating integrators (2) introduced in the controller aims at generating a rate limited control signal. Since the Lipschitz constant of the $sat(\cdot)$ function is equal to 1, it follows, $\forall\ i = 1,\ldots,m$, that

$$\begin{aligned} |\Delta u_{(i)}(t+1)| &= |sat_{\rho_{a(i)}}(v_{(i)}(t) + sat_{\rho_{r(i)}}(y_{c(i)}(t))) \\ &\qquad - sat_{\rho_{a(i)}}(v_{(i)}(t))| \\ &\leq |sat_{\rho_{r(i)}}(y_{c(i)}(t))| \leq \rho_{r(i)} \end{aligned} \tag{6}$$

Hence, it follows that the signal $u(t)$ to be delivered to the plant will respect the rate constraints.

The problem addressed in the sequel regards therefore the computation of matrices $A_c$, $B_c$, $C_c$, $D_c$, $E_c$ and $F_c$ of the linear dynamic compensator, in such a way that the domain of attraction of the closed-loop system is maximized under some performance constraints.

## III. Preliminaries

In this section we present some ideas which form the basis of the statement of the main result in Section IV. With this aim, consider a generic linear system driven by a saturating control law as follows:

$$z(t+1) = Uz(t) + V sat_\eta(Kz(t)) \tag{7}$$

with $z \in \Re^n$, $Kz \in \Re^m$ and matrices $U$ and $V$ with appropriate dimensions.

Based on the definition of a vector valued deadzone nonlinearity $\psi_\eta(Kz(t)) \triangleq Kz(t) - sat_\eta(Kz(t))$, i.e., $(\psi_\eta(Kz(t)))_{(i)} \triangleq K_{(i)}z -$

$sat_{\eta_{(i)}}(K_{(i)}z(t))$, $\forall\ i = 1,\ldots,m$, system (7) can be equivalently re-written as follows:

$$z(t+1) = (U+VK)z(t) - V\psi_\eta(Kz(t)). \tag{8}$$

Now, consider a diagonal matrix $0 \le \Lambda < I_m$. Provided $z$ belongs to a set

$$\mathcal{S}_\Lambda = \left\{ z \in \Re^n; |K_{(i)}z| \le \frac{\eta_{(i)}}{1-\Lambda_{(i,i)}},\ i=1,\ldots,m \right\} \tag{9}$$

it follows that each component of the nonlinearity $\psi_\eta(Kz)$ belongs to the sector $(0,\Lambda_{(i,i)})$, i.e., $0 \le (\psi_\eta(Kz))_{(i)}(K_{(i)}z) \le \Lambda_{(i,i)}(K_{(i)}z)^2$, and it follows that a classical sector condition

$$\psi_\eta(Kz)'T[\psi_\eta(Kz) - \Lambda Kz] \le 0, \tag{10}$$

is verified, $\forall\, T \in \Re^{m\times m}$ diagonal and positive definite.

Given a Lyapunov candidate function $V(z(t))$, and following an absolute stability approach (generalized circle criterion), the regional (local) asymptotic stability of system in a region $\mathcal{R} = \{z \in \Re^n; V(z) \le c\}$ is ensured if:

a) $\dot{V}(z(t)) - \psi_\eta(Kz)'T[\psi_\eta(Kz) - \Lambda Kz] < 0$,

b) $\mathcal{R} \subset \mathcal{S}_\Lambda$

This approach has been used in many works dealing with the analysis and the synthesis of systems presenting saturations, as is the case in [10], [14] and [21] among others. However, some drawbacks concerning the use of the classical sector condition (10) should be pointed out. First, stability is ensured not just for deadzone nonlinearities but for all nonlinearities belonging to the sector defined by $\Lambda$. Second, considering quadratic Lyapunov functions, it leads to stability conditions in the form of nonlinear matrix inequalities (basically due to the product of $T$, $\Lambda$ and $K$). Then, in order to deal with these conditions in an LMI framework, either the matrix $\Lambda$ or the multiplier matrix $T$, should be fixed a priori, which is another source of conservatism (see a discussion in [20]).

It has been shown in [20] that, for the particular case of deadzone nonlinearities, these drawbacks can be overcome. In this case, provided $z$ belongs to a set

$$\mathcal{S}_G = \{z \in \Re^n; |(K_{(i)} - G_{(i)})z| \le \eta_{(i)},\ i=1,\ldots,m\}, \tag{11}$$

the following generalized sector condition hold

$$\psi_\eta(Kz)'T[\psi_\eta(Kz) - Gz] \le 0, \tag{12}$$

$\forall\, T \in \Re^{m\times m}$ diagonal and positive definite. In this case, the set inclusion condition b) above will be replaced by $\mathcal{R} \subset \mathcal{S}_G$. Note that relation (10) can be seen as a particular case of (12) if we set $G = \Lambda K$. Hence, (12) will produce stability conditions which encompass the ones generated from (10). Furthermore, direct information about the nonlinearity type is present in (12), which is not the case when the relation (10) is considered, therefore reducing the conservatism of the conditions. On the other hand, as will be made clear in the sequel, since $G$ appears as a free variable, conditions in LMI form can be directly obtained, avoiding iterative schemes as proposed, for instance, in [14].

## IV. Main Results

First, note that the problem of computing the linear compensator matrices $A_c$, $B_c$, $C_c$, $D_c$, $E_c$ and $F_c$ can be seen as the synthesis of a dynamic linear output feedback compensator (3)–(4) for an augmented system (composed by the plant plus the controller integrators), given as follows:

$$\begin{aligned}\tilde{x}(t+1) &= \mathbf{A}\tilde{x}(t) - \mathbf{B}_1\psi_{\rho_a}(\mathbf{L}\tilde{x}(t)) + \mathbf{B}y_c(t) \\ &\quad - \mathbf{B}\psi_{\rho_r}(y_c(t)) \\ \tilde{y}(t) &= C\tilde{x}(t)\end{aligned} \tag{13}$$

with

$$\tilde{x} = \begin{bmatrix} x \\ v\end{bmatrix},\quad \tilde{y} = \begin{bmatrix} y \\ v\end{bmatrix},\quad \mathbf{A} = \begin{bmatrix} A & B \\ 0 & I_m\end{bmatrix},\quad \mathbf{B}_1 = \begin{bmatrix} B \\ 0\end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} 0 \\ I_m\end{bmatrix},\quad \mathbf{C} = \begin{bmatrix} C & 0 \\ 0 & I_m\end{bmatrix},\quad \mathbf{L} = [\,0\quad I_m\,],$$

$\psi_{\rho_a}(\mathbf{L}\tilde{x}) = \mathbf{L}\tilde{x} - sat_{\rho_a}(\mathbf{L}\tilde{x})$ and $\psi_{\rho_r}(y_c) = y_c(t) - sat_{\rho_r}(y_c)$.

Thus, the closed-loop system, obtained from the interconnection of the plant (1) and the controller (2)–(4) through (5), can be represented by the following nonlinear system:

$$\xi(t+1) = \mathcal{A}\xi(t) - \mathcal{B}_1\psi_{\rho_a}(\mathcal{L}\xi(t)) - \mathcal{B}\psi_{\rho_r}(\mathcal{K}\xi(t)) \tag{14}$$

with

$$\xi = \begin{bmatrix} \tilde{x} \\ x_c\end{bmatrix},\quad \mathcal{A} = \begin{bmatrix} \mathbf{A}+\mathbf{B}D_c\mathbf{C} & \mathbf{B}C_c \\ B_c\mathbf{C} & A_c\end{bmatrix},\quad \mathcal{B}_1 = \begin{bmatrix} \mathbf{B}_1 \\ E_c\end{bmatrix},$$
$$\mathcal{B} = \begin{bmatrix} \mathbf{B} \\ F_c\end{bmatrix},\quad \mathcal{L} = [\,\mathbf{L}\quad 0\,],\ \text{and}\ \mathcal{K} = [\,D_c\mathbf{C}\quad C_c\,].$$

We are now ready to state the main result of the paper. This result will follow the ideas discussed in Section III. Since the equivalent representation of the closed-loop system (14) presents two deadzone nonlinearities, two generalized sector conditions (12) and respective set inclusion conditions are considered in order to take into account the nonlinear behavior of the system. In order to compute the matrices of the dynamic compensator, the ideas proposed in [22], where appropriated change of variables allows the formulation of synthesis conditions in LMI form, are here adapted to also include the anti-windup gains ($E_c$ and $F_c$) and the variables associated to the sector conditions.

*Theorem 1:* If there exist symmetric positive definite matrices $X, Y \in \Re^{(n+m)\times(n+m)}$, positive definite diagonal matrices $S_r, S_a \in \Re^{m\times m}$, and matrices $\hat{A} \in \Re^{(n+m)\times(n+m)}$, $\hat{C} \in \Re^{m\times(n+m)}$, $\hat{B} \in \Re^{(n+m)\times(p+m)}$, $\hat{D} \in \Re^{m\times(p+m)}$, $Z_{r1}$, $Z_{r2}$, $Z_{a1}$, $Z_{a2} \in \Re^{m\times(n+m)}$, $Q_r, Q_a \in \Re^{(n+m)\times m}$ such that the inequalities[1](15)–(17) below are verified

$$\begin{bmatrix} X & \star & \star & \star & \star & \star \\ I_{n+m} & Y & \star & \star & \star & \star \\ Z_{r1} & Z_{r2} & 2S_r & \star & \star & \star \\ Z_{a1} & Z_{a2} & 0 & 2S_a & \star & \star \\ \mathbf{A}X+\mathbf{B}\hat{C} & \mathbf{A}+\mathbf{B}\hat{D}\mathbf{C} & \mathbf{B}S_r & \mathbf{B}_1S_a & X & \star \\ \hat{A} & Y\mathbf{A}+\hat{B}\mathbf{C} & Q_r & Q_a & I_{n+m} & Y \end{bmatrix} > 0 \tag{15}$$

$$\begin{bmatrix} X & \star & \star \\ I_{n+m} & Y & \star \\ \hat{C}_{(i)} - Z_{r1(i)} & \hat{D}_{(i)}\mathbf{C} - Z_{r2(i)} & \rho^2_{r(i)} \end{bmatrix} \ge 0 \tag{16}$$

$$\begin{bmatrix} X & \star & \star \\ I_{n+m} & Y & \star \\ X_{(n+i)} - Z_{a1(i)} & [0\ I_m]_{(i)} - Z_{a2(i)} & \rho^2_{a(i)} \end{bmatrix} \ge 0 \tag{17}$$

$\forall\, i = 1,\ldots,m$, then the dynamic compensator (2)–(4) with

$$\begin{aligned}\begin{bmatrix} A_c & B_c \\ C_c & D_c\end{bmatrix} &= \begin{bmatrix} N & Y\mathbf{B} \\ 0 & I_{n+m}\end{bmatrix}^{-1} \times \begin{bmatrix} \hat{A} - Y\mathbf{A}X & \hat{B} \\ \hat{C} & \hat{D}\end{bmatrix} \\ &\quad \times \begin{bmatrix} M' & 0 \\ \mathbf{C}X & I_{n+m}\end{bmatrix}^{-1} \\ F_c &= N^{-1}(Q_rS_r^{-1} - Y\mathbf{B}) \\ E_c &= N^{-1}(Q_aS_a^{-1} - Y\mathbf{B}_1)\end{aligned} \tag{18}$$

where matrices $M$ and $N$ verify $NM' = I_{n+m} - YX$, guarantees that the region $\mathcal{E}(P) \triangleq \{\xi \in \Re^{2(n+m)}; \xi'P\xi \le 1\}$, with $P \triangleq \begin{bmatrix} Y & N \\ N' & \bullet\end{bmatrix}$

[1] $\star$ stands for symmetric blocks; $\bullet$ stands for an element that has no influence on the development

and $P^{-1} \triangleq \begin{bmatrix} X & M \\ M' & \bullet \end{bmatrix}$, is a region of asymptotic stability for the closed-loop system (14).

*Proof:* Consider matrices $G_a, G_r \in \Re^{m\times(2(n+m))}$ and define the following sets:

$$\Xi(\rho_a) \triangleq \Big\{\xi \in \Re^{2(n+m)}; |\mathcal{L}_{(i)}\xi - G_{a(i)}\xi| \leq \rho_{a(i)},\ i=1,\dots,m\Big\}$$

$$\Xi(\rho_r) \triangleq \Big\{\xi \in \Re^{2(n+m)}; |\mathcal{K}_{(i)}\xi - G_{r(i)}\xi| \leq \rho_{r(i)},\ i=1,\dots,m\Big\}$$

Consider now the closed-loop system (14) and the candidate Lyapunov function $V(\xi(t)) = \xi(t)'P\xi(t)$, $P = P' > 0$. Define $\Delta V(\xi(t)) = V(\xi(t+1)) - V(\xi(t))$. From [20, Lemma 1], if $\xi(t) \in \Xi(\rho_a)\cap\Xi(\rho_r)$, it follows that[2] $\Delta V(\xi(t)) \leq \Delta V(\xi(t)) - 2\psi_{\rho_r}(t)'T_r[\psi_{\rho_r}(t) - G_r\xi(t)] - 2\psi_{\rho_a}(t)'T_a[\psi_{\rho_a}(t) - G_a\xi(t)]$, for any diagonal matrices $T_r, T_a > 0$, which can be re-writtten as $\Delta V(\xi(t)) \leq -\theta(t)'\Gamma\theta(t)$ with

$$\Gamma = \begin{bmatrix} P & \star & \star \\ -T_rG_r & 2T_r & \star \\ -T_aG_a & 0 & 2T_a \end{bmatrix} - \begin{bmatrix} -\mathcal{A}' \\ \mathcal{B}' \\ \mathcal{B}_1' \end{bmatrix} P \begin{bmatrix} -\mathcal{A} & \mathcal{B} & \mathcal{B}_1 \end{bmatrix}$$

and $\theta(t) = [\xi(t)'\ \psi_{\rho_r}(t)'\ \psi_{\rho_a}(t)']'$. Hence, from Schur's complement, it follows that $\Delta V(\xi(t)) < 0$ if $\xi(t) \in \Xi(\rho_a)\cap\Xi(\rho_r)$ and

$$\tilde{\Gamma} = \begin{bmatrix} P & \star & \star & \star \\ -T_rG_r & 2T_r & \star & \star \\ -T_aG_a & 0 & 2T_a & \star \\ -\mathcal{A} & \mathcal{B} & \mathcal{B}_1 & P^{-1} \end{bmatrix} > 0. \qquad (19)$$

Define now a matrix $\Pi = \begin{bmatrix} X & I_{n+m} \\ M' & 0 \end{bmatrix}$ [22]. Note that, from (15), it follows that $X - Y^{-1} > 0$. Therefore, $I_{n+m} - YX$ is nonsingular. Thus, it is always possible to compute square and nonsingular matrices $N$ and $M$ verifying the equation $NM' = I_{n+m} - YX$. From the nonsingularity of $M$ it is inferred that $\Pi$ is nonsingular. Consider now $S_a = T_a^{-1}$ and $S_r = T_r^{-1}$, and define $\Xi = blockdiag\,(-\Pi, S_r, S_a, P\Pi)$. From the definition of $\Pi$, and considering the following change of variables:

$$\begin{bmatrix} \hat{A} - Y\mathbf{A}X & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} N & Y\mathbf{B} \\ 0 & I_{n+m} \end{bmatrix} \times \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \times \begin{bmatrix} M' & 0 \\ \mathbf{C}X & I_{n+m} \end{bmatrix}$$

$$Q_r = Y\mathbf{B}S_r + NF_cS_r;$$

$$Q_a = Y\mathbf{B}_1S_a + NE_cS_a$$

$$G_r\Pi = [Z_{r1}\quad Z_{r2}];$$

$$G_a\Pi = [Z_{a1}\quad Z_{a2}]$$

it follows that $\Xi'\tilde{\Gamma}\Xi > 0$ corresponds to (15). Thus, since $X, Y, N, M, \Pi, S_r$ and $S_a$ are nonsingular, it follows that if (15) is verified, (19) holds with the matrices $A_c, B_c, C_c, D_c, E_c$ and $F_c$ defined as in (18).

On the other hand, it follows that $\mathcal{K}\Pi = [\hat{C}\quad D_c\mathbf{C}]$ and $\mathcal{L}\Pi = [[0\quad I_m]\quad 0]\Pi = [\tilde{X}\quad \tilde{I}]$, where $\tilde{X}$ and $\tilde{I}$ correspond respectively to the matrices composed by the last $m$ lines of matrices $X$ and $I_{n+m}$. Hence, left and right-multiplying inequalities (16) and (17) respectively by $blockdiag((\Pi^{-1})', 1)$ and its transpose, it is easy to see that the set of LMIs (16) and (17) ensures that $\mathcal{E}(P) \subset \Xi(\rho_r)$ and $\mathcal{E}(P) \subset \Xi(\rho_a)$ respectively [7].

[2]For ease of notation in the sequel we denote $\psi_r(t) = \psi_{\rho_r}(\mathcal{K}\xi(t))$ and $\psi_a(t) = \psi_{\rho_a}(\mathcal{L}\xi(t))$.

Thus, if (15)–(17) are satisfied, one obtains $\Delta V(\xi(k)) < 0$, $\forall\, \xi(k) \in \mathcal{E}(P)$, which means that $\mathcal{E}(P)$ is a contractive region for system (14), i.e., if $\xi(0) \in \mathcal{E}(P)$, then the corresponding trajectory converges asymptotically to the origin. ♢

*Remark 1:* Since a quadratic Lyapunov function is considered, the result of Theorem 1 guarantees regional exponential asymptotic stability. Note also that, due to the presence of integrators in the controller structure, our problem is equivalent to the synthesis of a linear saturating compensator for an open-loop system which will always present some eigenvalues equal to 1. From these facts, even if the open-loop system is asymptotic stable, it follows that it is not possible to achieve global asymptotic stabilization with the proposed approach.

*Remark 2:* As stated in Theorem 1, the use of the generalized sector condition, to take into account saturation effects, leads directly to LMI conditions for the synthesis of the controller. On the other hand, considering polytopic differential inclusions to model saturation (such as in [23] and [24]), it is worth noticing that it is not possible to achieve convex conditions for the synthesis of time-invariant controllers of type (2)–(4)

## V. Optimization Problems

According to Theorem 1, any feasible solution of the set of LMIs (15)–(17) provides a stabilizing, and probably different, dynamic controller. Among all these solutions, it is of interest to choose one that optimizes some particular objective. In this paper, the objective to be maximized will be the size of the domain of attraction of the closed-loop system, that is, the size of the projection of $\mathcal{E}(P)$ onto the states of the plant (i.e., $x$). This set is denoted as $\mathcal{E}_x(P)$ and is given by $\mathcal{E}_x(P) = \{x \in \Re^n; \exists v \in \Re^m, x_c \in \Re^{n+m}; [x'v'x_c']' \in \mathcal{E}(P)\} = \{x \in \Re^n; x'X_{11}^{-1}x \leq 1\}$, where $X_{11} \in \Re^{n\times n}$ is obtained from $X = \begin{bmatrix} X_{11} & \star \\ X_{21} & X_{22} \end{bmatrix}$.

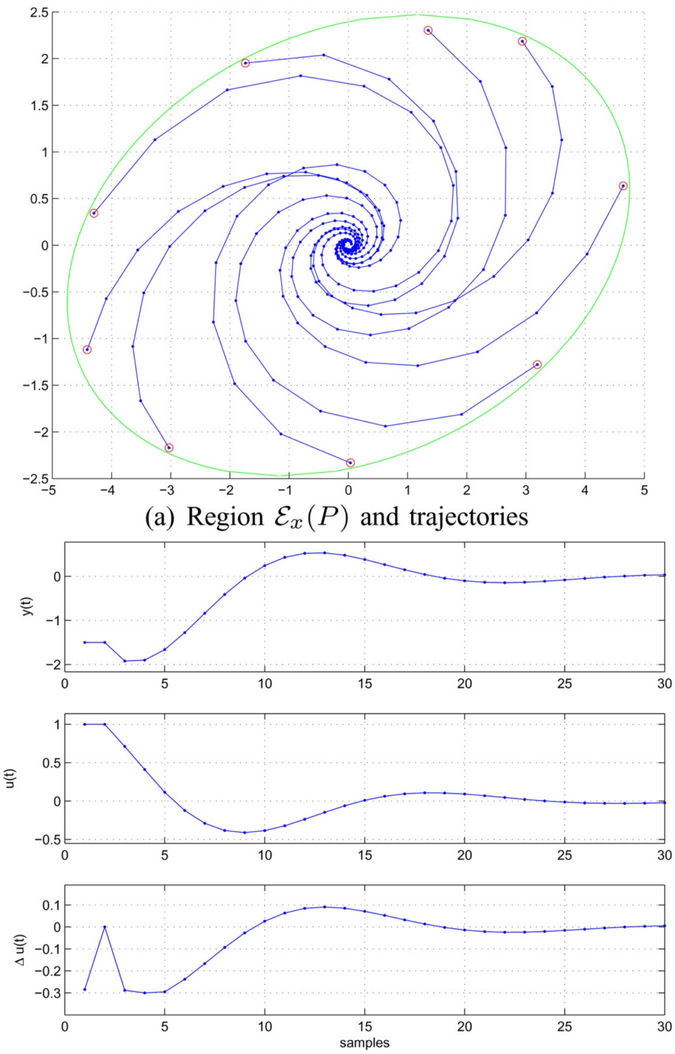
Note that for any initial state $x(0) \in \mathcal{E}_x(P)$, initial values of the states of the dynamic controller $v(0)$ and $x_c(0)$ can be found such that $\xi(0) \in \mathcal{E}(P)$, i.e., such that the asymptotically stability of the closed-loop system is ensured. In order to maximize the size of $\mathcal{E}_x(P)$, we can, for instance, consider the minimization of the trace of $X_{11}^{-1}$. Note that the trace of $X_{11}^{-1}$ is equal to the sum of the semi-axis of the $\mathcal{E}_x(P)$. This can be indirectly accomplished by considering an auxiliary matrix variable $R$ and the following optimization problem:

$$\min\ \text{trace}(R) \quad \text{s.t}\quad (15),(16),(17), \begin{bmatrix} R & I_n \\ I_n & X_{11} \end{bmatrix} > 0 \qquad (20)$$

Other optimization criteria on the size of $\mathcal{E}_x(P)$, leading to convex problems, can easily be stated, such as: minor axis maximization, maximization in certain directions and volume maximization (see details for instance in [23] and [24])

It is worth noticing that performance requirements such as contraction rate, pole placement of the closed-loop system or quadratic cost minimization can be added to the problem. In this case, the resulting optimization problem can also be written as an LMI optimization problem. For instance the controller can be designed in order to ensure some degree of time-domain performance in a neighborhood of the origin. In general we consider this neighborhood as the region of linear behavior of the closed-loop system [7], i.e., the region where saturation does not occur, which is defined as

$$R_L = \Big\{\xi \in \Re^{2(n+m)}; |\mathcal{K}_{(i)}\xi| \leq \rho_{r(i)},\ |\mathcal{L}_{(i)}\xi| \leq \rho_{a(i)},\quad i=1,\dots,m\Big\}$$




(a) Region $\mathcal{E}_x(P)$ and trajectories

(b) Evolution of $y(t)$, $u(t)$ and $\Delta u(t)$

Fig. 2. simulations.

In this case, the time-domain performance can be achieved if we consider the pole placement of matrix $\mathcal{A}$ in a suitable region inside the unit circle. Considering an LMI framework, the results stated in [22] can be used to place the poles in a so called LMI region in the complex plane. For example, if we verify the following LMI:

$$\begin{bmatrix} -\mu X & -\mu I_{n+m} & \star & \star \\ -\mu I_{n+m} & -\mu Y & \star & \star \\ \mathbf{A}X + \mathbf{B}\hat{C} & \mathbf{A} + \mathbf{B}\hat{D}\mathbf{C} & -\mu X & -\mu I_{n+m} \\ \hat{A} & Y\mathbf{A} + \hat{B}\mathbf{C} & -\mu I_{n+m} & -\mu Y \end{bmatrix} < 0 \quad (21)$$

it is easy to show that the poles of $\mathcal{A}$ will be placed in a disk centered on zero and with radius $0 < \mu < 1$. In this case, the smaller the $\mu$ the faster the decay rate of the time-response inside the linearity region.

*Remark 3:* If the open-loop system is stabilizable, LMIs (15)–(17) will always be feasible. Note that the solution of these LMIs with $Z_{1r} = Z_{2r} = Z_{1a} = Z_{2a} = 0$ (which corresponds to consider $G_a = G_r = 0$ in the generalized sector conditions) is feasible, since it corresponds to the synthesis of a dynamic linear compensator (as in [22]) leading to $\mathcal{E}(P) \subset R_L$. Of course, under performance constraints, this leads to small regions of stability (see discussion in [7]). The idea is therefore to explore the degrees of freedom in variables $Z_{1r}$, $Z_{2r}$, $Z_{1a}$, $Z_{1a}$ to obtain stability regions not included in $R_L$ (i.e., where saturations effectively occur).

TABLE I
TRADE-OFF: SATURATION × REGION OF STABILITY × PERFORMANCE

| | saturated solution | | linear solution | |
|---|---|---|---|---|
| $\mu$ | $trace(X_{11}^{-1})$ | $\sqrt{det(X_{11})}$ | $trace(X_{11}^{-1})$ | $\sqrt{det(X_{11})}$ |
| 0.99 | 0.0715 | 31.8254 | 0.0798 | 28.0904 |
| 0.9 | 0.2214 | 11.3793 | 0.3178 | 7.5111 |
| 0.8 | 0.7846 | 3.3509 | 1.0328 | 2.4725 |

## VI. NUMERICAL EXAMPLE

Consider the discrete-time linear system given by

$$A = \begin{bmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1]$$

with the following saturating limits: $|u(t)| \leq 1$ and $|\Delta u(t)| \leq 0.3$. In order to ensure a certain time-domain performance when the system is not saturated, we also consider that the poles of matrix $\mathcal{A}$ must be placed inside a disk of radius 0.9.

Solving the optimization problem (20) with the additional constraint (21), the following controller matrices are obtained:

$$A_c = \begin{bmatrix} 0.6326 & -0.0482 & 0.0166 \\ -0.8236 & 0.0628 & -0.0216 \\ 1.9616 & -0.1494 & 0.0515 \end{bmatrix},$$

$$E_c = \begin{bmatrix} 6.5054 \\ 30.9151 \\ -60.5314 \end{bmatrix},$$

$$B_c = \begin{bmatrix} 14.8846 & 8.1483 \\ 39.1828 & 24.2906 \\ -42.6207 & -142.5520 \end{bmatrix}, \quad F_c = \begin{bmatrix} 4.0208 \\ -16.2020 \\ -200.5261 \end{bmatrix},$$

$$C_c = [0.0079 \quad -0.0006 \quad 0.0002],$$

$$D_c = [-0.0582 \quad -0.5910].$$

The projection of the stability region $\mathcal{E}(P)$ onto the plant states is given by

$$\mathcal{E}_x(P) = \left\{ x \in \Re^2 : x' \begin{bmatrix} 0.0472 & -0.0223 \\ -0.0223 & 0.1742 \end{bmatrix} x \leq 1 \right\}$$

In Fig. 2(a) this contractive ellipsoid is shown as well as the trajectories of the controlled system for several initial states. For a given plant initial state, the initial controller states i.e $v(0)$ and $x_c(0)$, are chosen in such a way that $\xi(0) = [x(0)', v(0)', x_c(0)']'$ is in $\mathcal{E}(P)$. In Fig. 2(b) the evolution of the output system $y(t)$, control action $u(t)$ and increment of the control action $\Delta u(t)$ are depicted when the system starts from $x(0) = [2.8 - 1.5]'$, $v(0) = 1.2966$, $x_c(0) = [449.4707 - 709.7774 - 156.1419]'$. Notice that the limit requirements in $u(t)$ and $\Delta u(t)$ are satisfied thanks to the proposed saturating dynamic output feedback. Indeed, note that both $u(t)$ and $\Delta u(t)$ are effectively saturated in the first samples.

In Table I, the trade-off between saturation, the size of the stability region and the time-domain performance are illustrated in terms of the pole placement of matrix $\mathcal{A}$ in a disk of radius $\mu$. Considering different values of $\mu$, the values of $trace(X_{11}^{-1})$ and $\sqrt{det(X_{11})}$ (which is related to the volume of the ellipsoid), obtained from the solution of (20), are shown in two situations: the first one regarding the application of the results of Theorem 1, i.e., the saturation and the nonlinear behavior of the closed-loop system are effectively considered; the second one concerns the linear solution, i.e., the stability region is forced to be contained in the region of linear behavior of the closed-loop system $(R_L)$ in order to avoid saturation (i.e., $\mathcal{E}(P) \subset R_L$). As expected, the smaller the $\mu$ (i.e., more stringent is the performance requirement), the smaller the obtained stability region. On the other hand, the solutions considering saturation and nonlinear behavior lead to larger domains of stability.

## VII. Conclusion

In this paper a technique for the design of stabilizing dynamic output feedback controllers for discrete-time linear systems with rate and amplitude constrained actuators is proposed. This controller is composed of a classical linear dynamic compensator in cascade with an input saturating integrator system and two static anti-windup loops.

Theoretical conditions to ensure local (regional) stabilization of the closed-loop system, composed of the plant and the proposed controller, are formulate in LMI form. This allows the controller matrices to be computed in order to maximize the size of the domain of attraction, maintaining certain performance requirement, from the solution of convex optimization problems.

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# Output Feedback Stabilization for a Discrete-Time System With a Time-Varying Delay

Yong He, *Senior Member, IEEE*, Min Wu, *Senior Member, IEEE*, Guo-Ping Liu, *Senior Member, IEEE*, and Jin-Hua She, *Senior Member, IEEE*

***Abstract*—This study employs the free-weighting matrix approach to investigate the output feedback control of a linear discrete-time system with an interval time-varying delay. First, the delay-dependent stability is analyzed using a new method of estimating the upper bound on the difference of a Lyapunov function without ignoring any terms; and based on the results, a design criterion for a static output feedback (SOF) controller is derived. Since the conditions thus obtained for the existence of admissible controllers are not expressed strictly in terms of linear matrix inequalities, a modified cone complementarity linearization algorithm is employed to solve the nonconvex feasibility SOF control problem. Furthermore, the problem of designing a dynamic output feedback controller is formulated as one of designing an SOF controller. Numerical examples demonstrate the effectiveness of the method and its advantage over existing methods.**

***Index Terms*—Discrete-time systems, linear matrix inequality (LMI), output feedback, stabilization, time-varying delay.**

## I. Introduction

Increasing attention is being paid to the delay-dependent stability, stabilization, and $H_\infty$ control of linear systems with state delays (See,

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Y. He and M. Wu are with the School of Information Science and Engineering, Central South University, Changsha 410083, China (e-mail: min@csu.edu.cn).

G. -P. Liu is with the Faculty of Advanced Technology, University of Glamorgan, Pontypridd CF37 1DL, U.K. He is also with the CSIS Lab, Institute of Automation, Chinese Academy of Sciences, Beijing 100080, China.

J.-H. She is with the School of Computer Science, Tokyo University of Technology, Tokyo 192-0982, Japan.

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