

1
2

3

4

5

6

18
19
20
21

J.M.D.-B. is supported by project FEDER MEC MTM2009-08652 and ESF EUROCORES programme EuroGIGA - ComPoSe IP04 - MICINN Project EUI-EURC-2011-4306.

P.P.-L. is partially supported by project FEDER MEC MTM2009-08652 and CONICYT, FONDECYT/Iniciación 11110069.

R.F.-M partially supported by Conacyt of Mexico, grant 153984.

$$\begin{aligned}
f(n) &:= \max\{\rho(S) : S \text{ is a set of } n \text{ points}\} \\
f_{\text{col}}(n) &:= \max\{\rho_{\text{col}}(S) : n \text{ is even and } S \text{ is a set of } n \text{ red and } n \text{ blue points}\} \\
g(n) &:= \min\{\rho(S) : S \text{ is a set of } n \text{ points}\} \\
g_{\text{col}}(n) &:= \min\{\rho_{\text{col}}(S) : n \text{ is even and } S \text{ is a set of } n \text{ red and } n \text{ blue points}\}
\end{aligned}$$

22 In this paper we prove the following bounds.

$$\begin{aligned}
f(n) &= \Theta(n^4) \\
f_{\text{col}}(n) &= \Theta(n^4) \\
\Omega(n^3) &\leq g(n) \leq O(n^4) \\
\Omega(n) &\leq g_{\text{col}}(n) \leq O(n^2)
\end{aligned}$$

23 The first equality ($f(n) = \Theta(n^4)$) has been noted before in the literature. In
24 [8, 7, 15, 16] $f(n) = O(n^4)$ is proved. In [16] the author proves both $f(n) = O(n^4)$
25 and $f(n) = \Omega(n^4)$. As far as we know, all the other bounds are new.

26 A different problem but in the same setting has been studied recently in [15]. In
27 that paper, the authors study what a robot can infer from its environment when
28 all the information that is available is the cyclic positions of some landmarks as
29 they appear from the robot's position. Other authors have considered problems of
30 the same flavor, when a similar kind of information is available. See for example
31 [10, 11, 14]. In [8] the authors study the algorithmic problem of updating the radial
32 ordering of a moving observation point.

33 We point out that computing the radial ordering of S around every point *in* S
34 is an unavoidable step in some geometric algorithms, as for example, performing a
35 radial sweeping of a point set. Moreover, many optimization problems are solved
36 by considering the arrangement generated by every line passing through every pair
37 of points in S , and finding the optimum point inside each of the $O(n^4)$ cells in the
38 arrangement [12]. In many cases this is because the radial ordering of the points
39 in S around every point within a cell is the same. It could be interesting in this
40 scenario to know how many cells induce the same radial ordering.

41 For a bi-colored point set, a radial sweeping algorithm also requires the ordering
42 as an initial step, so it could be useful to know bounds on the number of different
43 colored radial ordering of S from points in the plane. From the combinatorial point
44 of view, this problem is related to partitioning bi-colored point sets with *k-fans*
45 [3, 4]. A *k-fan* in the plane is a point p (called the center) and k rays emanating
46 from p . This structure can be used to partition S into k monochromatic subsets and
47 it depends only on the colored radial ordering of S with respect to p . The existence
48 of balanced—each part having an equal number of red and blue points—*k-fans* for
49 colored point sets has been studied in recent papers [5, 6] but, as far as we know,
50 the number of different monochromatic partitions induced by *k-fans* has not yet
51 been considered.

52 The assumptions that S is in strong general position; that S has the same number
53 of red and blue points and that n is even, may seem arbitrary. However, the three
54 of them are crucial hypothesis in our results (see Section 4).

55 A preliminary version of this paper appeared in [9].

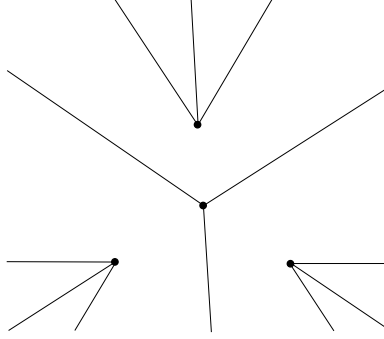


FIGURE 1. The order partition of a set of four points.

2. UNCOLORED CASE

56

57 In this section S is a set of n points in strong general position in the plane. We
 58 discretize the problem by partitioning the set of observation points into a finite
 59 number of sets so that points in the same set induce the same radial ordering. This
 60 partition is made by half-lines, which if crossed by an observation point, generate a
 61 transposition of two consecutive elements in the radial ordering. For every pair of
 62 points $x_1, x_2 \in S$, consider the line passing through them. Contained in this line we
 63 have two half-lines; one begins in x_1 and does not contain x_2 , while the other begins
 64 in x_2 and does not contain x_1 . Two observation points are in the same element
 65 of the partition if they can be connected by a path which does not intersect any
 66 half-line. We call this partition the *order partition* (see Figure 1). Since it induces
 67 a decomposition of the plane, we refer to its elements as *cells*. The order partition
 68 is used (under different names) also in [8, 7, 15]. Note that if a point moves in a
 69 path not crossing any half-line, the radial ordering with respect to this point is the
 70 same throughout the motion. Thus points in the same cell induce the same radial
 71 ordering. As a set of three points already shows, the converse is not true in general;
 72 two observation points may lie in different cells of the order partition and induce
 73 the same radial ordering of S .

74 As mentioned before, the following two bounds on $f(n)$ have been proved before;
 75 we provide proofs for completeness.

76 **Theorem 2.1.** $f(n) \leq O(n^4)$.

77 *Proof.* The order partition cannot have more cells than the arrangement induced
 78 by the lines passing through each pair of points in S . Such an arrangement has
 79 $O(n^4)$ cells. $\square$

80 **Theorem 2.2.** $f(n) \geq \Omega(n^4)$.

81 *Proof.* Follows from $f(n) \geq f_{\text{col}}(n)$ and Theorem 3.2 $\square$

82 We now prove an upper and a lower bound on $g(n)$. The upper bound follows
 83 from the upper bound on $f(n)$.

84 **Theorem 2.3.** $g(n) \leq O(n^4)$.

85 The lower bound on $g(n)$ is far more elaborate. First we show a lower bound of
 86 $\Omega(n^4)$ on the number of cells in the order partition. As far as we know, this is the
 87 first lower bound ever given on the size of the order partition.

88 **Theorem 2.4.** *The number of cells in the order partition of S is $\Omega(n^4)$.*

89 *Proof.* Let $\mathcal{L}$ be the set of lines passing through each pair of points in S . First we
 90 show that in the line arrangement $\mathcal{A}$ generated by $\mathcal{L}$ there are $\frac{n^4}{8} - O(n^3)$ cells.
 91 We may regard $\mathcal{A}$ as a plane graph G with an extra vertex v^* placed at infinity. So
 92 that v^* is contained in all the unbounded faces of G and twice in all the lines in $\mathcal{L}$.
 93 Let V , E and F be the number of vertices, edges and faces of G respectively. Note
 94 that by our strong general assumption G has exactly: one vertex of degree $n(n-1)$;
 95 n vertices of degree $2(n-1)$ and $V-n-1$ vertices of degree 4. Thus E equals
 96 $n(n-1)/2 + n(n-1) + 2V - 2n - 2$ which is $2V + O(n^2)$. By Euler's formula F equals
 97 $V + O(n^2)$. Let p and q be any two points in S . Let L_p be the set of lines in $\mathcal{L}$ that
 98 contain p but not q . Let L_q be the set of lines in $\mathcal{L}$ that contain q but not p . Note
 99 that each line in L_p intersects each line in L_q in a point not in S , with the exception
 100 of only two cases. The case in which both lines are parallel or the case in which
 101 they both contain the same point in $S \setminus \{p, q\}$. In total each case occurs at most
 102 $(n-2)$ times; thus p and q induce at least $(n-2)(n-2) - 2(n-2) = (n-2)(n-4)$
 103 vertices of degree 4. Doing this for every pair of vertices, we count each 4-degree
 104 vertex exactly four times. In total there are $\frac{n^4}{8} - O(n^3)$ vertices and $\frac{n^4}{8} - O(n^3)$
 105 faces in G (and the same number of cells in $\mathcal{A}$). The order partition can be obtained
 106 by removing from $\mathcal{A}$ each line segment joining a pair of points of S . Let $e_1, \dots, e_{\binom{n}{2}}$
 107 be these line segments in any given order. Let $\text{cr}(S)$ be the number of pairs of these
 108 edges that intersect in their interior. Let M be the number of vertices of $\mathcal{A}$ lying in
 109 the interior of any of the e_i 's. Note that for each set of four points of S we obtain:
 110 one vertex of M if the set is in convex position and three vertices otherwise. The
 111 number of sets of four elements of S that are in convex position is precisely $\text{cr}(S)$.
 112 Thus $M = 3\binom{n}{4} - 2\text{cr}(S)$. It is known that $\text{cr}(S)$ is bounded from below by $\frac{3}{8}\binom{n}{4}$
 113 (see [1, 2, 13]). Thus M is at most $\frac{9}{4}\binom{n}{4} = \frac{3}{32}n^4 - O(n^3)$. We remove each e_i
 114 in order, and show that at the end $\Omega(n^4)$ cells remain. Let d_i be the number of
 115 vertices in M lying in the interior of e_i just before it is removed. Note that when e_i
 116 is removed, in the worst case, $d_i + 1$ cells of $\mathcal{A}$ are lost. Thus when all of the e_i 's are
 117 removed at least $\frac{n^4}{8} - O(n^3) - \sum_{i=1}^{\binom{n}{2}} (d_i + 1) = \frac{n^4}{8} - O(n^3) - M - \binom{n}{2} \geq \frac{n^4}{32} - O(n^3)$
 118 cells remain. $\square$

119 We now prove a useful lemma for finding distinct radial orderings of S .

120 **Lemma 2.5** (Partition Lemma). *Let (R, B) be a partition of S into non-empty*
 121 *subsets. Let p and q be two points in different cells of the order partition. If no*
 122 *half-line spanned by a point in R and point in B intersects the line segment with*
 123 *endpoints p and q , then the radial orderings of S as seen from p and q are distinct.*

124 *Proof.* Since p and q lie in different cells of the order partition, the line segment
 125 joining them must intersect at least one half-line. Let $x_1, x_2 \in S$ be the points
 126 defining this half-line. Note that x_1 and x_2 are both in R or both in B . Let
 127 $x_3 \in S$ be a point in the element of the partition not containing x_1 and x_2 . Assume
 128 without loss of generality that the radial ordering of $\{x_1, x_2, x_3\}$ with respect to
 129 p is $[x_1, x_2, x_3]$. Since the line segment joining p and q crosses the half-line only

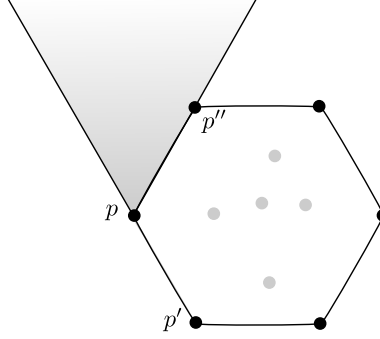


FIGURE 2. The cone in the proof of Theorem 2.7.

130 once, the radial ordering of $\{x_1, x_2, x_3\}$ with respect to q is $[x_2, x_1, x_3]$. Therefore,
 131 the radial ordering of S with respect to p is different from the radial ordering with
 132 respect to q . $\square$

133 The following Lemma is also proved in [7](Theorem 4.2) and in [15](Theorem 1),
 134 we include a proof for completeness.

135 **Lemma 2.6.** *Let p and q be two observation points in the interior of the convex*
 136 *hull of S , lying in different cells of the order partition. Then the radial orderings*
 137 *of S with respect to p and q are distinct.*

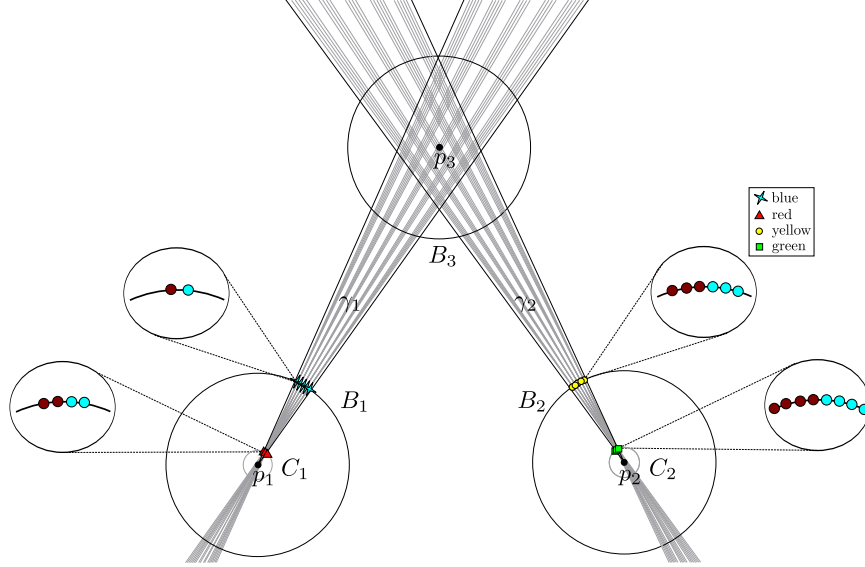
138 *Proof.* Let ℓ be the straight line containing p and q . Since p and q are in the
 139 interior of the convex hull of S , ℓ partitions S into two sets that together with p
 140 and q satisfy the conditions of the Partition Lemma. $\square$

141 Finally, we combine the above results to prove the lower bound on $g(n)$.

142 **Theorem 2.7.** $g(n) \geq \Omega(n^3)$.

143 *Proof.* An open cell of the order partition is *interior* if it intersects the interior of
 144 the convex hull of S and it is *exterior* if it has a point not in the interior of the
 145 convex hull of S . Note that a cell can be both interior and exterior. If less than
 146 half of the the cells are exterior then at least half of them are interior and we are
 147 done by Theorem 2.4 and Lemma 2.6. Assume then, that at least half of the cells
 148 are exterior. Thus there are $\Omega(n^4)$ exterior cells.

149 Let C be the convex hull of S and m be its number of vertices. Let p be one
 150 of these vertices. Let p' and p'' be the vertices previous and next to p in C in
 151 clockwise order respectively. Let Γ_p be the convex cone with apex p , bounded by:
 152 the infinite ray with apex p and passing through p'' and the infinite ray with apex p'
 153 and passing through p (see Figure 2). Let $R := \{p\}$ and $B := S \setminus \{p\}$. Note that any
 154 two points in Γ_p lying in different cells of the order partition, together with R and
 155 B , satisfy the conditions of the Partition Lemma. Therefore, the radial orderings of
 156 S with respect to any two points in Γ_p lying in different cells of the order partition
 157 are distinct. For each vertex of C define such a cone. Every exterior cell intersects
 158 one of these cones. Therefore there is a cone intersecting $\Omega(n^4/m) = \Omega(n^3)$ of them
 159 and the result follows. $\square$

FIGURE 3. A bi-colored set with $\Omega(n^4)$ different colored radial orderings.

3. COLORED CASE

In this section n is even, and S is a set of n red and n blue points in strong general position in the plane. We prove upper and lower bounds on $f_{\text{col}}(n)$ and $g_{\text{col}}(n)$.

Theorem 3.1. $f_{\text{col}}(n) \leq O(n^4)$.

Proof. Follows from $f_{\text{col}}(n) \leq f(n)$ and Theorem 2.1 □

Theorem 3.2. $f_{\text{col}}(n) \geq \Omega(n^4)$.

Proof. Assume that $n \geq 20$. We start by constructing a four-colored set of points S' with $\Omega(n^4)$ distinct colored radial orderings; afterwards we obtain S by replacing each point of a given color with a suitable “pattern” of red and blue points. These four patterns are chosen so that if they appear consecutively in a radial ordering, then any other equivalent radial ordering must match patterns of the same type. Since the patterns behave like the original four colors, the new set also has $\Omega(n^4)$ colored radial orderings.

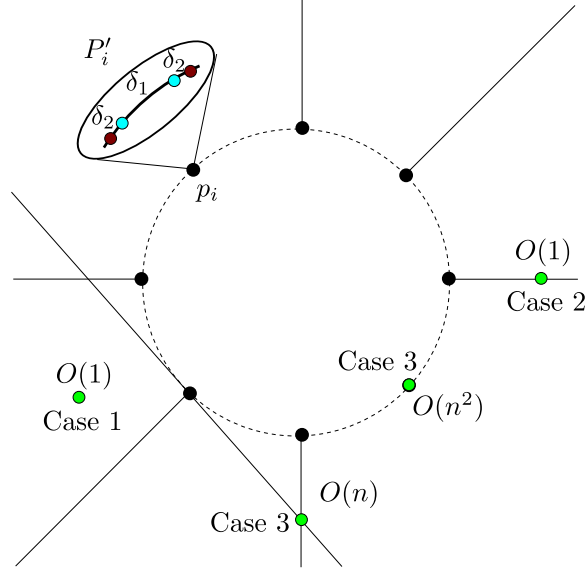
Let B_1 , B_2 and B_3 be three disks of radius $1/4$, whose centers p_1 , p_2 , and p_3 are the vertices of an equilateral triangle of side length equal to one. Let $\varepsilon, \alpha > 0$. Let C_1 and C_2 be circles of radius ε centered at p_1 and p_2 , respectively. Let γ_1 and γ_2 be infinite wedges of angle α , with apices p_1 and p_2 respectively. Assume that γ_1 is bisected by the line segment joining p_1 and p_3 , while γ_2 is bisected by the line segment joining p_2 and p_3 . Refer to Figure 3. Let m and r be the only natural numbers such that $n = 10m + r$ and $10 \leq r \leq 19$. Partition γ_1 with m infinite rays emanating from p_1 , such that the angle between two consecutive rays equals $\alpha/(m+1)$. Do likewise for γ_2 , with m infinite rays emanating from p_2 . At every point of intersection of these rays with the boundary of B_1 , place a blue point; at

every point of intersection with C_1 a red point; at every point of intersection with the boundary of B_2 a yellow point; finally, at every point of intersection with C_2 a green point. Thus m points of each color are placed. This ends the construction of S' .

Let L be the set of lines passing through a red and a blue point. Let L' be the set of lines passing through a yellow and a green point. Choose α and ε small enough so that the following conditions are met: (1) Every line in $L \cup L'$ intersects the interior of B_3 and these are the only lines passing through two points of S' that do. (2) Neither two lines in L nor two lines in L' intersect at a point in the interior of B_3 . (3) Every line in L intersects every line in L' at a point in B_3 . By the previous conditions and the fact that $|L| = m^2$ and $|L'| = m^2$, $L \cup L'$ splits B_3 into precisely $(m+1)^4$ cells. For each of these cells choose a point q_i in its interior. We show that the colored radial orderings of S' as seen from each of these points is different. Note that for each point in B_3 there exists a line separating the red and blue points from the green and yellow points. Thus we may assume, that the colored radial orderings as seen from points in B_3 are written so that all the blue and red points appear before the green and yellow points. Let q_i and q_j be two points in B_3 , belonging to different cells of the order partition. Consider the colored radial ordering when walking from q_i to q_j in a straight line. By conditions (1) and (2), the only transpositions that occur when a half-line is crossed is between a red and a blue point or between a yellow and a green point. This implies that the k -th red point is always the same red point and that the number of blue points after the k -th red point is either increasing or decreasing monotonically; the same observation holds for the green and yellow points. Therefore, in the walk once a line in $L \cup L'$ is crossed, all colored radial orderings afterwards will be distinct. Thus the number of different colored radial orderings of S' is at least $(m+1)^4$, which is $\Omega(n^4)$.

To construct S , we replace the points in S' by patterns of red and blue points, in such a way that the colored radial orderings at points q_i remain different. The points in the patterns replacing a point $p \in S'$ are placed consecutively in the same circle containing p . If these points are placed close enough to p , then they will appear consecutively in the colored radial ordering with respect to every point q_i . The points of S' are replaced in the following way: every blue point with a pattern of one red and one blue point; every red point with a pattern of two red and two blue points; every yellow point with a pattern of three red and three blue points; and every green point with a pattern of four red and four blue points. Refer to Figure 3. Note that our choice of patterns implies that two equivalent radial orderings must match patterns of the same type. So far, $10m$ red and $10m$ blue points have been placed. The remaining $2r$ points can be placed in such a way that in the radial ordering with respect to every point q_i the r red points appear consecutively followed by the r blue points (for example in the line segment joining p_1 and p_2). This final condition guarantees that the colored radial orderings at each q_i remain different. $\square$

To prove the lower bound on $g(n)$, we construct a two-colored set, P , of $2n$ points, in strong general position, with at most $O(n^2)$ colored radial orderings. The construction is somewhat involved. We employ a similar technique as in the proof of Theorem 3.2. We start with a set of $n/2$ points in strong general position in the unit circle. All the points have the same color and thus the colored radial

FIGURE 4. A bi-colored set with $O(n^2)$ different colored radial orderings.

232 orderings are all equivalent. Afterwards, we replace each point with a symmetric
 233 pattern of red and blue points. This is done in such a way that the new number of
 234 distinct colored radial orderings increases to at most $O(n^2)$.

235 Before detailing the construction of P , we present some definitions and a helpful
 236 lemma. A set of lines is in *general position* if not three lines share a common point
 237 and every two lines intersect. Every line ℓ in the plane is defined as $\ell := \{t \in \mathbb{R} : p + t \cdot v\}$,
 238 for some point p in the plane and some unit vector v in $\mathbb{R}^2$. Define the
 239 distance between two lines ℓ_1 and ℓ_2 as the minimum of $\|p_1 - p_2\| + \|v_1 - v_2\|$, taken
 240 over all pairs $(p_1, v_1), (p_2, v_2)$ that define ℓ_1 and ℓ_2 , respectively. Let L be a set
 241 of lines in the plane. For $\delta > 0$, a δ -*perturbation* of L is any set of lines obtained
 242 from L by replacing each line by a line at a distance of at most δ . Likewise S' is a
 243 δ -*perturbation* of S if it can be obtained from S by replacing each point by a point
 244 at a distance of at most δ .

245 **Lemma 3.3.** *Let L be a set of lines in general position and L' a δ -perturbation of L .
 246 If δ is small enough then the line arrangement $\mathcal{A}$ generated by L is combinatorially
 247 equivalent to the line arrangement $\mathcal{A}'$ generated by L' .*

248 *Proof.* Let ℓ_1 and ℓ_2 be two lines in L and ℓ'_1 and ℓ'_2 be the corresponding lines in
 249 L' . Let p be the point of intersection of ℓ_1 and ℓ_2 and let p' be the intersection of
 250 ℓ'_1 and ℓ'_2 . Note that if δ is small enough, p' is a point. Let $\varepsilon > 0$. Choose δ small
 251 enough so that the intersection of any two lines in L' is a point, and such that its
 252 distance to the corresponding intersection point in L is at most ε .

253 Now, choose ε small enough so that the intersection points of any two pairs of
 254 lines in L' are at a distance greater than 0. Move each line of L' continuously
 255 to its corresponding line in L , monotonically decreasing their distance throughout
 256 the process. Note that since through out the motion not three lines can share a

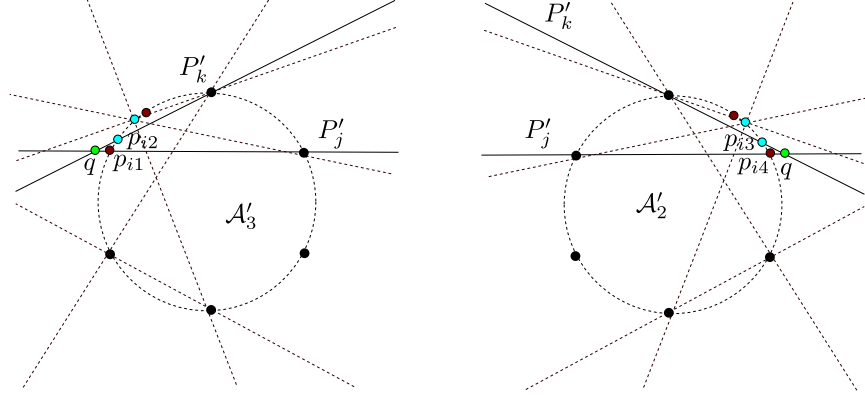


FIGURE 5. Schematic depiction of conditions (2) and (3).

257 point there are no combinatorial changes in $\mathcal{A}'$. Therefore, $\mathcal{A}'$ is combinatorially
 258 equivalent to $\mathcal{A}$. $\square$

259 We now proceed with the construction of P . Let P'' be a set of $n/2$ points in
 260 strong general position on the unit circle C , centered at the origin. Let L'' be the set
 261 of lines passing through every pair of points in P'' , together with the lines passing
 262 through every point of P'' and tangent to C . Let $\mathcal{A}''$ be the line arrangement
 263 generated by L'' .

264 Let $\delta_1, \delta_2 > 0$, such that $\delta_1 > 2\delta_2$. We form a new set of points P' , by replacing
 265 each point p_i of P'' with a set of four points $P'_i := \{p_{i1}, p_{i2}, p_{i3}, p_{i4}\}$, placed clockwise
 266 consecutively in C , and colored “red, blue, blue, red” respectively. Place: p_{i1} at p ;
 267 p_{i4} at distance δ_1 from p_{i1} ; p_{i2} at distance δ_2 from p_{i1} , and p_{i3} at distance δ_2 from
 268 p_{i4} . See Figure 4.

269 For $1 \leq k \leq 4$, let L'_k be the set of lines passing through every pair of points
 270 (p_{ik}, p_{jk}) , together with the lines passing through every point of p_{ik} and tangent
 271 to C . Let $\mathcal{A}'_k$ be the line arrangement generated by L'_k . Note that $\mathcal{A}'', \mathcal{A}'_1, \mathcal{A}'_2, \mathcal{A}'_3$
 272 and $\mathcal{A}'_4$ are all combinatorially equivalent.

273 Choose δ_1 small enough so that the following condition is met.

- 274 (1) Let C_1, C_2 and C_3 be three closed elements (vertices, edges or cells) in
 275 $\mathcal{A}'_i, \mathcal{A}'_j$ and $\mathcal{A}'_k$, respectively. If C_1, C_2 and C_3 have a non empty intersection,
 276 their corresponding elements in $\mathcal{A}''$ have a non empty intersection.

277 Choose δ_2 small enough with respect to δ_1 so that the following conditions are
 278 met:

- 279 (2) Let ℓ_1 be a line passing through a point of P'_j and p_{i1} ($i \neq j$). Let ℓ_2 be
 280 a line passing through a point of P'_k and p_{i2} ($i \neq k$ and $j \neq k$). If q is the
 281 intersection point of ℓ_1 and ℓ_2 , then q is in the same cell of $\mathcal{A}'_3$ as p_{i1} . (See
 282 Figure 5)
 283 (3) Let ℓ_1 be a line passing through a point of P'_j and p_{i4} ($i \neq j$). Let ℓ_2 be
 284 a line passing through a point of P'_k and p_{i3} ($i \neq k$ and $j \neq k$). If q is the
 285 intersection point of ℓ_1 and ℓ_2 , then q is in the same cell of $\mathcal{A}'_2$ as p_{i4} . (See
 286 Figure 5)

Finally, let P be a δ_3 -perturbation of P' , with $\delta_3 < \delta_2$, such that P is in strong general position and its elements still belong to C . Relabel the points so that p_{ik} is now in P . Define L_k , P_k and $\mathcal{A}_k$ with respect to P . We choose δ_3 small enough so that: Every $\mathcal{A}_k$ is combinatorially equivalent to A'' (Lemma 3.3) and conditions (1), (2) and (3) are still met (with $\mathcal{A}_2, \mathcal{A}_3, P_j, P_k$ taking the place of $\mathcal{A}'_2, \mathcal{A}'_3, P'_j, P'_k$).

Theorem 3.4. $g_{\text{col}}(n) \leq O(n^2)$.

Proof. We prove that P has $O(n^2)$ colored radial orderings. Let q be an observation point in the plane. Since there is only one cell of the order partition of P inside its convex hull, we assume that q lies outside the convex hull of P . Let C_k be the cell containing q in $\mathcal{A}_k$. Let D_k be the cell corresponding to C_k in $\mathcal{A}''$. Let $D := D_1 \cap D_2 \cap D_3 \cap D_4$. By condition (1) and Helly's Theorem, D is non-empty.

There are three cases depending on whether D is a cell, an edge or a vertex of $\mathcal{A}''$ (see Figure 4):

Case 1. D is a cell. In this case, every pattern replacing a point of P'' will appear consecutively in the colored radial ordering around q . Moreover, by the symmetry of the patterns they are all “red, blue, blue, red”. So in this case there is only one possible colored radial ordering.

Case 2. D is an edge. We distinguish two sub-cases, whether the edge is contained in a line passing through two points p_i and p_j of P'' or whether it is contained in one of the tangent lines passing through a point p_k of P'' .

In the first sub-case, the patterns at points different from p_i and p_j will appear consecutively in the colored radial ordering (thus each pattern appears as “red, blue, blue, red”). However, the points in the patterns at p_i and p_j will appear together, but they may be intermixed. Since there are only 8 points involved, there is only a constant number of ways in which this can happen. The second sub-case is similar.

Case 3. D is a vertex. We split the analysis on whether D is in P'' or not.

- D is a point in P'' . Let p_i be the point in P'' such that p_i equals D . First suppose that q is neither in the cell of $\mathcal{A}_3$ that contains p_{i1} nor in the cell of $\mathcal{A}_2$ that contains p_{i4} . By conditions (2) and (3) in the colored radial ordering with respect to q , points p_{i1} and p_{i2} appear consecutively or the points between them all belong to some P_j . Similarly for points p_{i3} and p_{i4} and some P_k . There are at most a constant number of ways in which the points of P_j and P_k can be intermixed with $\{p_{i1}, p_{i2}\}$ and $\{p_{i3}, p_{i4}\}$ respectively. The patterns replacing other points of P'' appear consecutively (thus each pattern appears as “red, blue, blue, red”). Summarizing, the points in $\{p_{i1}, p_{i2}\} \cup P_j$ appear consecutively as do the points in $\{p_{i3}, p_{i4}\} \cup P_k$ and the remaining points appear alternatively as “red, blue, blue, red”. There are only $O(n)$ ways in which $\{p_{i1}, p_{i2}\} \cup P_j$ can appear with respect to $\{p_{i3}, p_{i4}\} \cup P_k$.

Now suppose that q is in the cell of $\mathcal{A}_2$ that contains p_{i4} . By conditions (2) and (3) in the colored radial ordering of $P \setminus \{p_{i3}, p_{i4}\}$ with respect to q , p_{i1} and p_{i2} appear consecutively; the remaining points of $P \setminus \{p_{i3}, p_{i4}\}$, appear alternatively as “red, blue, blue, red”. Thus there are only two possible radial orderings (depending on the position of p_{i2} with respect to p_{i1}) of $P \setminus \{p_{i3}, p_{i4}\}$ with respect to q . The only thing remaining to consider are the $O(n^2)$ ways in which p_{i3} may appear with respect to p_{i4} .

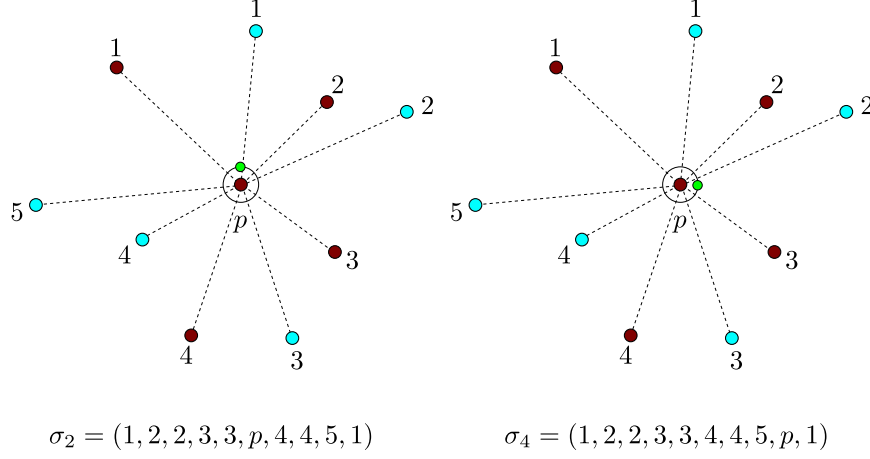


FIGURE 6. A walk showing a linear lower bound on the number of colored radial orderings.

The case where q is in the cell of $\mathcal{A}_3$ that contains p_{i1} is similar to the case where q is in the cell of $\mathcal{A}_2$ that contains p_{i4} . In this case there are also at most $O(n^2)$ colored radial orderings.

- D is not a point in P'' .

In this case D is the intersection point of two lines ℓ_1 and ℓ_2 in L'' . These lines may be defined by two points or one point of P'' (depending on whether they are tangent lines or not). In both situations, we have that the patterns at the points of P'' defining each line appear together but intermixed. The patterns at any other points will appear consecutively (thus each pattern appears as “red, blue, blue, red”). Since there are at most four points defining ℓ_1 and ℓ_2 , the number of ways in which their respective pattern points can appear is at most a constant. The only thing left to consider is the $O(n)$ number of ways in which the patterns at the points defining ℓ_1 can appear with respect to those of ℓ_2 .

There are at most $O(n^2)$ distinct colored radial orderings in every case. $\square$

We now give a linear lower bound for the number of colored radial orderings of S . Some notation is required. For a given radial ordering σ of S , let $\sigma(i)$ ($0 \leq i < 2n$) be its $(i + 1)$ th-element (where addition is taken modulo $2n$). Thus, two radial orderings σ and ρ are equivalent whenever there exists a natural number j such that $\sigma(i) = \rho(i + j)$ for all i ; they are equivalent as colored radial orderings when the color of $\sigma(i)$ is equal to the color of $\rho(i + j)$ for all i .

Theorem 3.5. $g_{\text{col}}(n) \geq n$.

Proof. Throughout the proof we use both *colored* and *non-colored* radial orderings. In each instance we explicitly mention to which of the two types of radial orderings we are referring to. To obtain the claimed lower bound, we show a walk in which n distinct *colored* radial orderings are seen. First, we choose a red point p of S and let $\mathcal{C}$ be a circle centered at p . After that, we walk once clockwise around $\mathcal{C}$.

361 We choose $\mathcal{C}$ to be small enough so that the only half-lines crossed in the walk are
 362 those involving p (see Figure 6).

363 Consider the *non-colored* radial orderings seen in this walk. Note that since $\mathcal{C}$
 364 does not cross any half-line defined by points of $S \setminus \{p\}$, the points of $S \setminus \{p\}$ remain
 365 fixed in these *non-colored* radial orderings. The only point that changes position
 366 is p ; it moves counter clockwise, transposing an element of $S \setminus \{p\}$ every time a
 367 half-line is crossed. We prove that every time that p transposes a blue point, a new
 368 different *colored* radial ordering is seen.

369 Since in the walk around $\mathcal{C}$, the blue points of S always appear in the same
 370 *non-colored* radial ordering, we assume that all the *non-colored* radial orderings of
 371 S as seen from points in $\mathcal{C}$ are written starting at the same blue point. Among
 372 these radial orderings, for each $k = 0, \dots, n-1$, let σ_k be the radial ordering in
 373 which p is just after the $(k+1)$ -th blue point; refer to Figure 6. We show that all
 374 the *colored* radial orderings associated with these n *non-colored* radial orderings
 375 are distinct.

376 Let then σ_k and σ_l be two such *non-colored* radial orderings encountered at
 377 points q_1 and q_2 in the walk, respectively. Suppose that σ_k and σ_l are equivalent
 378 as *colored* radial orderings. Then there exist a fixed natural number j such that for
 379 all $i = 0, \dots, 2n-1$, the color of $\sigma_k(i)$ is equal to the color of $\sigma_l(i+j)$.

380 We now define a directed graph that captures the relationship between σ_k and
 381 σ_l ; we employ the structure of this graph to conclude that k must equal l . Let G
 382 be the directed graph whose vertex set is S and in which, for all $i = 0, \dots, 2n-1$
 383 there is an arc from $\sigma_k(i)$ to $\sigma_l(i+j)$; see Figure 7. Note that every vertex in G has
 384 indegree and outdegree equal to one. Therefore, G is the union of pairwise disjoint
 385 directed cycles of points of the same color.

386 Let Γ be the cycle containing p and let $S' := S \setminus V(\Gamma)$. Let ρ_1 and ρ_2 be the
 387 *non-colored* radial orderings of S' as seen from q_1 and q_2 , respectively. We make
 388 the additional assumption that ρ_1 is written starting at $\sigma_k(0)$ while ρ_2 is written
 389 starting at $\sigma_l(2n-j)$ (note that being blue, these points are not in Γ). Since in
 390 particular p is not in S' , these radial orderings are equivalent. Therefore, there
 391 exists a fixed natural number j' such that $\rho_1(i) = \rho_2(i+j')$, for all i . Since S'
 392 comes from removing the points of Γ , ρ_1 can be formed by removing the elements
 393 of Γ from σ_k and ρ_2 can be formed by writing σ_l starting at $\sigma_l(2n-j)$ and then
 394 removing the elements of Γ (See Figure 7). Thus the color of $\rho_1(i)$ is equal to the
 395 color of $\rho_2(i)$ for all i .

396 Let G' be the directed graph whose vertex set is S' and in which there is an
 397 arc from $\rho_1(i)$ to $\rho_2(i)$. As before, every vertex in G' has indegree and outdegree
 398 equal to one. Therefore G' is the union of disjoint cycles of vertices of the same
 399 color (in fact G' is the subgraph of G induced by S'). Since ρ_2 is just a “shift” of
 400 j' places to the right of ρ_1 , all of these cycles have the same length m . Therefore,
 401 both the number of red and blue points in S' are multiples of m . This implies that
 402 the number of vertices in Γ is also a multiple of m .

403 Let $r \cdot m$ be the length of Γ , since Γ is not empty, then $r \geq 1$. Assume that Γ , start-
 404 ing from p is given by $(p = v_1, v_2, \dots, v_m, \dots, v_{2m}, \dots, v_{rm})$. Let b_k be the $(k+1)$ -th
 405 blue point and $\Gamma' := (b_k = u_1, \dots, u_m)$ be the cycle in G containing b_k . Consider
 406 the following sequence of pairs of vertices $(u_1, v_1), (u_2, v_2), \dots, (u_m, v_m)$. Note that
 407 in σ_k , the point $v_1 = p$ is just after the point $u_1 = b_k$; afterwards, for $2 \leq i \leq m$, the
 408 point v_i is just after the point u_i in both σ_l and σ_k . (Recall that the order of $S \setminus \{p\}$

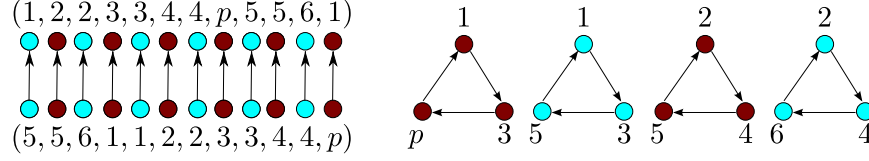


FIGURE 7. Two equivalent colored radial orderings and their corresponding graph.

in σ_k and σ_l is the same.) Suppose that $r > 1$, then the point v_{m+1} is just after u_1 in σ_l while in σ_k it is just after the point v_1 (which is equal to p). Consider now the following sequence of vertices $(v_1, v_{m+1}), (v_2, v_{m+2}), \dots, (v_{(r-1)m+1}, v_{rm+1} = p)$. From the same arguments as before, for $m+2 \leq i \leq rm$, the point v_i is just after the point v_{i-m} in both σ_k and σ_l . For $i = rm+1$, the point v_{rm+1} is just after the point $v_{(r-1)m+1}$ in σ_l , but $v_{rm+1} = p$ and $v_{(r-1)m+1}$ is red, a contradiction since p is just after a blue point in σ_l . Thus $r = 1$. This implies that $\sigma_k = \sigma_l$, since v_1 (which is equal to p) is after u_1 (which is equal to b_k) in both σ_l and σ_k . $\square$

4. CONCLUSIONS

We proved an upper bound of $O(n^4)$ and a lower bound of $\Omega(n^3)$ on the number of radial orderings that every set of n points in strong general position in the plane must have. The upper bound was first given in [7]. As a corollary in the same paper it was noted that every set of n points in the plane contains $O(n^4)$ different star shaped polygonizations. Our result implies that every set of n points in strong general position in the plane has $\Omega(n^3)$ different star shaped polygonizations. We leave the closing of this gap as an open problem:

Conjecture 4.1. $g(n) = \Theta(n^4)$.

We used the assumption that S is in strong general position heavily on the proof of the lower bound on $g(n)$. However, we believe that it is not needed and that only general position (no three collinear points) is sufficient.

Conjecture 4.2. *If S is a set of n points in general position in the plane, then it has at least $\Omega(n^4)$ distinct radial orderings.*

For colored point sets the situation is far more intriguing, here we have been able to prove that such a gap exists. Mainly that there are bi-colored sets of $2n$ points with $\Theta(n^4)$ colored radial orderings and sets with only $\Theta(n^2)$. The best lower bound we have been able to provide is of $\Omega(n)$. We make the following conjecture.

Conjecture 4.3. $g_{\text{col}}(n) = \Theta(n^2)$.

Note that we used the assumption that n is even heavily in the proof of Theorem 3.4. In fact, it can be shown that the number of colored radial orderings may increase to $\Theta(n^3)$ if a red and a blue point are added to the construction in the proof of Theorem 3.4. Also in the proof of Theorem 3.5 we relied on the fact that the number of red points equals the number of blue points. It is possible to construct a set of n red and $n-1$ blue points such that a walk like the one described in Theorem 3.5 yields only one colored radial ordering. It may be the case that the bounds given in Theorems 3.4 and 3.5 no longer hold when either one of these two

hypothesis is dropped.

Acknowledgments. The problems studied here, were introduced and partially solved during a stay at Universidad de la Habana, Cuba, January 2010. The authors would like to thank our host Carlos Ochoa during this visit, and Clemens Huemer; Merce Claverol and David Wood for helpful comments.

REFERENCES

- [1] B. M. Ábrego and S. Fernández-Merchant. A lower bound for the rectilinear crossing number. *Graphs Combin.*, 21(3):293–300, 2005.
- [2] J. Balogh and G. Salazar. k -sets, convex quadrilaterals, and the rectilinear crossing number of K_n . *Discrete & Comput. Geom.*, 35(4):671–690, 2006.
- [3] I. Bárány and I. Matoušek. Equipartition of two measures by a 4-fan. *Discrete & Comput. Geom.*, 27:293–301, 2002.
- [4] I. Bárány and J. Matoušek. Simultaneous partitions of measures by k -fans. *Discrete & Comput. Geom.*, 25:317–334, 2001.
- [5] S. Bereg. Equipartitions of measures by 2-fans. *Discrete & Comput. Geom.*, 34:87–96, 2005.
- [6] S. Bespamyatnikh, D. Kirkpatrick, and J. Snoeyink. Generalizing ham-sandwich cuts to equitable subdivisions. *Discrete & Comput. Geom.*, 24:605–622, 2000.
- [7] L. Deneen and G. Shute. Polygonizations of point sets in the plane. *Discrete & Comput. Geom.*, 3(1):77–87, 1988.
- [8] O. Devillers, V. Dujmović, H. Everett, S. Hornus, S. Whitesides, and S. Wismath. Maintaining visibility information of planar point sets with a moving viewpoint. *Internat. J. Comput. Geom. Appl.*, 17(4):297–304, 2007.
- [9] J. M. Díaz-Báñez, R. Fabila-Monroy, and P. Pérez-Lantero. On the number of radial orderings of colored planar point sets. In *Computational Geometry (XIV Spanish Meeting on Computational Geometry)*, volume 7579 of *LNCS*, pages 126–137, Alcalá de Henares, Spain, 2011.
- [10] B. Gfeller, M. Mihalák, S. Suri, E. Vicari, and P. Widmayer. Counting targets with mobile sensors in an unknown environment. In *Proceedings of the 3rd international conference on Algorithmic aspects of wireless sensor networks, ALGOSENSORS’07*, pages 32–45, Berlin, Heidelberg, 2008. Springer-Verlag.
- [11] O. Kannianen and T. M. R. Allho. Minimalist navigation for a mobile robot based on a simple visibility sensor information. In *AI and Machine Consciousness, Proceedings of the 13th Finnish Artificial Intelligence Conference STeP 2008*, pages 68–75, 2008.
- [12] S. Langerman and Steiger W. Optimization in arrangements. In *In Proc. 20th Sympos. Theoret. Aspects Comput. Sci.*, Lect. Notes in Comput. Sci., vol. 2607, pages 50–61, Berlin, Heidelberg, 2003. Springer-Verlag.
- [13] L. Lovász, K. Vesztegombi, Uli Wagner, and Emo Welzl. Convex quadrilaterals and k -sets. In *Towards a theory of geometric graphs*, volume 342 of *Contemp. Math.*, pages 139–148. Amer. Math. Soc., Providence, RI, 2004.
- [14] S. Suri, E. Vicari, and P. Widmayer. Simple robots with minimal sensing: From local visibility to global geometry. *Int. J. Rob. Res.*, 27:1055–1067, September 2008.
- [15] B. Tovar, L. Freda, and S. M. LaValle. Using a robot to learn geometric information from permutations of landmarks. In *Topology and robotics*, volume 438 of *Contemp. Math.*, pages 33–45. Amer. Math. Soc., Providence, RI, 2007.
- [16] R. Ulber. On the number of star-shaped polygons and polyhedra. In *Proc. 11th Canadian Conference on Computational Geometry CCCG’99*, Vancouver, Canada, 1999.

490

491 DEPARTAMENTO MATEMÁTICA APLICADA II.

492 UNIVERSIDAD DE SEVILLA

493 SEVILLE, SPAIN

494 *E-mail address:* `dbanez@us.es`

495

496 DEPARTAMENTO DE MATEMÁTICAS

497 CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL INSTITUTO POLITÉCNICO NACIONAL.

498 MEXICO CITY, MEXICO

499 *E-mail address:* `ruyfabila@math.cinvestav.edu.mx`

500

501 ESCUELA DE INGENIERÍA CIVIL EN INFORMÁTICA.

502 UNIVERSIDAD DE VALPARAÍSO

503 VALPARAISO, CHILE

504 *E-mail address:* `pablo.perez@uv.cl`