Rotation sets for graph maps of degree 1*

Lluís Alsedà Sylvie Ruette

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Abstract

A *topological graph* is a compact connected set X containing a finite subset E such that each connected component of $X \setminus E$ is homeomorphic to an open interval. A *tree* is a topological graph with no loop (i.e., no subset homeomorphic to a circle). Thanks to their one-dimensional character, dynamical systems on topological graphs have some properties which are similar to those of interval maps. For example a transitive graph map with at least one periodic point has a positive topological entropy and a dense set of periodic points [1, Corollary 5.3.11]; note that a transitive graph map with no periodic point is topologically conjugate to an irrational rotation on the circle [2].

A point x is periodic of period n for the map f if $f^n(x) = x$ and $f^i(x) \neq x$ for all $1 \leq i \leq n-1$. Call $Per(f)$ the set of periods:

$$Per(f) = \{n \in \mathbb{N}^* \mid \text{there exists a periodic point of period } n\}.$$

For interval maps the striking Sharkovskii Theorem states that the existence of a periodic point of a given period implies the existence of other periods which are determined by the following order:

$$3 \rhd 5 \rhd 7 \rhd 9 \rhd \cdots \rhd 2{\cdot}3 \rhd 2{\cdot}5 \rhd 2{\cdot}7 \rhd \cdots \rhd 2^2{\cdot}3 \rhd 2^3{\cdot}5 \rhd \cdots \rhd 2^\infty \rhd \cdots 2^3 \rhd 2^2 \rhd 2 \rhd 1;$$

if the continuous map $f\colon [0,1] \to [0,1]$ has a periodic point of period n then it has periodic points of period m for all integers $m \lhd n$. In other words, the set $Per(f)$ is equal to $\{n \in \mathbb{N}^* \mid m \unlhd n\}$ for some $n \in \mathbb{N}^* \cup \{2^\infty\}$ (f is said of *type* n). Moreover all the possibilities occur, that is, for every $n \in \mathbb{N}^* \cup \{2^\infty\}$ there exists a continuous interval map of type n.

Finding a generalisation of the Sharkovskii Theorem for graph maps is a big challenge and in general it is not known what the sets of periods may be. The characterisation of $Per(f)$ was given for some classes of graphs, in particular for n-stars (an n-star is a tree composed of n intervals with a common endpoint) [3] and for continuous maps on trees such that all the branching points are fixed [4]. For a continuous map on the circle the set of periods depends on the degree d of f (see e.g. [1]). If $d \neq -1, 0, 1$ then $Per(f) = \mathbb{N}^*$ (if $d = -2$ the case $Per(f) = \mathbb{N}^* \setminus \{2\}$ is also possible). If $d \in \{0, -1\}$ then $Per(f)$ is ruled by the Sharkovskii order, that is,

*oral communication.

$Per(f) = \{m \in \mathbb{N}^* \mid m \trianglelefteq n\}$ for some $n \in \mathbb{N}^* \cup \{2^\infty\}$. The case $d = 1$ is more complex; the answer is given by the rotation set theory, that we expose briefly below.

Let $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ be the circle and $\pi: \mathbb{R} \to \mathbb{S}^1$ the natural projection. Consider a continuous map $f: \mathbb{S}^1 \to \mathbb{S}^1$ of degree 1 and $F: \mathbb{R} \to \mathbb{R}$ a lifting of f, that is, $f \circ \pi = \pi \circ F$. The rotation number of $x \in \mathbb{S}^1$ is defined as

$$\rho(x) = \lim_{n \to +\infty} \frac{F^n(\hat{x}) - \hat{x}}{n},$$

where $\pi(\hat{x}) = x$; the limit (when it exists) does not depend on the choice of $\hat{x}$. It is the asymptotic rotation speed of $f^n(x)$ on the circle. Let $Rot(f) = \{\rho(x) \mid x \in \mathbb{S}^1\}$ be the rotation set of f. Then $Rot(f)$ is a compact nonempty interval. A periodic point of period q has a rational rotation number of the form p/q for some $p \in \mathbb{Z}$. Reciprocally, if $p/q \in Rot(f)$ there exists a periodic point of rotation number p/q. More precisely, if $p/q \in Int(Rot(f))$ then for every integer $n \geq 1$ there exists a periodic point x of period nq with $\rho(x) = p/q$; if $\min Rot(f) = p/q$ with p, q coprime then there exists $s \in \mathbb{N}^* \cup \{2^\infty\}$ such that the set of periods of periodic points of rotation number p/q is equal to $\{nq \mid q \trianglelefteq s\}$ (where $\trianglelefteq$ is the Sharkovskii order). The same result holds for $\max Rot(f)$. Finally we get the following characterisation of the set of periods of f: if $Rot(f) = [a, b]$ then

$$Per(f) = \{n \in \mathbb{N}^* \mid \exists k \in \mathbb{Z},\ a < \tfrac{k}{n} < b\} \cup S(a, s_a) \cup S(b, s_b)$$

where $s_a, s_b \in \mathbb{N}^* \cup \{2^\infty\}$, $S(r, s) = \emptyset$ if $r \notin \mathbb{Q}$ and $S(p/q, s) = \{nq \mid n \trianglelefteq s\}$ if p, q are coprime.

The aim of this work is to generalise the rotation set theory to continuous maps $f: G \to G$ of degree 1 where G is a topological graph with a unique loop S. There is no difficulty to extend the definition of rotation numbers, and a periodic point still has a rational rotation number. In this setting the rotation set $Rot(f)$ may not be connected and presently we do not know if it is closed. However the subset $Rot_S(f)$ of rotation numbers of points belonging to the loop S has properties which are similar to, although weaker than, those of the rotation interval for a circle. The set $Rot_S(f)$ is a compact non empty interval and, if $\bigcup_{n \geq 0} f^n(S)$ is dense in G (in particular if f is transitive) then $Rot_S(f) = Rot(f)$. If $p/q \in Rot_S(f)$ then there exists a periodic point of rotation number p/q, and if $p/q \in Int(Rot_S(f))$ then for all integers n great enough there exists a periodic point of period nq of rotation number p/q. We derive from these results that if $Rot_S(f)$ is not reduced to a single point then $\mathbb{N} \setminus Per(f)$ is finite. Actually most of these results are valid in a more general class of spaces that we will define below.

References

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