

# Ordered Median Tree Location Problem (OMT)

Advances on data analysis, logistics and transportation problems on complex networks

*Group reunion*

(Ongoing work)

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Justo Puerto Albandoz<sup>1,2</sup>

Miguel A. Pozo Montaño<sup>1</sup>

Alberto Torrejón Valenzuela<sup>2</sup>

IMUS 21/12/2021



<sup>1</sup> Department of Statistics and Operational Research, University of Seville (Seville, Spain).

<sup>2</sup> Institute of Mathematics of the University of Seville (Seville, Spain).

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## **Introduction**

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$$|V| = 8 \quad p = 3$$

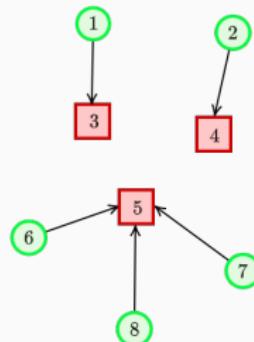


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Following the structure in *Pozo, Puerto, Rodríguez-Chía (2020): The ordered median tree of hubs location problem.*

$$|V| = 8 \quad p = 3$$

PMED

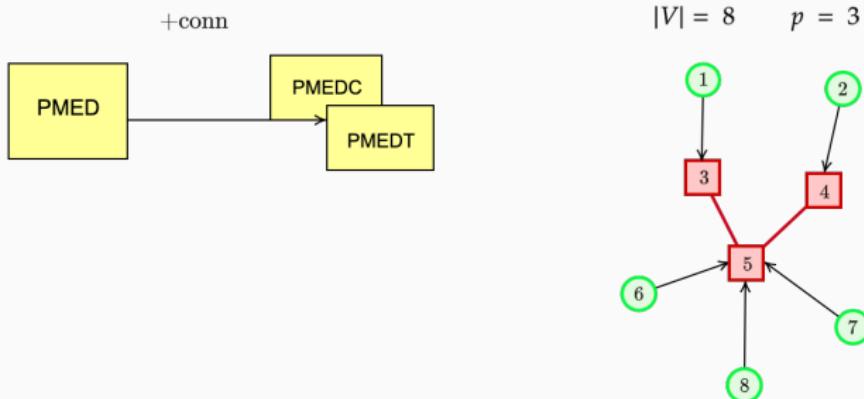


**PMED** *p*-median location problem<sup>1</sup>.

Find the location of  $p$  servers and assignments that minimizes the sum of the weighted allocation cost of each node to a server.

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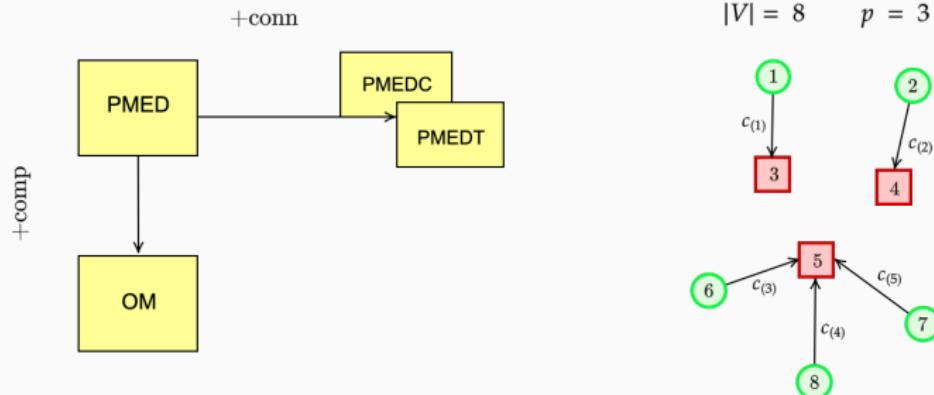
<sup>1</sup>Hakimi (1964)



**PMEDC/PMEDT**

*p*-median location problem with inner (tree) connection structure.

Solve the *p*-median problem connecting servers with a (tree) network.

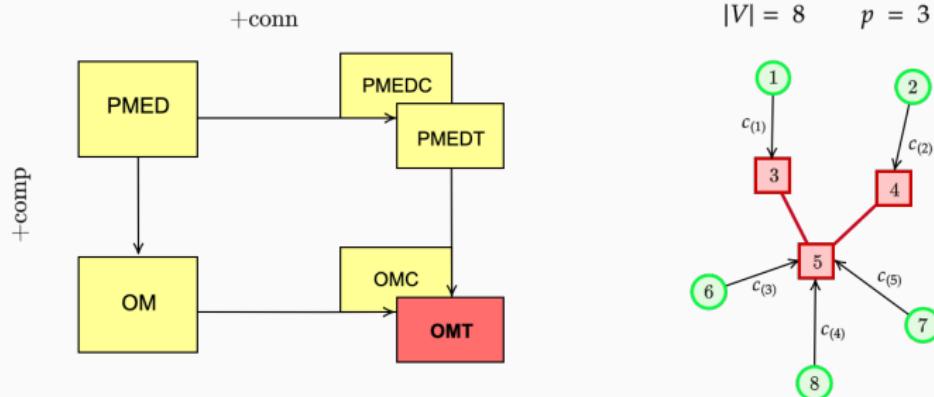


## OM Ordered Median location problem<sup>2</sup>.

Find the location of  $p$  servers and assignments that minimize the sum of the sorted weighted allocation cost of each node to a server.

<sup>2</sup>Nikel, Puerto (2005)

## OMT framework: problem case



OMC/OMT

Ordered Median location problem with inner (tree) connection structure.

Solve the OM connecting servers with a (tree) network.

## **Problem description**

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# Problems involved and considerations

## - Allocation

- Locate  $p$  servers
- Single allocation, each client is allocated to exactly one server
- All clients can potentially become servers
- Is costless for a server to allocate to itself
- Is costless to use a node as a server

## - Connection

- Servers are connected using an inner tree structure

## - Compensation

- Allocation to servers is compensated using scaling factor parameters

### Problem objective

Minimize the total rank dependant compensated allocation cost of the system (weighted by the number of allocations considered) plus the routing through the tree of servers cost (weighted by the number of edges in the tree of servers).

## Notation

$G$	undirected weighted network
$V$	set of network nodes
$E$	set of edges connecting network nodes
$i, j \in V$	indexes for the network nodes
$p$	fixed number of servers to locate
$c_{ij}$	$(i, j) \in E$ (allocation/design) cost
$\ell \in V$	index for the $\ell$ -th position of the sorted sequence allocation costs
$\lambda_\ell$	scaling factor for the $\ell$ -th allocation cost
$D = (V, A)$	the directed network of $G$ of set of arcs $A$
$S \subset V$	subset of nodes
$\delta(S)$	cut-set of $S$ , edges with one node in $S$ and other node outside $S$
$\delta^-(S)$	cut-set directed into $S$
$\delta^+(S)$	cut-set directed out of $S$

An **arborescence** rooted at  $r \in V$  of  $D$  is a subgraph  $D' = (V, A', r)$  where  $A' \subseteq A$  such that each non-root node has exactly one incoming/outcoming edge (thus  $|A'| = |V| - 1$ ) and  $D'$  has no cycles.

## Compensation situations

Several situations can be model depending on  $\lambda$ :

- **median criterion:**  $\lambda = (\underbrace{1, 1, \dots, 1}_{|V|})$
- **$k$ -centrum criterion:**  $\lambda = (\underbrace{0, \dots, 0}_{\lfloor \alpha |V| \rfloor}, \underbrace{1, \dots, 1}_{|V| - \lfloor \alpha |V| \rfloor})$
- **$k$ -trimmed mean criterion:**  $\lambda = (\underbrace{0, \dots, 0}_{\lfloor \alpha |V| \rfloor}, \underbrace{1, \dots, 1}_{|V| - \lfloor (\alpha + \beta) |V| \rfloor}, \underbrace{0, \dots, 0}_{\lfloor \beta |V| \rfloor})$

## **Problem formulation**

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# Variables declaration

## Variables (general)

- $x_{ij} \in \{0, 1\} \forall i, j \in V$ , is 1 if client  $i$  assigned to server  $j$ .
- $z_{ij} \in \{0, 1\} \forall i, j \in V$ , is 1 if  $(i, j)$  is an edge connecting servers.
- $x_{ij}^\ell \in \{0, 1\} \forall i, j, \ell \in V$ , is 1 if client  $i$  is assigned to server  $j$  and assignation cost  $(i, j)$  is ranked in the  $\ell$ -th position.

## Variables (MTZ)

- $y_{ij} \in \{0, 1\} \forall i, j \in V$ , is 1 iff  $(i, j)$  is an arc of the arborescence.
- $l_i \geq 0 \forall i \in V$ , is the position that node  $i$  occupies in the arborescence respect to the root node.

## Variables (flow based)

- $f_{ij} \geq 0 \forall i, j \in V$  is the amount of flow through arc  $(i, j)$  in the tree of servers.
- $r_i \in \{0, 1\} \forall i \in V$ , is 1 if the node  $i \in V$  is selected as the source node for the tree of servers.

## Subtour elimination formulation

$$F^{sub} : \min \quad \frac{1}{\sum_{\ell \in V} \lambda_{\ell}} \sum_{\ell \in V} \sum_{i,j \in V} \lambda_{\ell} c_{ij} x_{ij}^{\ell} + \frac{1}{p-1} \sum_{i,j \in V} c_{ij} z_{ij} \quad (1a)$$

$$\text{s.t.} \quad \sum_{i \in V} x_{ii} = p \quad (1b)$$

$$\sum_{j \in V} x_{ij} = 1 \quad \forall i \in V \quad (1c)$$

$$x_{ij} \leq x_{jj} \quad \forall i, j \in V \quad (1d)$$

$$2z_{ij} \leq x_{ii} + x_{jj} \quad \forall i, j \in V \quad (1e)$$

$$\sum_{i,j \in V: i < j} z_{ij} = p - 1 \quad (1f)$$

$$\sum_{i,j \in S} z_{ij} \leq |S| - 1 \quad \forall S \neq \emptyset, S \subset V \quad (1g)$$

$$\sum_{i,j \in V} x_{ij}^{\ell} \leq 1 \quad \forall \ell \in V \quad (1h)$$

$$\sum_{i,j \in V} c_{ij} x_{ij}^{\ell} \leq \sum_{i,j \in V} c_{ij} x_{ij}^{\ell+1} \quad \forall \ell \in V : \ell < |V| \quad (1i)$$

$$\sum_{\ell \in V} x_{ij}^{\ell} = x_{ij} \quad \forall i, j \in V \quad (1j)$$

$$x_{ij}, z_{ij}, x_{ij}^{\ell} \in \{0, 1\} \quad \forall i, j, \ell \in V \quad (1k)$$

Implemented through a branch-and-cut subtour elimination algorithm via connected components identification

# MTZ formulation

$$F^{mtz} : \min \quad \frac{1}{\sum_{\ell \in V} \lambda_\ell} \sum_{\ell \in V} \sum_{i,j \in V} \lambda_\ell c_{ij} x_{ij}^\ell + \frac{1}{p-1} \sum_{i,j \in V} c_{ij} z_{ij} \quad (2a)$$

$$\text{s.t.} \quad \sum_{i \in V} x_{ii} = p \quad (2b)$$

$$\sum_{j \in V} x_{ij} = 1 \quad \forall i \in V \quad (2c)$$

$$x_{ij} \leq x_{jj} \quad \forall i, j \in V \quad (2d)$$

$$2z_{ij} \leq x_{ii} + x_{jj} \quad \forall i, j \in V \quad (2e)$$

$$\sum_{i,j \in V : i < j} z_{ij} = p - 1 \quad (2f)$$

$$\sum_{(j,i) \in \delta^-(i)} y_{ji} = 1 \quad \forall i \in V \setminus \{r\} \quad (2g)$$

$$y_{ij} + y_{ji} = z_{ij} \quad \forall i, j \in V : i < j \quad (2h)$$

$$l_j \geq l_i + 1 - n(1 - y_{ij}) \quad \forall i, j \in V \quad (2i)$$

$$l_r = 1 \quad (2j)$$

$$2 < l_i < p \quad \forall i \in V \setminus \{r\} \quad (2k)$$

$$\sum_{i,j \in V} x_{ij}^\ell \leq 1 \quad \forall \ell \in V \quad (2l)$$

$$\sum_{i,j \in V} c_{ij} x_{ij}^\ell \leq \sum_{i,j \in V} c_{ij} x_{ij}^{\ell+1} \quad \forall \ell \in V : \ell < |V| \quad (2m)$$

$$\sum_{\ell \in V} x_{ij}^\ell = x_{ij} \quad \forall i, j \in V \quad (2n)$$

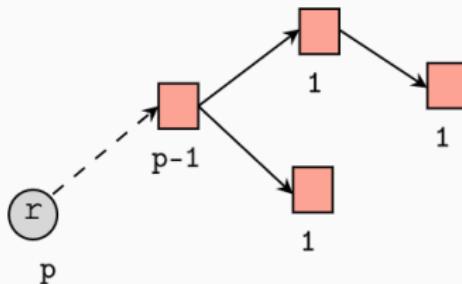
$$x_{ij}, y_{ij}, z_{ij}, x_{ij}^\ell \in \{0, 1\}, l_i \geq 0 \quad \forall i, j, \ell \in V \quad (2\tilde{n})$$

## Flow based formulations

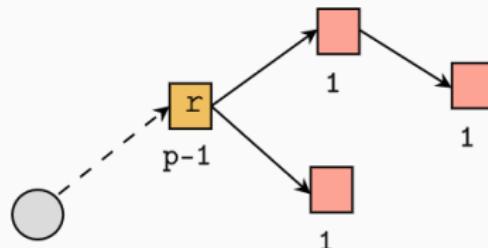
This formulation relies on a source node  $r \in V$  which distributes the flow.

Contrary to the MTZ formulation, the choice of the root node is very influential as care must be taken when spreading the flow. This selection can be done in two ways:

- Adding a set of variables  $r_i \in \{0, 1\} \forall i \in V$ , is 1 if the node  $i \in V$  is selected as the source node for the tree of servers.
- Arbitrarily selecting the source node and distributing the flow along the tree, distinguishing whether the node selected as the source is a server or not (figure below).



Non-server source node



Server source node

## Flow based formulations

$$F_1^{\text{flow}} : \min \quad \frac{1}{\sum_{\ell \in V} \lambda_\ell} \sum_{\ell \in V} \sum_{i,j \in V} \lambda_\ell c_{ij} x_{ij}^\ell + \frac{1}{p-1} \sum_{i,j \in V} c_{ij} z_{ij} \quad (3a)$$

$$\text{s.t.} \quad \sum_{i \in V} x_{ii} = p \quad (3b)$$

$$\sum_{j \in V} x_{ij} = 1 \quad \forall i \in V \quad (3c)$$

$$x_{ij} \leq x_{jj} \quad \forall i, j \in V \quad (3d)$$

$$2z_{ij} \leq x_{ii} + x_{jj} \quad \forall i, j \in V \quad (3e)$$

$$\sum_{i,j \in V : i < j} z_{ij} = p - 1 \quad (3f)$$

$$\sum_{i \in V} r_i = 1 \quad (3g)$$

$$r_i \leq x_{ii} \quad \forall i \in V \quad (3h)$$

$$\sum_{(i,j) \in \delta^+(i)} f_{ij} - \sum_{(j,i) \in \delta^-(i)} f_{ji} = (p-1)r_i - (x_{ii} - r_i) \quad \forall i \in V \quad (3i)$$

$$f_{ij} \leq (p-1)z_{ij} \quad \forall i, j \in V : i < j \quad (3j)$$

$$f_{ji} \leq (p-1)z_{ij} \quad \forall i, j \in V : i < j \quad (3k)$$

$$\sum_{i,j \in V} x_{ij}^\ell \leq 1 \quad \forall \ell \in V \quad (3l)$$

$$\sum_{i,j \in V} c_{ij} x_{ij}^\ell \leq \sum_{i,j \in V} c_{ij} x_{ij}^{\ell+1} \quad \forall \ell \in V : \ell < |V| \quad (3m)$$

$$\sum_{I \in V} x_{ij}^\ell = x_{ij} \quad \forall i, j \in V \quad (3n)$$

$$r_i, x_{ij}, z_{ij}, x_{ij}^\ell \in \{0, 1\} \quad \forall i, j, \ell \in V \quad (3\tilde{n})$$

## Flow based formulations

$$F_2^{\text{flow}} : \min \quad \frac{1}{\sum_{\ell \in V} \lambda_\ell} \sum_{\ell \in V} \sum_{i,j \in V} \lambda_\ell c_{ij} x_{ij}^\ell + \frac{1}{p-1} \sum_{i,j \in V} c_{ij} z_{ij} \quad (4a)$$

$$\text{s.t.} \quad \sum_{i \in V} x_{ii} = p \quad (4b)$$

$$\sum_{j \in V} x_{ij} = 1 \quad \forall i \in V \quad (4c)$$

$$x_{ij} \leq x_{jj} \quad \forall i, j \in V \quad (4d)$$

$$2z_{ij} \leq x_{ii} + x_{jj} \quad \forall i, j \in V \quad (4e)$$

$$\sum_{i,j \in V: i < j} z_{ij} = p - 1 \quad (4f)$$

$$\sum_{(i,j) \in \delta^+(i)} f_{ij} - \sum_{(j,i) \in \delta^-(i)} f_{ji} = px_{ri} - x_{ii} \quad \forall i \in V \quad (4g)$$

$$f_{ij} \leq (p-1)z_{ij} \quad \forall i, j \in V : i < j \quad (4h)$$

$$f_{ji} \leq (p-1)z_{ij} \quad \forall i, j \in V : i < j \quad (4i)$$

$$\sum_{i,j \in V} x_{ij}^\ell \leq 1 \quad \forall \ell \in V \quad (4j)$$

$$\sum_{i,j \in V} c_{ij} x_{ij}^\ell \leq \sum_{i,j \in V} c_{ij} x_{ij}^{\ell+1} \quad \forall \ell \in V : \ell < |V| \quad (4k)$$

$$\sum_{\ell \in V} x_{ij}^\ell = x_{ij} \quad \forall i, j \in V \quad (4l)$$

$$x_{ij}, z_{ij}, x_{ij}^\ell \in \{0, 1\} \quad \forall i, j, \ell \in V \quad (4m)$$

## **Computational experience I**

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$F^{subelim}$				$F^{mtz}$				$F_1^{flow}$			$F_2^{flow}$		
N	p	cpu	gap	opt	cpu	gap	opt	cpu	gap	opt	cpu	gap	opt
20	5	12.42	0.00	5	9.67	0.00	5	6.12	0.00	5	5.37	0.00	5
20	6	14.95	0.00	5	9.16	0.00	5	5.10	0.00	5	6.31	0.00	5
20	10	12.19	0.00	5	3.78	0.00	5	2.92	0.00	5	2.63	0.00	5
30	7	201.43	0.00	5	97.38	0.00	5	135.67	0.00	5	54.01	0.00	5
30	10	170.04	0.00	5	58.26	0.00	5	37.26	0.00	5	29.44	0.00	5
30	15	289.87	0.00	5	16.37	0.00	5	17.79	0.00	5	15.00	0.00	5
40	10	1068.18	0.00	5	344.56	0.00	5	366.53	0.00	5	291.57	0.00	5
40	13	2731.51	3.18	3	333.48	0.00	5	350.11	0.00	5	202.05	0.00	5
40	20	2709.80	0.75	3	128.97	0.00	5	215.18	0.00	5	37.86	0.00	5
50	12	3600.17	26.91	0	1903.51	0.00	5	2576.82	1.68	3	1726.42	0.00	5
50	16	3500.90	28.94	1	425.58	0.00	5	367.01	0.00	5	463.44	0.00	5
50	25	3600.30	757.68	0	261.16	0.00	5	119.20	0.00	5	67.12	0.00	5
60	15	3600.16	727.30	0	2450.88	1.33	4	3036.14	3.92	2	2181.56	1.02	4
60	20	3463.82	714.32	1	794.80	0.00	5	663.83	0.00	5	394.66	0.00	5
60	30	3600.43	1422.85	0	932.32	0.52	4	951.71	0.05	4	141.86	0.00	5
70	17	3600.25	853.96	0	2882.69	9.05	3	2091.34	0.00	5	2133.86	0.22	4
70	23	3600.29	1220.63	0	2761.70	0.43	4	2980.91	0.35	2	2210.71	0.38	4
70	35	3600.46	1584.22	0	1047.98	0.26	4	1558.11	0.00	4	251.31	0.00	5
80	20	3600.34	1104.53	0	3600.34	10.23	0	3600.97	10.56	0	3124.01	16.44	1
80	26	3600.42	1323.93	0	2885.87	1.46	2	3247.65	2.18	2	2434.59	0.73	3
80	40	3600.68	1924.47	0	2176.22	0.22	3	3600.25	1.18	0	1362.71	0.00	5
90	22	3605.03	1287.25	0	3603.49	54.55	0	3602.80	33.68	0	3039.30	10.74	1
90	30	3602.36	1547.73	0	2777.30	1.64	2	2791.34	0.90	3	1511.64	0.00	5
90	45	3601.45	1934.19	0	2737.79	1.08	2	2270.14	0.38	3	691.77	0.00	5
100	25	3603.60	1390.43	0	3601.19	71.17	0	3600.59	18.23	0	3601.91	18.30	0
100	33	3601.25	1633.03	0	3279.68	25.10	1	3146.47	6.43	1	2670.32	0.81	3
100	50	3603.37	2072.06	0	2563.29	0.73	2	2369.44	0.60	2	2186.59	0.32	3

## **Improvements**

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## Initial solution (heuristic)

- PMEDT + OM heuristic algorithm

- Solve the PMEDT
  - Fix variables  $\bar{x}_{ij}, \bar{z}_{ij}$  of the PMEDT solution
  - Solve the OM over the fixed variables, i.e., compute  $\bar{x}_{ij}^\ell$  solution

- OM + MST heuristic algorithm

- Solve the OM
  - Fix variables  $\bar{x}_{ij}, \bar{x}_{ij}^\ell$  of the OM solution and the set of servers identified

$$\bar{x} = \{x_{ii} \mid x_{ii} = 1, \forall i \in V\}$$

- Solve MST problem over the set of servers  $\bar{x}$  (*Kruskal or Prim algorithm*)

# Covering formulation

Consider the ordered sequence of unique costs:

$$c_{(0)} = 0 < c_{(1)} < \dots < c_{(|H|)} = \max_{i,j \in V} c_{ij}$$

## Variables (covering)

- $u_{\ell h} \in \{0, 1\} \forall i, j \in V$  if the  $\ell$ -th allocation cost is at least  $c_{(h)}$ .

$$F^{cov} : \min \quad \frac{1}{\sum_{\ell \in V} \lambda_{\ell}} \sum_{\ell \in V} \sum_{h \in H} \lambda_{\ell} u_{\ell h} (c_{(h)} - c_{(h-1)}) + \frac{1}{p-1} \sum_{i,j \in V} c_{ij} z_{ij} \quad (5a)$$

$$\text{s.t.} \quad \sum_{i \in V} x_{ii} = p \quad (5b)$$

$$\sum_{j \in V} x_{ij} = 1 \quad \forall i \in V \quad (5c)$$

$$x_{ij} \leq x_{jj} \quad \forall i, j \in V \quad (5d)$$

$$2z_{ij} \leq x_{ii} + x_{jj} \quad \forall i, j \in V \quad (5e)$$

$$z_{ij} \in \mathcal{T} \quad \forall i, j \in V \quad (5f)$$

$$\sum_{\ell \in V} u_{\ell h} = \sum_{i,j \in V: c_{ij} \geq c_{(h)}} x_{ij} \quad \forall h \in H \quad (5g)$$

$$u_{\ell h} \leq u_{\ell+1h} \quad \forall h \in H, \forall \ell \in V : \ell < |V| \quad (5h)$$

$$x_{ij}, z_{ij}, u_{\ell h} \in \{0, 1\} \quad \forall i, j, \ell \in V, \forall h \in H \quad (5i)$$

## Covering formulation: preprocessings

Fixing  $u$ -variables to 1

$$u_h^\ell = 1 \quad \forall \ell \in \{H_h^1 + 1, \dots, N\}$$

$$F_1^{pre} : \max H_h^1 := \sum_{i \in V} \sum_{j \in V} x_{ij}$$

$$\text{s.t. } \sum_{i \in V} x_{ii} = p$$

$$\sum_{j \in V} x_{ij} \leq 1$$

$$x_{ij} \leq x_{jj}$$

$$i \in V \quad (6c)$$

$$c_{ij} x_{ij} \leq c_{(h-1)}$$

$$x_{ij} \in \{0, 1\}$$

Fixing  $u$ -variables to 0

$$u_h^\ell = 0 \quad \forall \ell \in \{1, \dots, N - H_h^0 + p\}$$

$$F_0^{pre} : \max H_h^0 := \sum_{i \in V} \sum_{j \in V} x_{ij}$$

$$\text{s.t. } \sum_{i \in V} x_{ii} = p$$

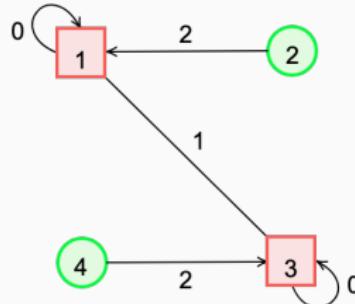
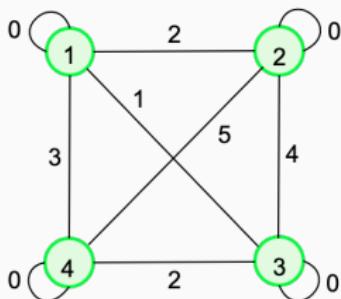
$$\sum_{j \in V} x_{ij} \leq 1 \quad i \in V \quad (7c)$$

$$x_{ij} \leq x_{jj} \quad i, j \in V : i \neq j \quad (7d)$$

$$c_{ij} \geq c_{(h)} x_{ij} \quad i, j \in V : i \neq j \quad (7e)$$

$$x_{ij} \in \{0, 1\} \quad i, j \in V. \quad (7f)$$

## Covering formulation: example



$h$	$\overbrace{\# \text{ alloc}   c_{ij} \leq c_{(h-1)} }^{H_h^1}$	$N - H_h^1$	$\overbrace{\# \text{ alloc}   c_{ij} \geq c_{(h-1)} }^{H_h^0}$	$N - H_h^0 + p$
1	2	2	4	2
2	3	1	4	2
3	4	0	4	2
4	4	0	4	2
5	4	0	3	3

$$preproc(u_{\ell h}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & NF & NF & NF & 0 \\ 1 & 1 & NF & NF & NF \end{pmatrix} \quad u_{\ell h} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

## **Computational experience II**

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$F^{cov}$  with fixing preprocessings

N	p	CPU	gap	opt	nod
20	5	0.34	0.00	5	1
20	6	0.49	0.00	5	105.4
20	10	0.24	0.00	5	26.8
30	7	2.09	0.00	5	81
30	10	1.12	0.00	5	80.6
30	15	0.7	0.00	5	283.6
40	10	2.95	0.00	5	503.4
40	13	4.51	0.00	5	1650.2
40	20	1.54	0.00	5	1606.8
50	12	10.21	0.00	5	1976.2
50	16	3.65	0.00	5	411.8
50	25	2.08	0.00	5	2077
60	15	11.95	0.00	5	1898.4
60	20	5.73	0.00	5	974.6
60	30	4.61	0.00	5	9334.6
70	17	18.44	0.00	5	1468.6
70	23	8.1	0.00	5	905.2
70	35	4.65	0.00	5	4601.6
80	20	33.43	0.00	5	3375.4
80	26	20.29	0.00	5	10820.2
80	40	7.78	0.00	5	5441.8
90	22	26.27	0.00	5	2045.4
90	30	19.3	0.00	5	5147
90	45	574.7	0.00	5	1829827.4
100	25	52.44	0.00	5	7089
100	33	105.29	0.00	5	161173.2
100	50	185	0.00	5	955019

## Benders decomposition

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## Benders decomposition

$$F^{MP} : \min \quad \frac{1}{\sum_{\ell \in V} \lambda_\ell} \sum_{\ell \in V} \sum_{i,j \in V} \lambda_\ell c_{ij} x_{ij}^\ell + \frac{1}{p-1} \sum_{i,j \in V} c_{ij} z_{ij} + \mu \quad (8a)$$

$$\text{s.t.} \quad \sum_{i \in V} \sum_{\ell \in V} x_{ii}^\ell = p \quad (8b)$$

$$\sum_{j \in V} \sum_{\ell \in V} x_{ij}^\ell = 1 \quad \forall i \in V \quad (8c)$$

$$\sum_{\ell \in V} x_{ij}^\ell \leq \sum_{\ell \in V} x_{jj}^\ell \quad \forall i, j \in V \quad (8d)$$

$$2z_{ij} \leq \sum_{\ell \in V} x_{ii}^\ell + \sum_{\ell \in V} x_{jj}^\ell \quad \forall i, j \in V \quad (8e)$$

$$z_{ij} \in \mathcal{T} \quad \forall i, j \in V \quad (8f)$$

$$\sum_{i,j \in V} x_{ij}^\ell \leq 1 \quad \forall \ell \in V \quad (8g)$$

$$\sum_{i,j \in V} c_{ij} x_{ij}^\ell \leq \sum_{i,j \in V} c_{ij} x_{ij}^{\ell+1} \quad \forall \ell \in V : \ell < |V| \quad (8h)$$

$$z_{ij}, x_{ij}^\ell \in \{0, 1\} \quad \forall i, j, \ell \in V, \mu \geq 0 \quad (8i)$$

$F^{SP}$  : Minimum Spanning Tree  $\longrightarrow$  Kruskal algorithm

$$Opt.cut : \frac{obj_{SP}}{p-1} \left[ \sum_{i \in V} \bar{x}_{ii} - (p-1) \right] \leq \mu$$

Feasibility cuts are not needed

# Classical Benders decomposition algorithm

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**Algorithm 1:** OMT classical Benders decomposition

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```
1  $UB := \infty$ 
2  $LB := 0$ 
3 while  $UB \leq LB$  do
4   Solve to optimality  $F^{MP} \rightarrow (obj^{MP}, \bar{x})$ , where  $obj^{MP} = obj^{OM} + \mu$ 
5   if  $obj^{MP} > LB$  then
6     update LB
7   Solve to optimality  $F^{SP} \rightarrow obj^{SP}$ 
8   if  $obj^{OM} + obj^{SP} < UB$  then
9     update UB
10  Add optimality constraint Opt.cut to  $F^{MP}$ 
```

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# Modern Benders decomposition algorithm

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**Algorithm 2:** OMT branch-and-Benders-cut

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```
1 Set tree  $\mathcal{T} = \{o\}$ , where  $o = F^{MP}$  has no branching constraints
2 Initialize a pool of cuts  $\mathcal{P} = \{\}$ 
3 while  $\mathcal{T}$  is nonempty do
4   Select a node  $o' \in \mathcal{T}$ 
5    $\mathcal{T} := \mathcal{T} \setminus \{o'\}$ 
6   Solve  $o' \rightarrow (obj_{o'}^{MP}, \bar{x})$ , where  $obj_{o'}^{MP} = obj_{o'}^{OM} + \mu$ 
7   if  $\bar{x}$  is fractional then
8     Branch, resulting in nodes  $o''$  and  $o'''$ 
9      $\mathcal{T} := \mathcal{T} \cup \{o'', o'''\}$ 
10    else
11       $LB := obj_{o'}^{MP}$ 
12      Solve  $F^{SP} \rightarrow obj_{o'}^{SP}$ 
13      if  $obj_{o'}^{OM} + obj_{o'}^{SP} < UB$  then
14        update UB
15        Add optimality constraint  $Opt.cut$  to  $\mathcal{P}$ 
16     $\mathcal{T} := \mathcal{T} \cup \{o'\}$ 
```

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## Problems in Benders decomposition algorithms

- **Classical:** every *Opt.cut* added implies solving to optimality a MP, computationally costly.
- **Modern:** the pool of cuts  $\mathcal{P}$  considered in a specific node of the B&C procedure may not contain cuts that have already been identified, slowing down the algorithm efficiency.

### Warm-start phase

Introduce a certain number of cuts in the initial pool of cuts  $\mathcal{P}$  preventing that the cuts already introduced are considered later again in the algorithm.

Parameters to tune for an optimal performance:

- $\text{max.time}^{MP}$  → Time limit that every solution computation of the MP in the warm-start phase can take to obtain the best feasible solution possible.
- $\text{max.gap}^{MP}$  → Gap percentage allowed between  $LB^{MP}$  and  $UB^{MP}$  in every MP of the warm-start phase<sup>3</sup>.
- $\text{max.time}$  → Total time limit that the entire warm-start phase can take.
- $\text{max.gap}$  → Gap percentage allowed between  $LB$  and  $UB$  in the entire warm-start phase.

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<sup>3</sup> $gap(UB, LB) = 100 \cdot \frac{UB - LB}{LB}$

## **Computational experience III**

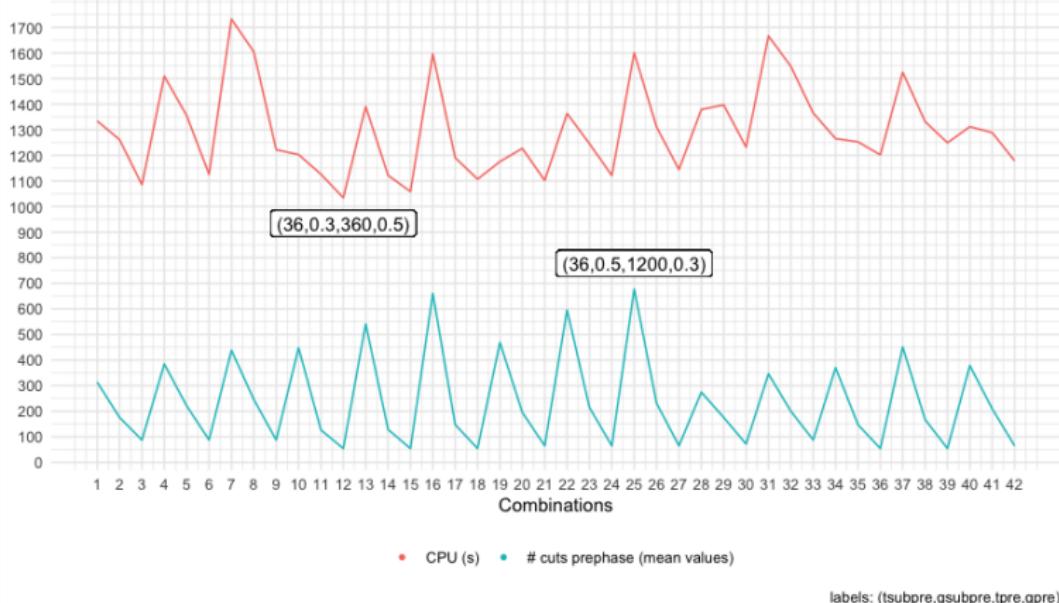
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		classical			modern		
N	p	cpu	gap	opt	cpu	gap	opt
20	5	3600.02	41.03	0	783.63	0.00	5
20	6	3600.02	43.38	0	3237.50	15.42	2
20	10	3600.03	66.38	0	3600.25	198.40	0
30	7	3600.04	60.67	0	3600.23	105.68	0
30	10	3600.04	74.28	0	3600.41	182.57	0
30	15	3600.03	86.99	0	3600.77	447.03	0
40	10	3600.06	71.97	0	3600.29	147.91	0
40	13	3600.04	92.89	0	3600.65	284.48	0
40	20	3600.05	88.37	0	3600.97	508.95	0
50	12	3600.05	126.50	0	3600.51	239.44	0
50	16	3600.06	99.69	0	3600.75	272.37	0
50	25	3600.06	105.59	0	3602.07	616.07	0

# Tuning a warm-start phase for modern Benders decomposition

Tuning Benders modern decomposition parameters

N=20, p=5, density=1



## **Conclusions and further work**

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# Conclusions and further work

## Conclusions

- OMT is a new case of study.
- A modeling approach for solving the OMT is presented along this work.
- Several competitive formulations have obtained optimal solutions up to 50 nodes, and with small gap up to 100 nodes.
- A Benders decomposition approach arises straightforward adapted to our problem structure.
- The covering formulation with fixing preprocessings provides better results so far, solving to optimality up to 100 nodes.

## Further work

- Improve efficiency of the subtour elimination algorithm via connected components (in progress).
- Improve presented formulations using other sorting alternatives, valid inequalities, etc.
- Tune a warm-start phase for the modern Benders decomposition approach (in progress).
- Consider introducing fractional cuts in the Benders warm-start phase.
- Test considering different criterion.
- ...

*All models are wrong, but some are useful.*

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GEORGE E. P. BOX

## Thanks for your attention

Questions, comments, suggestions and improvements are welcome

### Acknowledgements

This research was supported by the project "*Nuevos resultados sobre los problemas de diseño y optimización en redes complejas: Aplicaciones al diseño de ciudades inteligentes*" (FEDER-US-1256951). This support is gratefully acknowledged.