Applications, properties and curiosities of the WRP and related problems

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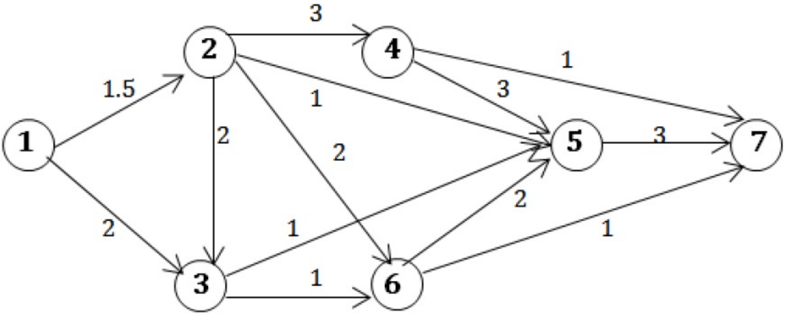
OUTLINE:

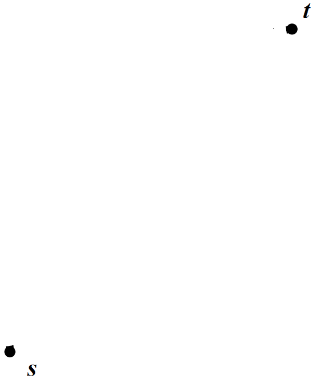
1. From Shortest Paths in Graphs to Geometric Shortest Paths
2. The Weightred Region Problem (WRP)
   - Applications
   - Properties
   - Curiosities
3. The Simple-Path ℓ_p-WRP
   - Local optimality condition for gate points and relation with Snell's law

Shortest Path Problem



El camino más corto que lleva del nodo 1 al 7 es
$1 \longrightarrow 3 \longrightarrow 6 \longrightarrow 7$ y su longitud es 4.

Geometric Shortest Path Problems



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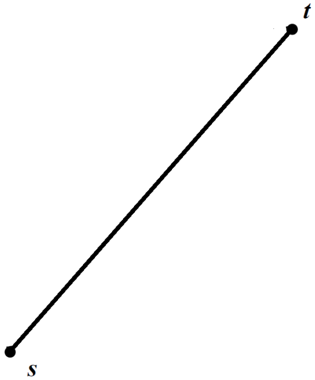

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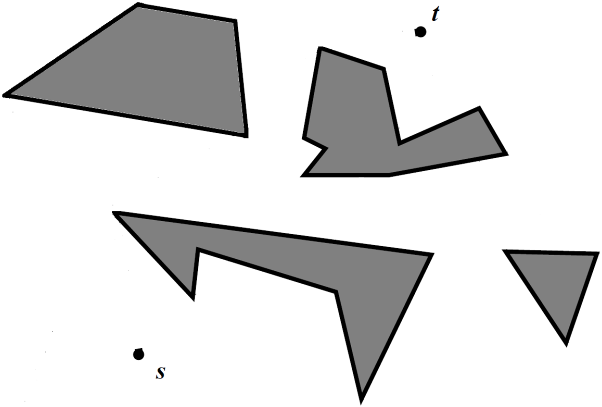

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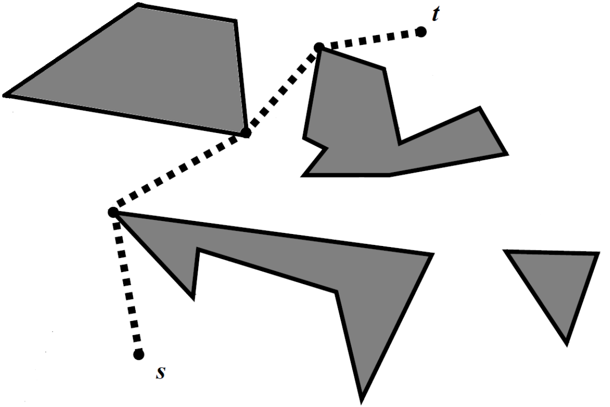
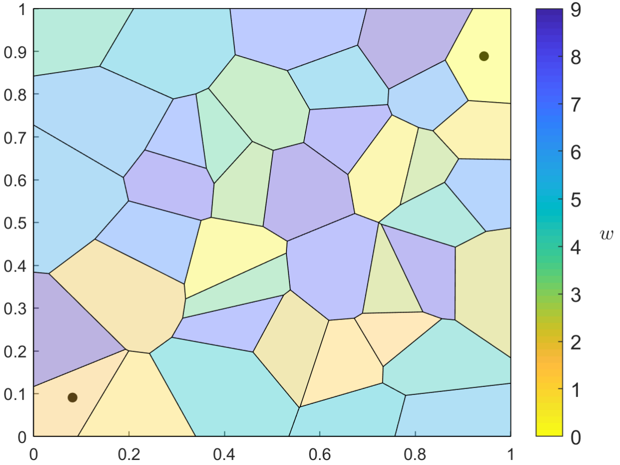


Weighted Region Problem (WRP)



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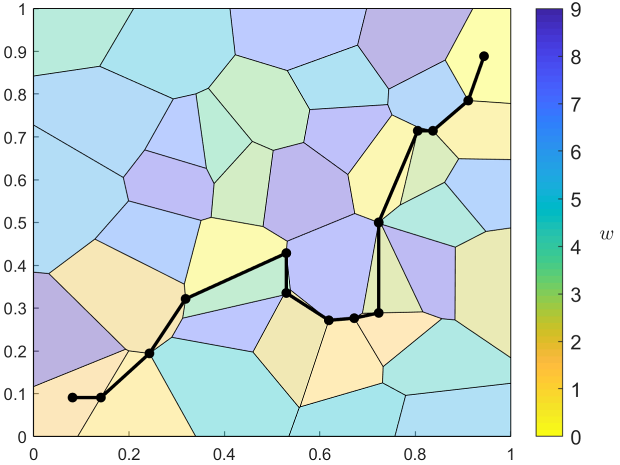
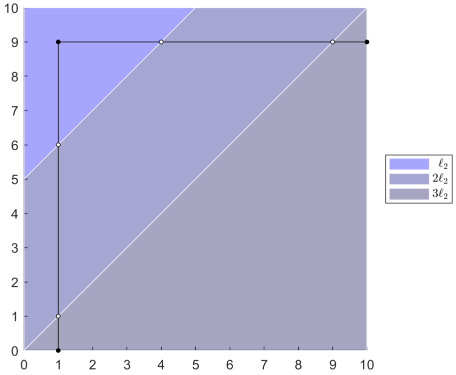


WRP and triangle inequality



$$3\|(10,9)-(1,0)\|_2 = 38.1837\ldots$$
$$3\|(1,1)-(1,0)\|_2 + 2\|(1,6)-(1,1)\|_2 + \|(1,9)-(1,6)\|_2 = 16$$
$$\|(4,9)-(1,9)\|_2 + 2\|(9,9)-(4,9)\|_2 + 3\|(10,9)-(9,9)\|_2 = 16$$

Applications of the WRP

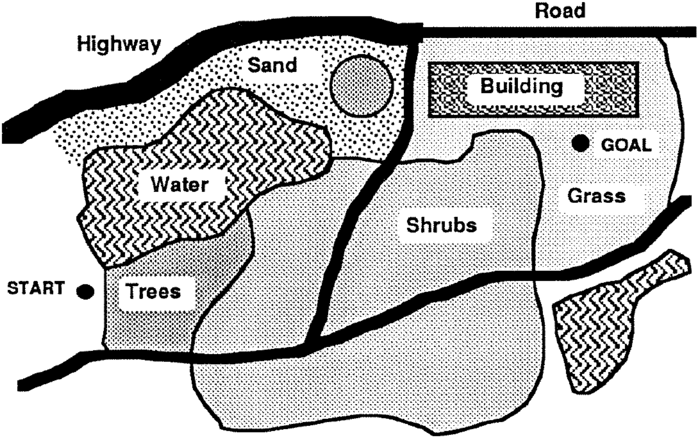


Figure: Mitchell and Papadimitriou (1991)

Applications of the WRP

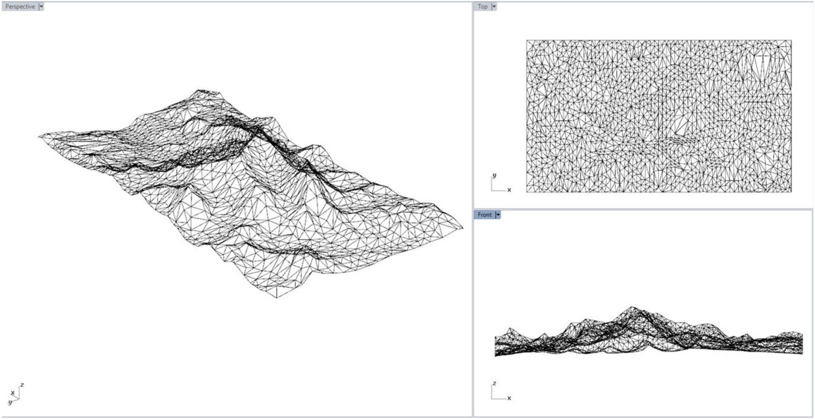
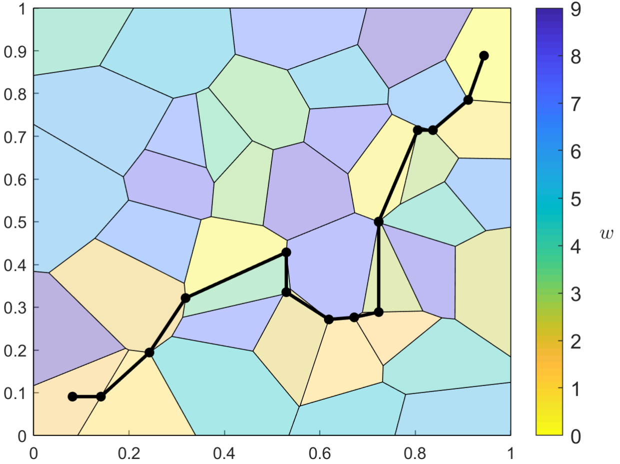


Figure: Gheibi, Maheshwari, Sack and Scheffer (2018)

Local optimality condition for gate points



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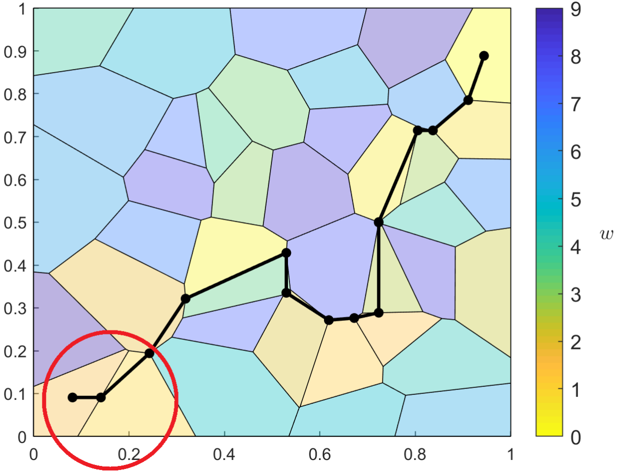
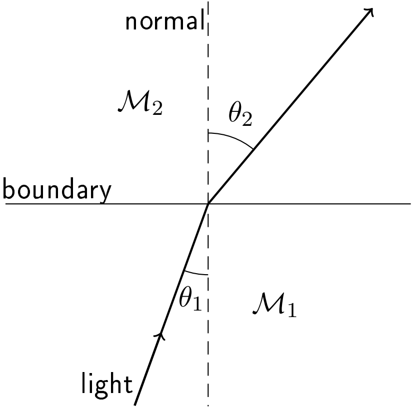


Snell's law



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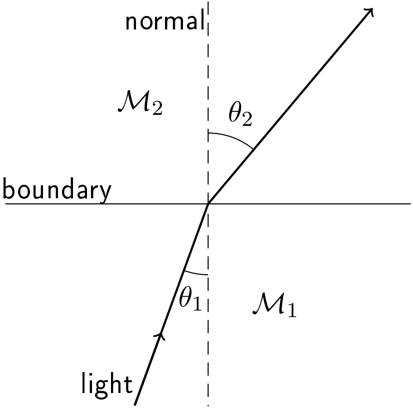

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$

Some references on the WRP and related problems

- J.S.B. Mitchell, C.H. Papadimitriou: The weighted region problem: finding shortest paths through a weighted planar subdivision. J. Assoc. Comput. Mach. 38, 18–73 (1991)
- C. S. Mata, J.S.B. Mitchell: A new algorithm for computing shortest paths in weighted planar subdivisions. In Proceedings of the thirteenth annual symposium on Computational geometry, 264-273 (1997)
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- S.-W. Cheng, H.-S. Na, A. Vigneron, Y. Wang: Approximate shortest paths in anisotropic regions. SIAM J. Comput. 38, 802–824, (2008)
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Unsolvability of the WRP

De Carufel, Grimm, Maheshwari, Owen and Smid (2014)

In general, the exact solution of WRP cannot be computed in $\mathbb{Q}$ using a finite number of the operations $+$, $-$, $\times$, $\div$, $\sqrt[k]{\ }$, for any $k \geq 2$.

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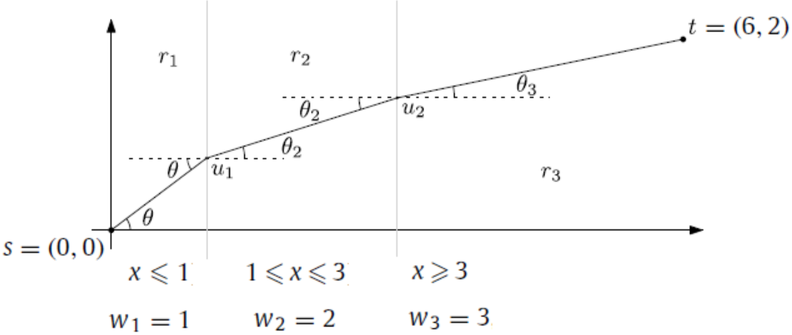
Gheibi, Maheshwari, Sack and Scheffer (2018):

- "It is unlikely that WRP can be solved in polynomial time".
- "To the best of our knowledge, still no FPTAS is known for WRP".

Counterexample in De Carufel, Grimm, Maheshwari, Owen and Smid (2014)



Counterexample in De Carufel, Grimm, Maheshwari, Owen and Smid (2014)

For simplicity, we let $\theta = \theta_1$. Hence, we must have $\sin(\theta_2) = \frac{w_1}{w_2}\sin(\theta)$ and $\sin(\theta_3) = \frac{w_1}{w_3}\sin(\theta)$.

Since the sum of the vertical distances travelled in all regions must be equal to the y-coordinate of t, we need to solve

$$\tan(\theta) + 2\tan(\theta_2) + 3\tan(\theta_3) = 2.$$

Since $\tan(\theta) = \frac{\sin(\theta)}{\sqrt{1-\sin^2(\theta)}}$ for $0 \leqslant \theta < \frac{1}{2}\pi$, this can be rewritten as

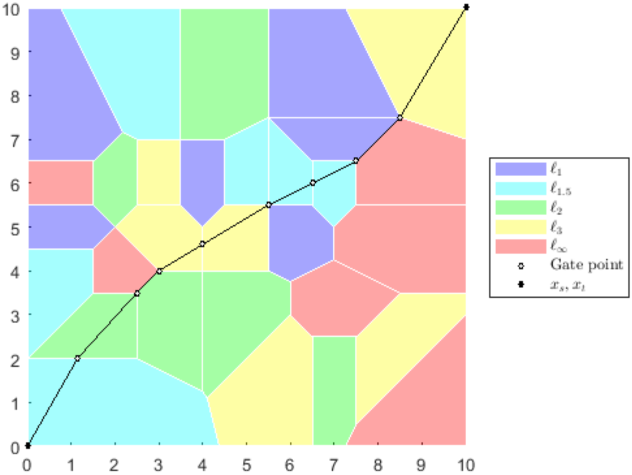
$$\phi(X) = \frac{X}{\sqrt{1-X^2}} + 2\frac{\frac{w_1}{w_2}X}{\sqrt{1-(\frac{w_1}{w_2}X)^2}} + 3\frac{\frac{w_1}{w_3}X}{\sqrt{1-(\frac{w_1}{w_3}X)^2}} = 2,$$

where $X = \sin(\theta)$. By appropriately squaring three times, this can be transformed into

$$p_{12}(u) = 419904 - 3545856u + 12394944u^2 - 24006816u^3 + 28904608u^4 - 22882588u^5 + 12204109u^6 - 4396586u^7 + 1060979u^8 - 168272u^9 + 16843u^{10} - 970u^{11} + 25u^{12} = 0,$$

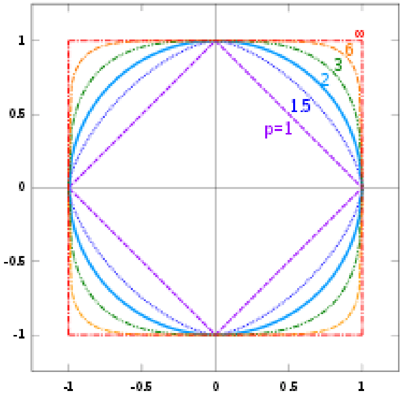
where $u = X^2$.

Simple-Path ℓ_p-WRP



Balls of the ℓ_p-norms

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \quad \forall x \in \mathbb{R}^n \quad \text{where } p \in [1, \infty)$$



Balls of the "pasted" ℓ_p-norms

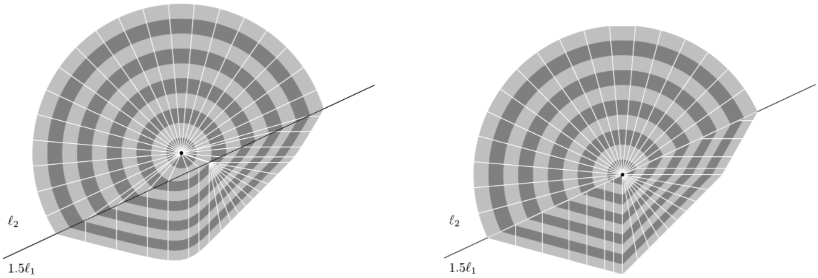
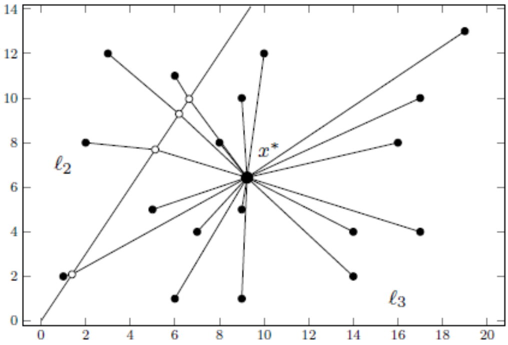
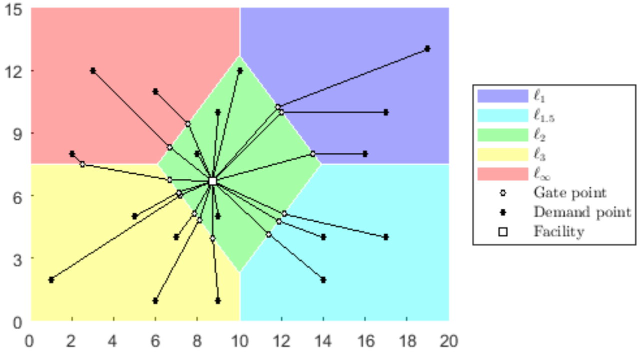


Figure: Plastria (2019)

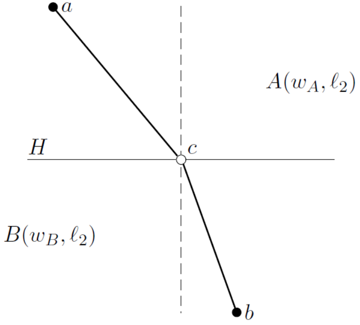
Blanco, Puerto and Ponce (2017)



Simple-Path ℓ_p-Weighted-Region Location Problem



Classical Snell's law



Snell's law - Classical form

$$w_A \sin\theta(c-a,v) = w_B \sin\theta(b-c,v)$$

for all non-zero vector $v \perp H$.

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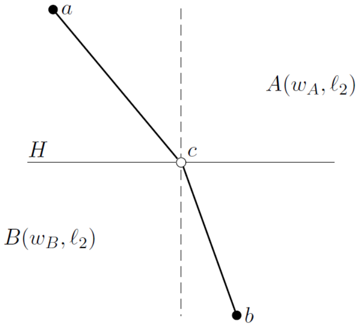

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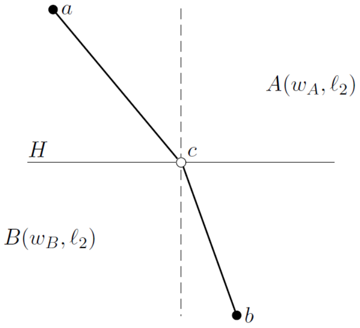
Snell's law - Cosine form

$$w_A \cos\theta(c-a, v) = w_B \cos\theta(b-c, v)$$

for all non-zero vector $v \in V(H)$.

Classical Snell's law



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Snell's law - Cosine form

$$w_A \cos\theta(c-a, v) = w_B \cos\theta(b-c, v)$$

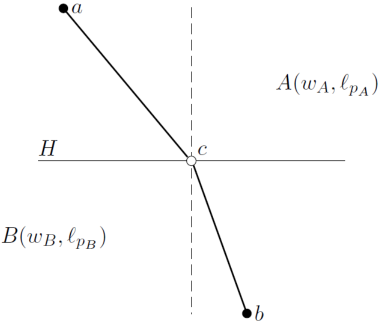
for all non-zero vector $v \in V(H)$.

Snell's law - Dot product form

$$w_A \left(\frac{c-a}{\|c-a\|_2}\right)^T v = w_B \left(\frac{b-c}{\|b-c\|_2}\right)^T v$$

for all non-zero vector $v \in V(H)$.

Generalized Snell's law



Optimality condition for the gate point c

$$w_A u_A^T v = w_B u_B^T v$$

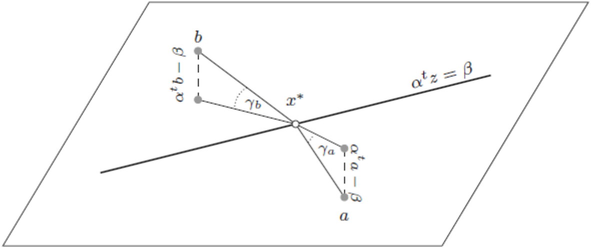
for all non-zero vector $v \in V(H)$, where

$$u_A = \left(\left[\frac{|c_1 - a_1|}{\|c - a\|_{p_A}} \right]^{p_A - 1} \text{sign}(c_1 - a_1), \ldots, \left[\frac{|c_n - a_n|}{\|c - a\|_{p_A}} \right]^{p_A - 1} \text{sign}(c_n - a_n) \right)^T$$

and

$$u_B = \left(\left[\frac{|b_1 - c_1|}{\|b - c\|_{p_B}} \right]^{p_B - 1} \text{sign}(b_1 - c_1), \ldots, \left[\frac{|b_n - c_n|}{\|b - c\|_{p_B}} \right]^{p_B - 1} \text{sign}(b_n - c_n) \right)^T.$$

Blanco, Puerto and Ponce (2017)



$$\sin_{p_A} \gamma_a = \frac{|\alpha^t a - \beta|}{\|a - x^*\|_{p_A}} \quad \left(\text{analogously } \sin_{p_B} \gamma_b = \frac{|\alpha^t b - \beta|}{\|b - x^*\|_{p_B}}\right)$$

Polarity correspondence of ℓ_p-norms

Definition (Polar norm)

Consider an ℓ_p-norm with $p \in (1, +\infty)$ and let B_p be its unit ball. Then, there exists a unique ℓ_{p°-norm with $p^\circ \in (1, +\infty)$ whose unit ball B_{p° is the polar set of B_p, where the polar set B_p° of B_p is given by

$$B_p^\circ = \left\{ x' \in \mathbb{R}^n : x^T x' \leq 1, \forall x \in B_p \right\}.$$

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The norms ℓ_p and ℓ_{p° are polar to each other iff $\frac{1}{p} + \frac{1}{p^\circ} = 1$.

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Notation:

Some considerations

- The standard angle between two non-zero vectors v and v' is defined as the real number $\theta \in [0, \pi]$ satisfying the equality $\cos\theta = \frac{v^T v'}{\|v\|_2 \|v'\|_2}$.

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- More generally, recall that in all normed spaces $(\mathbb{R}^n, \|\cdot\|)$ where the norm $\|\cdot\|$ can be defined from an inner product $\langle \cdot, \cdot \rangle$ as $\|\tilde{v}\| = \sqrt{\langle \tilde{v}, \tilde{v} \rangle}$ for each $\tilde{v} \in \mathbb{R}^n$, the Cauchy-Schwarz inequality $|\langle v, v' \rangle| \leq \|v\| \|v'\|$ is satisfied.

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- Consider now a normed space $(\mathbb{R}^n, \|\cdot\|_p)$ with $p \in (1, +\infty)$. It is known that when $p \neq 2$ there is not an inner product $\langle\cdot,\cdot\rangle$ from which the norm $\|\cdot\|_p$ can be defined as indicated above. Moreover, the Cauchy-Schwarz inequality is not satisfied in $(\mathbb{R}^n, \|\cdot\|_p)$ when $p \neq 2$.

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- Hölder inequality states:

$$\sum_{k=1}^{n} |v_k v'_k| \leq \|v\|_p \|v'\|_{p^\circ}$$

for all $v = (v_1, \cdots, v_n)^T, v' = (v'_1, \cdots, v'_n)^T \in \mathbb{R}^n$. Hölder inequality ensures $-1 \leq \frac{v^T v'}{\|v\|_p \|v'\|_{p^\circ}} \leq 1$ for all non-zero vectors $v, v' \in \mathbb{R}^n$.

Polar vector

Definition (ℓ_p-angle)

Let $p \in (1, +\infty)$. Given two non-zero vectors $v, v' \in \mathbb{R}^n$, the ℓ_p-angle between v and v', which we denote by $\theta_p(v, v')$, is the real number $\theta_p(v, v') \in [0, \pi]$ such that $\cos \theta_p(v, v') = \frac{v^T v'}{\|v\|_p \|v'\|_{p^\circ}}$.

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Remark: When $p \neq 2$, in general $\theta_p(v, v') \neq \theta_p(v', v)$, but it is satisfied $\theta_p(v, v') = \theta_{p^\circ}(v', v)$.

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Remark: When $p \neq 2$, in general $\theta_p(v, v') \neq \theta_p(v', v)$, but it is satisfied $\theta_p(v, v') = \theta_{p^\circ}(v', v)$.

Proposition 1

Assume $\mathbb{R}^n$ is endowed with an ℓ_p-norm with $p \in (1, +\infty)$. Consider the map between normed spaces $(\mathbb{R}^n, \|\cdot\|_p) \to (\mathbb{R}^n, \|\cdot\|_{p^\circ})$ that associates to $v = (v_1, \ldots, v_d)^T$ the vector $v^\circ = \left(\left(\frac{|v_1|}{\|v\|_p}\right)^{p-1} \mathrm{sign}(v_1)\|v\|_p, \ldots, \left(\frac{|v_n|}{\|v\|_p}\right)^{p-1} \mathrm{sign}(v_n)\|v\|_p \right)^T$ if v is not the zero vector, otherwise v° is the zero vector. Then, given $v \in \mathbb{R}^n$, the vector v° is the unique vector in $\mathbb{R}^n$ satisfaying $\|v\|_p = \|v^\circ\|_{p^\circ}$ and $\theta_p(v, v^\circ) = 0$.

Polar vector

Definition (ℓ_p-angle)

Let $p \in (1, +\infty)$. Given two non-zero vectors $v, v' \in \mathbb{R}^n$, the ℓ_p-angle between v and v', which we denote by $\theta_p(v, v')$, is the real number $\theta_p(v, v') \in [0, \pi]$ such that $\cos\theta_p(v, v') = \frac{v^T v'}{\|v\|_p \|v'\|_{p^\circ}}$.

Remark: When $p \neq 2$, in general $\theta_p(v, v') \neq \theta_p(v', v)$, but it is satisfied $\theta_p(v, v') = \theta_{p^\circ}(v', v)$.

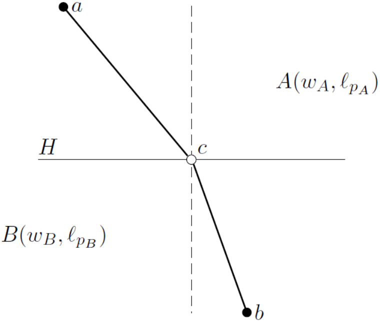
Proposition 1

Assume $\mathbb{R}^n$ is endowed with an ℓ_p-norm with $p \in (1, +\infty)$. Consider the map between normed spaces $(\mathbb{R}^n, \|\cdot\|_p) \to (\mathbb{R}^n, \|\cdot\|_{p^\circ})$ that associates to $v = (v_1, \dots, v_d)^T$ the vector $v^\circ = \left(\left(\frac{|v_1|}{\|v\|_p}\right)^{p-1} \text{sign}(v_1)\|v\|_p, \dots, \left(\frac{|v_n|}{\|v\|_p}\right)^{p-1} \text{sign}(v_n)\|v\|_p\right)^T$ if v is not the zero vector, otherwise v° is the zero vector. Then, given $v \in \mathbb{R}^n$, the vector v° is the unique vector in $\mathbb{R}^n$ satisfaying $\|v\|_p = \|v^\circ\|_{p^\circ}$ and $\theta_p(v, v^\circ) = 0$.

Definition (Polar vector)

Assume $\mathbb{R}^n$ is endowed with an ℓ_p-norm with $p \in (1, +\infty)$ and let $v \in \mathbb{R}^n$. The polar vector of v is the vector v° given in Proposition 1.

Generalized Snell's law

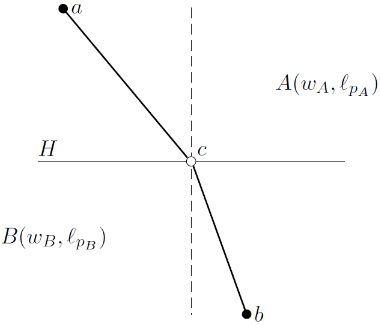


Generalized Snell's law - Dot product form

$$w_A \left(\frac{(c-a)^\circ}{\|(c-a)^\circ\|_{p_A^\circ}} \right)^T v$$
$$= w_B \left(\frac{(b-c)^\circ}{\|(b-c)^\circ\|_{p_B^\circ}} \right)^T v$$

for all non-zero vector $v \in V(H)$.

Generalized Snell's law



Generalized Snell's law - Dot product form

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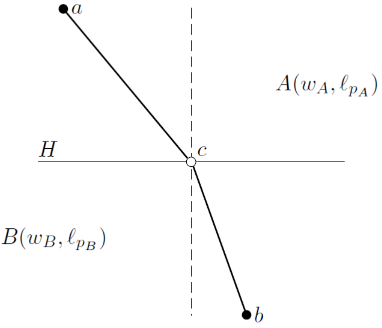
for all non-zero vector $v \in V(H)$.

Generalized Snell's law - Cosine form

$$w_A \|v\|_{p_A} \cos\theta_{p_A^\circ}((c-a)^\circ, v) = w_B \|v\|_{p_B} \cos\theta_{p_B^\circ}((b-c)^\circ, v)$$

for all non-zero vector $v \in V(H)$.

Generalized Snell's law



Generalized Snell's law - Dot product form

$$w_A \left(\frac{(c-a)^\circ}{\|(c-a)^\circ\|_{p_A^\circ}} \right)^T v$$
$$= w_B \left(\frac{(b-c)^\circ}{\|(b-c)^\circ\|_{p_B^\circ}} \right)^T v$$

for all non-zero vector $v \in V(\text{aff}(H))$.

Generalized Snell's law - Cosine form

$$w_A \|v\|_{p_A} \cos\theta_{p_A^\circ}((c-a)^\circ, v)$$
$$= w_B \|v\|_{p_B} \cos\theta_{p_B^\circ}((b-c)^\circ, v)$$

for all non-zero vector $v \in V(\text{aff}(H))$.

Many thanks for your attention.