Hierarchies, prices and portfolios

Marina Leal Palazón (m.leal@umh.es)

Advances on data analysis, logistics and transportation problems on complex networks

Outline

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Introduction

PORTFOLIO OPTIMIZATION

the process of choosing the proportions of various assets to be held in a portfolio, in such a way as to make the portfolio better than any other according to some criterion.

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Introduction

Markowitz, H.M. (1952). Portfolio selection. Journal of Finance, 7, 77–91.

Through two criteria:

- the expected return,
- the risk,
  - variance

as a measure of the variability of the return.

Introduction

PORTFOLIO OPTIMIZATION

the process of choosing the proportions of various assets to be held in a portfolio, in such a way as to make the portfolio better than any other according to some criterion.

$$+$$

TRANSACTION COSTS

costs incurred by the investors when buying and selling assets on the markets, that are charged by the brokers or the financial institutions playing the role of intermediary.

Introduction

R. Mansini, W. Ogryczak, M.G. Speranza. (2014). Twenty years of linear programming based on portfolio optimization. European Journal of Operational Research, Vol. 234, Issue 2, 518-535.

R. Mansini, W. Ogryczak, M.G. Speranza. (2015) Chapter 8: Portfolio Optimization and Transaction Costs. In Quantitative Financial Risk Management: Theory and Practice. C. Zopounidis and E. Galariotis, John Wiley and Sons, Inc, Hoboken, NJ, USA.

- Transaction Costs are assumed to be given: fixed cost applied to each security, variable depending on the amount, etc.

Outline

Discrete Pricing Portfolio.

M. Leal, D. Ponce and J. Puerto
"Portfolio problems with two levels decision-makers: Optimal portfolio selection with Pricing decisions on transaction costs".
European Journal of Operations Research, 2020.

Discrete Pricing Portfolio. *Contributions*

Contributions:

- Turning transaction costs into decision variables.
- Incorporating two levels of decision-makers in Portfolio Problems (incorporating the Broker as a decision-maker).
- Developing different bilevel programming formulations to obtain optimal solutions for the considered models.

\+ Discrete set of prices

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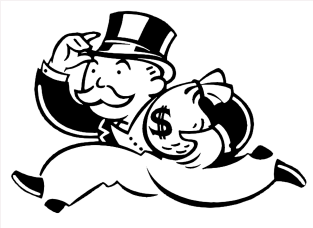
\+ Discrete set of prices

Investor



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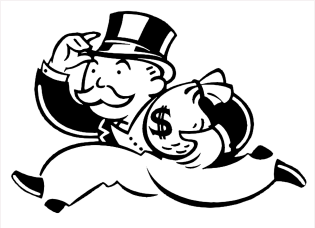
Broker-dealer

Investor



- $\{1, ..., n\}$ be the set of securities considered for an investment.
- $B \subseteq \{1, ..., n\}$ a subset in which the Broker can charge a transaction cost.

The Broker has to decide a price P_j for each security $j \in B$ from a discrete set of possible costs $C_j = \{c_{j1}, ..., c_{js_j}\}$, maximizing its benefits.

Discrete Pricing Portfolio. *The models. Broker problem*

- $x = (x_j)_{j=1,\ldots,n}$ vector of decision variables x_j expressing the weights defining a portfolio.
- Binary decision variables $a_{jk} = 1$ if price c_{jk} is assigned to P_j.

$$\max \quad \textit{benefits}$$

st.

the prices are chosen
from the set of possible prices.

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\begin{aligned}
\max \quad & \sum_{j \in B} P_j x_j \\
\text{st.} \quad & \\
& P_j = \sum_{k=1}^{s_j} c_{jk} a_{jk} \quad \forall j \in B, \\
& \sum_{k=1}^{s_j} a_{jk} = 1 \qquad \forall j \in B, \\
& a_{jk} \in \{0,1\} \qquad \forall j \in B, k = 1, \dots, s_j, \\
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Discrete Pricing Portfolio. *The models. Investor problem*

Two criteria:

- **the expected return**,
- the risk,
  - **conditional value at risk (CVaR)**

Fomulated as a LP:

$$
\begin{aligned}
\max \quad & \eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t \\
\text{st.} \quad & \\
& d_t \geq \eta - y_t, && \forall t, \\
& y_t = \textstyle\sum_{j=1}^{n} r_{jt}x_j, && \forall t, \\
& \sum_{j=1}^{n} x_j = 1, && \\
& x_j \geq 0, && \forall j, \\
& d_t \geq 0, && \forall t.
\end{aligned}
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Two criteria:

- **the expected return**,
- the risk,
  - **conditional value at risk (CVaR)**

CVaR(α): aims to avoid large losses. Measures the conditional expectation of the smallest returns with a cumulative probability α. (Average return of the given size (quantile) of worst realizations.

Fomulated as a LP:

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\end{aligned}
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Discrete Pricing Portfolio. *The models. Investor problem*

$$\max \quad \eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t$$

st.

$$d_t \geq \eta - y_t, \quad \forall t,$$
$$d_t \geq 0, \quad \forall t$$

CVaR

$$y_t = \sum_{j=1}^{n} r_{jt}x_j - \left(\sum_{j\in B} P_j x_j\right), \quad \forall t,$$

Expected return in each scenario

$$\sum_{j=1}^{n} x_j = 1,$$
$$x_j \geq 0, \quad \forall j,$$

Portfolio constraints

$$\sum_{t=1}^{T}\pi_t y_t \geq \mu_0.$$

Expected return

Discrete Pricing Portfolio. *The models. Investor problem*

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CVaR

Expected return in each scenario

Portfolio constraints

Expected return

Discrete Pricing Portfolio. *The models. Investor problem*

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\end{aligned}
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CVaR

Expected return in each scenario

Portfolio constraints

Expected return

$$\max \quad \eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t$$

st.

$$d_t \geq \eta - y_t, \quad \forall t,$$
$$d_t \geq 0, \quad \forall t$$

$$y_t = \sum_{j=1}^{n} r_{jt}x_j - \left(\sum_{j\in B} P_j x_j\right), \quad \forall t,$$

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CVaR

Expected return in each scenario

Portfolio constraints

Expected return

$$\max \quad \eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t$$

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$$d_t \geq \eta - y_t, \quad \forall t,$$
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CVaR

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Expected return in each scenario

$$\sum_{j=1}^{n} x_j = 1,$$
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Portfolio constraints

$$\sum_{t=1}^{T}\pi_t y_t \geq \mu_0.$$

Expected return

Discrete Pricing Portfolio. *The models*

Broker-leader
INVESTOR-follower

Broker-leader
INVESTOR-follower

$$
\begin{array}{ll}
\max & \textit{Broker objective function} \\
\text{st.} & \\
 & \textit{Broker constraints} \\
 & \\
 & x \in \arg\max \ \textit{investor objective function} \\
 & \qquad \text{st.} \\
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Discrete Pricing Portfolio. *The models*

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INVESTOR-leader
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\end{array}
$$

Discrete Pricing Portfolio. *The models*

Broker-leader
INVESTOR-follower

INVESTOR-leader
Broker-follower

SOCIAL WELFARE

SOCIAL WELFARE

$$
\begin{array}{ll}
\max & \textit{investor + Broker obj. functs.} \\
\text{st.} & \\
 & \textit{investor constraints} \\
 & \textit{Broker constraints}
\end{array}
$$

Discrete Pricing Portfolio. *The models*

Broker-leader
INVESTOR-follower

INVESTOR-leader
Broker-follower

SOCIAL WELFARE

Discrete Pricing Portfolio. *Broker-Investor model*

$$\max \quad \sum_{j \in B} P_j x_j$$

st.

Broker constraints

$$x \in arg \max \quad \{\eta - \frac{1}{\alpha} \sum_{t=1}^{T} \pi_t d_t\}$$

st.

$$\begin{aligned}
&\textstyle\sum_{j=1}^{n} x_j = 1, && \\
&d_t \geq \eta - y_t, && \forall t, \\
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\end{aligned}$$

Discrete Pricing Portfolio. *Broker-Investor model*

PRIMAL:

$$\begin{aligned}
\max \quad & \{\eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t\} \\
& \text{st.} \\
& \textstyle\sum_{j=1}^{n} x_j = 1, \\
& d_t \geq \eta - y_t, \forall t, \\
& y_t = \textstyle\sum_{j=1}^{n} r_{jt}x_j - \left(\sum_{j\in B} P_j x_j\right), \forall t, \\
& \textstyle\sum_{t=1}^{T} \pi_t y_t \geq \mu_0 \\
& x_j \geq 0, \forall j, \\
& d_t \geq 0, \forall t.
\end{aligned}$$

DUAL:

$$\begin{aligned}
\min \quad & \beta + \mu_0\mu \\
\text{st.} \quad & \\
& \beta - \sum_{t=1}^{T}(r_{jt} - P_j)\delta_t \geq 0, \forall j, \\
& \beta - \sum_{t=1}^{T} r_{jt}\delta_t \geq 0, \forall j, \\
& -\sum_{t=1}^{T}\gamma_t = 1, \\
& \gamma_t \geq -\frac{\pi_t}{\alpha}, \forall t, \\
& \gamma_t + \delta_t + \pi_t\mu \geq 0, \forall t, \\
& \gamma_t \leq 0, \forall t, \\
& \mu \leq 0.
\end{aligned}$$

Discrete Pricing Portfolio. *Broker-Investor model*

$$\max \quad \sum_{j \in B} P_j x_j$$

st.

Broker constraints

$$\eta - \frac{1}{\alpha} \sum_{t=1}^{T} \pi_t d_t = \beta + \mu_0 \mu$$

investor problem constraints,

dual problem constraints.

Linearising the products of variables

$\Downarrow$

MILP formulation

Discrete Pricing Portfolio. *Broker-Investor model*

$$\max \quad \sum_{j \in B} P_j x_j$$

st.

Broker constraints

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Linearising the products of variables

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MILP formulation

Discrete Pricing Portfolio. *Investor-Broker model*

$$\max \quad \{\eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t\}$$

st.

Investor constraints,

$$P \in arg\max \sum_{j\in B} P_j x_j$$

$$\begin{aligned} \text{s.t.} \quad & P_j = \sum_{k=1}^{s_j} c_{jk}a_{jk} && \forall j \in B, \\ & \sum_{k=1}^{s_j} a_{jk} = 1 && \forall j \in B, \\ & a_{jk} \in \{0,1\} && \forall j \in B, k = 1, ..., s_j, \\ & P \in \mathbb{P}. \end{aligned}$$

Discrete Pricing Portfolio. *Investor-Broker model*

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$$\max \quad \{\eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t\}$$

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$$\text{s.t.} \quad P_j = \sum_{k=1}^{s_j} c_{jk}a_{jk} \quad \forall j \in B,$$

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$$a_{jk} \in \{0,1\} \quad \forall j \in B, k = 1, ..., s_j,$$

$$P \not\in \mathbb{P}.$$

Given a solution x, fixing the prices to their maximum possible values is always an optimal solution of the Broker problem.

Discrete Pricing Portfolio. *Investor-Broker model*

$$\max \quad \{\eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t\}$$

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$$P \notin \mathbb{P}.$$

Given a solution x, fixing the prices to their maximum possible values is always an optimal solution of the Broker problem.

If we denote by $P_j^+ = \max_{k=1,\dots,s_j} c_{jk}\ \forall j \in B$, the Investor-leader Broker-follower Problem can be formulated as::

$$
\begin{aligned}
\max \quad & \{\eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t\} \\
\text{st.} \quad & \\
& \sum_{j=1}^{n} x_j = 1, \\
& d_t \geq \eta - y_t, && \forall t = 1,\dots,T, \\
& y_t = \textstyle\sum_{j=1}^{n} r_{jt}x_j - \left(\sum_{j\in B} P_j^+ x_j\right), && \forall t = 1,\dots,T, \\
& \textstyle\sum_{t=1}^{T} \pi_t y_t \geq \mu_0 \\
& x_j \geq 0, && \forall j = 1,\dots,n, \\
& d_t \geq 0, && \forall t = 1,\dots,T,
\end{aligned}
$$

Discrete Pricing Portfolio. *Investor-Broker model*

$$\max \quad \{\eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t\}$$

st.

Investor constraints,

$$P \in arg\max \sum_{j\in B} P_j x_j$$

$$\begin{aligned} \text{s.t.} \quad & P_j = \sum_{k=1}^{s_j} c_{jk}a_{jk} && \forall j \in B, \\ & \sum_{k=1}^{s_j} a_{jk} = 1 && \forall j \in B, \\ & a_{jk} \in \{0,1\} && \forall j \in B, k = 1, ..., s_j, \\ & P \in \mathbb{P}. \end{aligned}$$

Theorem

Let $\vartheta = \sum_{j \in B} P_j x_j$, and denote by Ω the set containing the points of the problem in $\mathbb{P}$. The ILBFP is equivalent to:

$$
\begin{aligned}
& \max\{\eta - \frac{1}{\alpha}\sum_{t=1}^{T} \pi_t d_t\} \\
& st. \quad \sum_{j=1}^{n} x_j = 1, \\
& \qquad d_t \geq \eta - y_t, && t = 1, ..., T, \\
& \qquad y_t = \sum_{j=1}^{n} r_{jt} x_j - (\vartheta), && t = 1, ..., T, \\
& \qquad \sum_{t=1}^{T} \pi_t y_t \geq \mu_0 \\
& \qquad x_j \geq 0, && j = 1, ..., n, \\
& \qquad d_t \geq 0, && t = 1, ..., T, \\
& \qquad \vartheta \geq \sum_{j \in B} P_{int,j} x_j, && P_{int} \in \Omega.
\end{aligned}
$$

Algorithm

Initialization:

Choose a feasible portfolio x^0. Set $CVaR^0 = +\infty$.

Iteration: $\tau = 1, 2, \ldots$

Solve the Broker (follower) problem for $x^{\tau-1}$. Let p^τ be an optimal solution.

Solve the incomplete formulation **ILBFP-Incomplete**$^\tau$.

Let $\chi^\tau = (x^\tau, y^\tau, \eta^\tau, d^\tau)$, and let $(\chi^\tau, \vartheta^\tau)$ be an optimal solution and $CVaR^\tau$ the optimal value.

if $(\chi^\tau, \vartheta^\tau)$ is feasible **then**

    $(\chi^{\tau-1}, p^\tau)$ are optimal solutions, and $CVaR^\tau$ the optimal value. END.

else if $(\chi^\tau, \vartheta^\tau)$ is not feasible in **then**

    go to iteration $\tau := \tau + 1$.

end if

Discrete Pricing Portfolio. *Social Welfare model*

$$\max \quad \xi \sum_{j \in B} P_j x_j + (1-\xi)\left(\eta - \frac{1}{\alpha}\sum_{t=1}^{T} \pi_t d_t\right)$$

st.

Broker constraints

CVaR constraints

- Linearising the product of variables ⇒ **MILP formulation**.
- No linearising ⇒ **Algorithm based on Benders cuts.**

Discrete Pricing Portfolio. *Social Welfare model*

$$\max \quad \xi \sum_{j \in B} P_j x_j + (1-\xi)\left(\eta - \frac{1}{\alpha}\sum_{t=1}^{T} \pi_t d_t\right)$$

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Discrete Pricing Portfolio. *Social Welfare model*

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\begin{aligned}
\max \quad & \xi \sum_{j \in B} P_j x_j + (1-\xi)\left(\eta - \frac{1}{\alpha}\sum_{t=1}^{T} \pi_t d_t\right) \\
\text{st.} \quad & \\
& \text{Broker constraints} \\
& \\
& \text{CVaR constraints}
\end{aligned}
$$

- Linearising the product of variables $\Rightarrow$ **MILP formulation**.
- No linearising $\Rightarrow$ **Algorithm based on Benders cuts.**

Discrete Pricing Portfolio. *Social welfare model*

$$\max \quad \sum_{j \in B} P_j x_j + \left(\eta - \frac{1}{\alpha} \sum_{t=1}^{T} \pi_t d_t \right)$$

st.

Broker constraints

CVaR constraints

Proposition

An optimal solution of the unweighted maximum social welfare model induces an objective value that is greater than or equal to the sum of the optimal returns of the two parties in any of the hierarchical models.

Discrete Pricing Portfolio. *Computational study*

- Historical data from **Dow Jones** Industrial Average.
- Daily returns of the 30 assets during one year ($T = 251$ scenarios)
- Different type of instances and prices.

- Comparing solution methods.
- Comparing solutions and risk-profiles.
- Comparing solutions across models.

Discrete Pricing Portfolio. *Computational study*

- Historical data from **Dow Jones** Industrial Average.
- Daily returns of the 30 assets during one year ($T = 251$ scenarios)
- Different type of instances and prices.

- Comparing solution methods.
- Comparing solutions and risk-profiles.
- Comparing solutions across models.

Discrete Pricing Portfolio. *Comparing solutions and profiles within models*

	$K = 5$	$K = 15$	$K = 50$
$\lvert B\rvert = 30$	**A**	**B**	**C**
$\lvert B\rvert = 20$	**D**	**E**	**F**
$\lvert B\rvert = 10$	**G**	**H**	**I**

Table: Types of instances for the sets of possible costs depending on the values of $|B|$ and K

- Different risk profiles

Discrete Pricing Portfolio. *Comparing solutions and profiles within models*

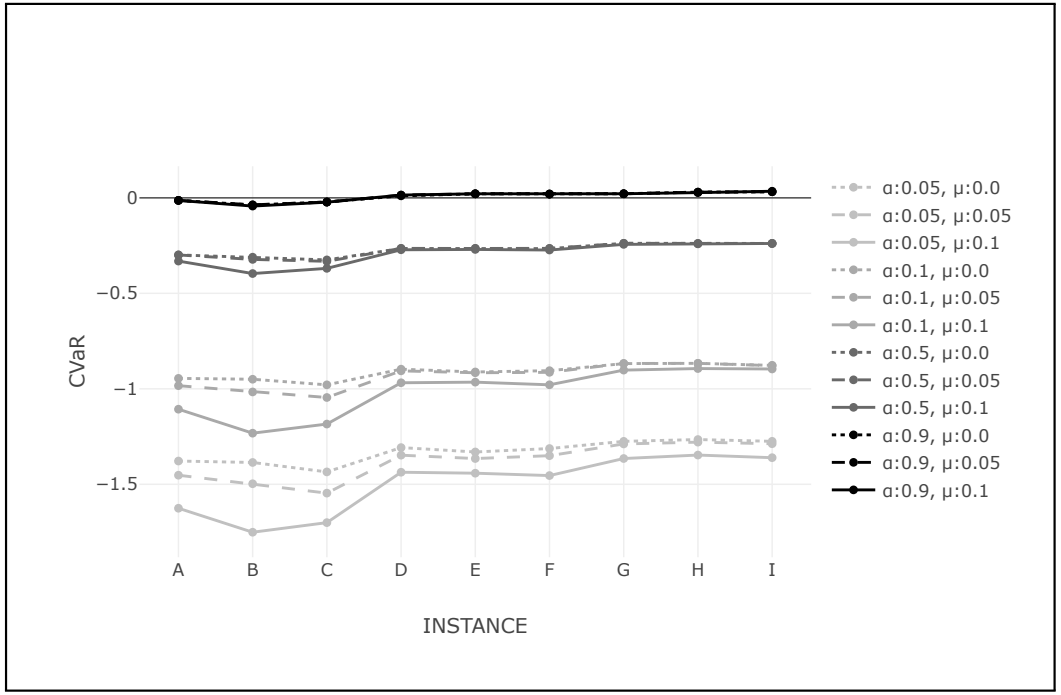


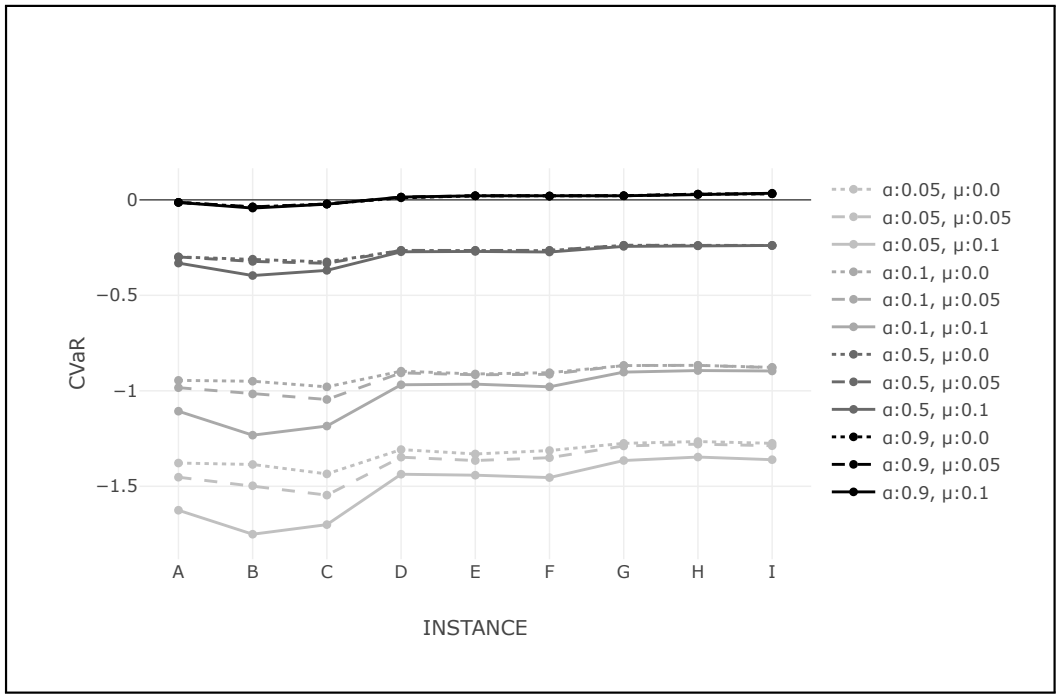
Figure: Values of the CVaR for **BLIFP**, for different α and μ_0 levels

- CVaR always increases with the value of α (more risk)

- When α increases, CVaR for different μ_0 becomes closer

- CVaR for smaller μ_0 is higher (larger feasible region)

Discrete Pricing Portfolio. *Comparing solutions and profiles within models*



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Discrete Pricing Portfolio. *Comparing solutions and profiles within models*

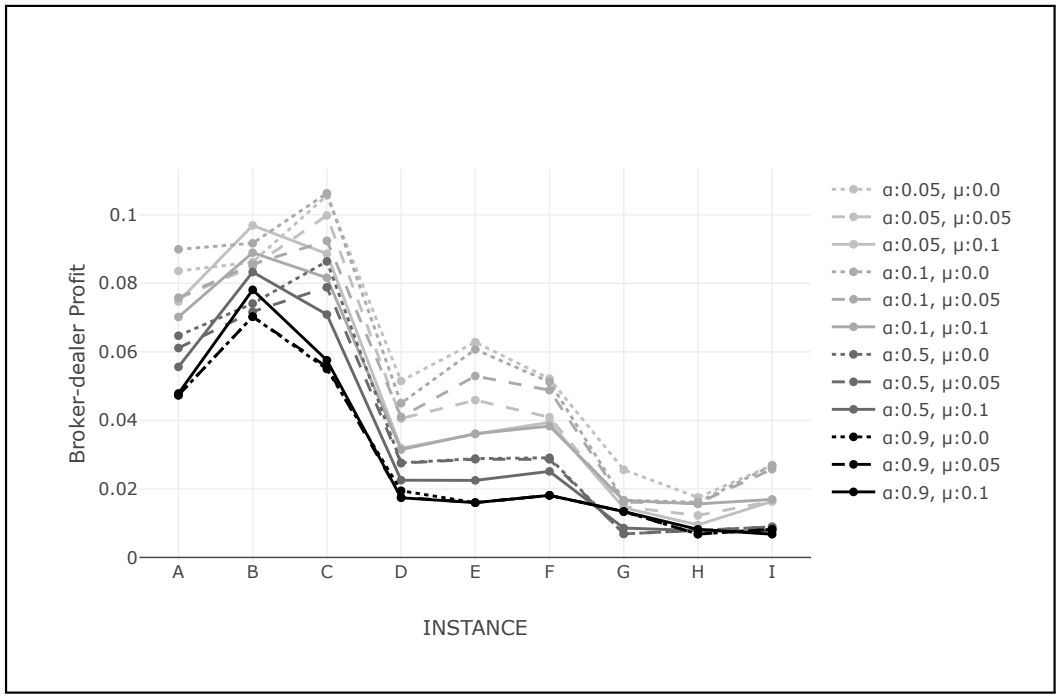
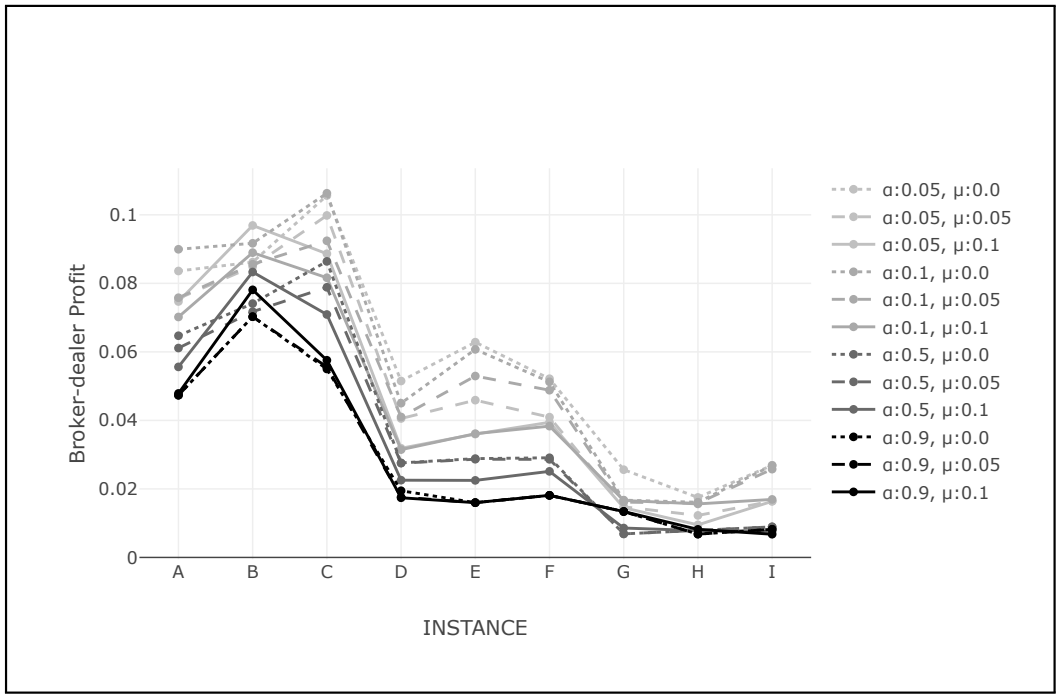


Figure: Values of the broker-dealer profit for **BLIFP**, for different values α and μ_0 levels

- Higher profit for more risk-averse investment (smaller α)

Discrete Pricing Portfolio. *Comparing solutions and profiles within models*



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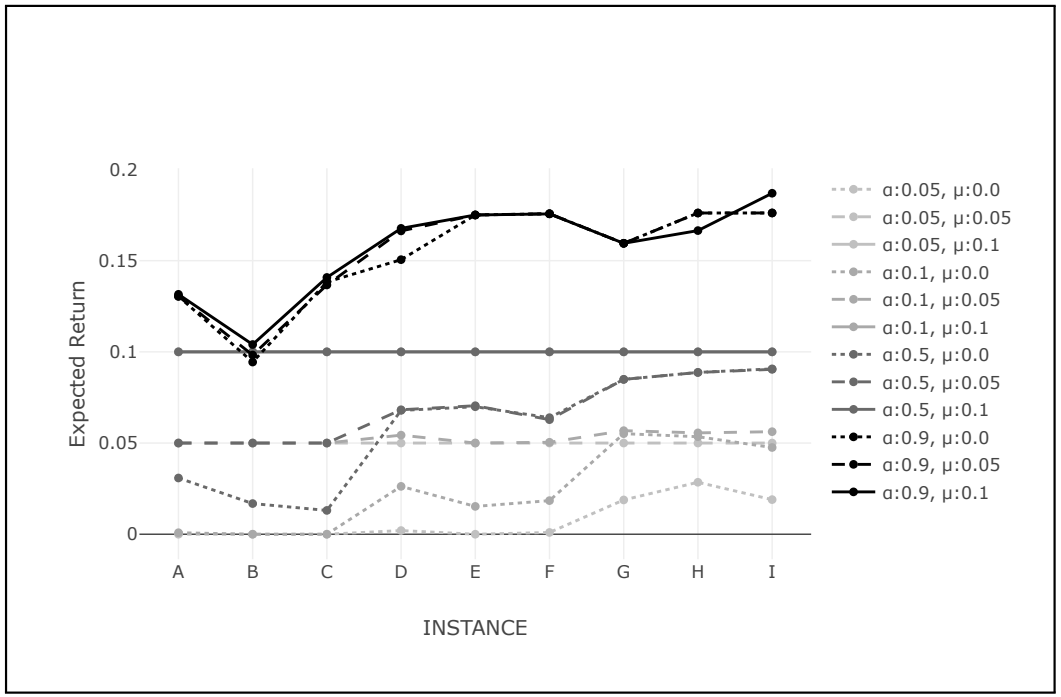
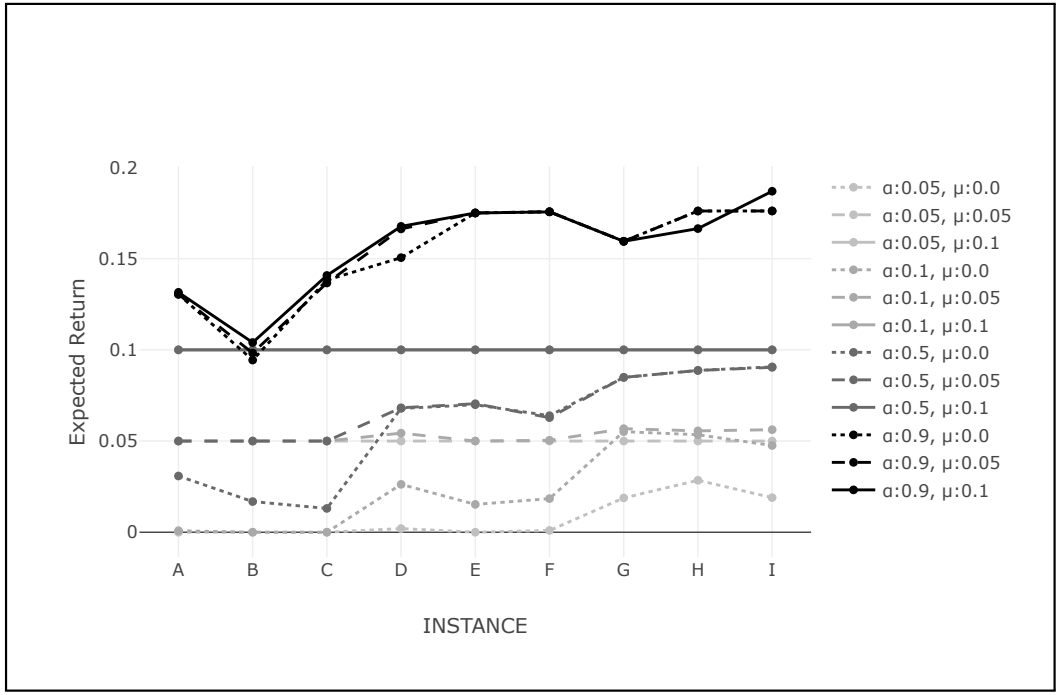


Figure: Values of the expected return for **BLIFP**, for different α and μ_0 levels

- Bigger expected return for higher values of α (considering a wider range of values to compute CVaR).

Discrete Pricing Portfolio. *Comparing solutions and profiles within models*



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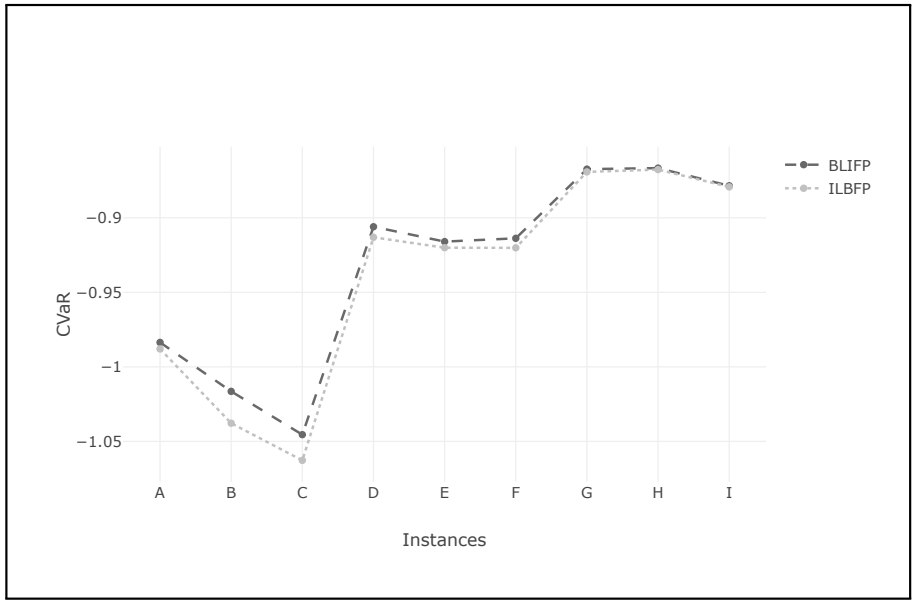
Discrete Pricing Portfolio. *Comparing solutions across models*



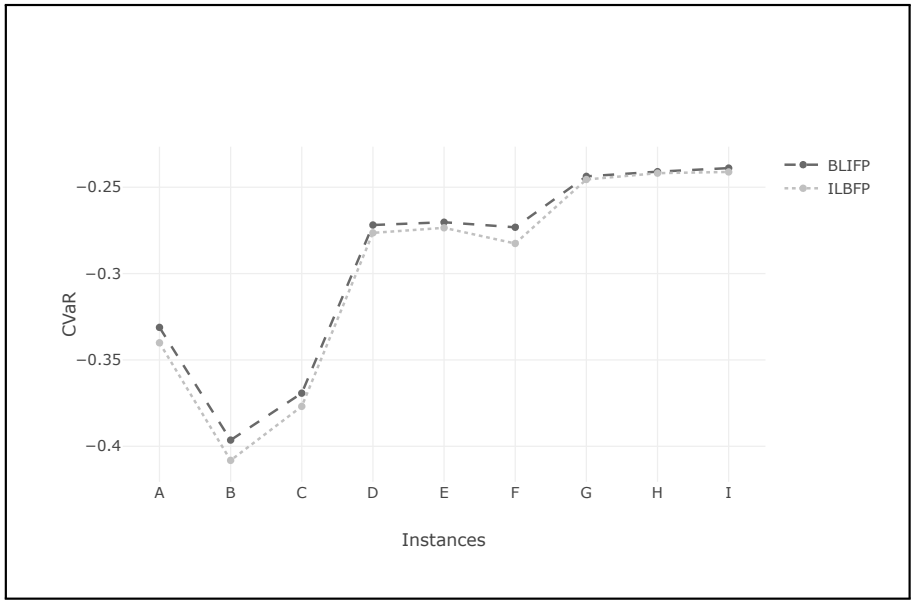
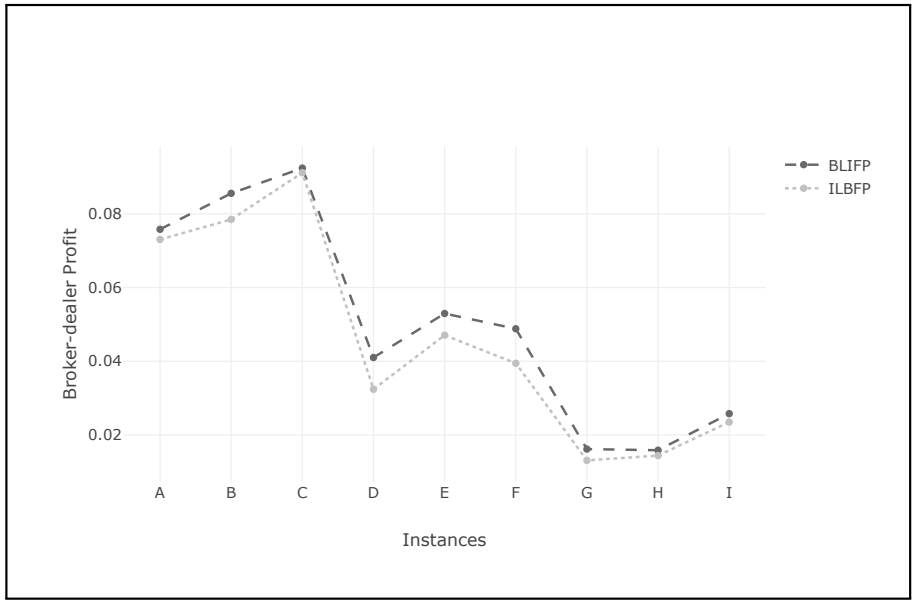


Figure: Values of the CVaR for **BLIFP** and **ILBFP** for $\alpha = 0.1$ and $\mu_0 = 0.05$ (left) and for $\alpha = 0.5$ and $\mu_0 = 0.1$ (right)

Discrete Pricing Portfolio. *Comparing solutions across models*



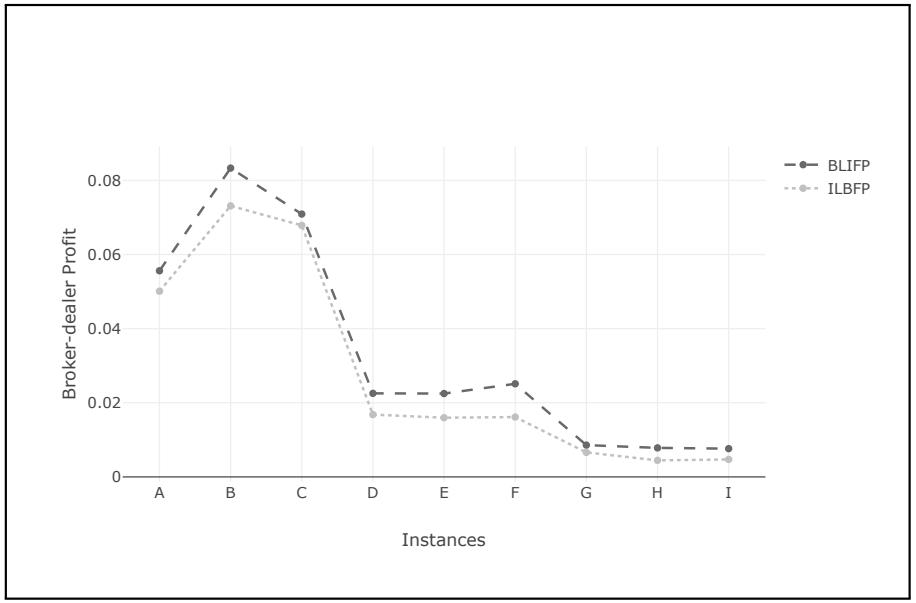
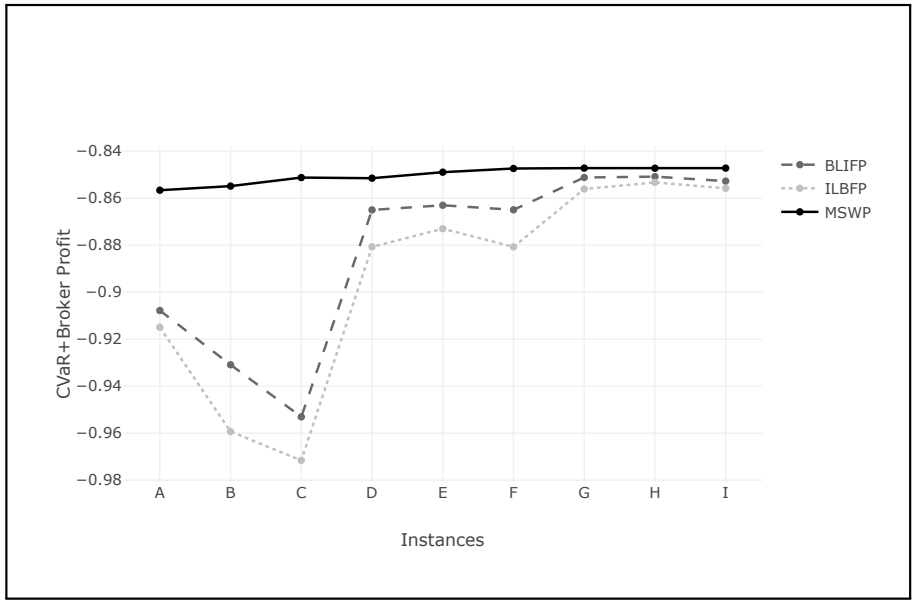


Figure: Values of the broker-dealer profit for **BLIFP** and **ILBFP**, for $\alpha = 0.1$ and $\mu_0 = 0.05$ (left) and for $\alpha = 0.5$ and $\mu_0 = 0.1$ (right)

Discrete Pricing Portfolio. *Comparing solutions across models*



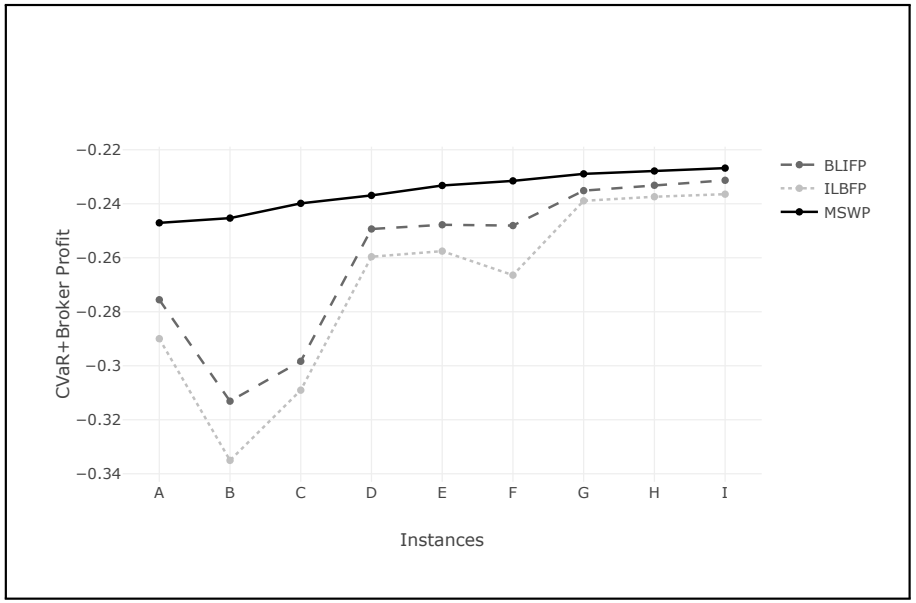


Figure: Values of the broker-dealer profit + CVaR for the three problems, for $\alpha = 0.1$ and $\mu_0 = 0.05$ (left) and for $\alpha = 0.5$ and $\mu_0 = 0.1$ (right)

Outline

Continuous Pricing Portfolio.

J. González, B. González, M. Leal and J. Puerto
"Global optimization for bilevel portfolio design: Economic insights from the Dow Jones index".
Omega, 2021.

Continuous Pricing Portfolio. *Contributions*

Contributions:

- Extension of the models to the case of continuous sets of prices.
- Extension to several leader and followers.
- More detailed economical case study

Continuous Pricing Portfolio. *Models*

The broker-dealer has to decide a price $0 \leq P_j \leq b_j, \forall j \in S$, in a set $\mathbb{P}$.

Continuous Pricing Portfolio. *Models*

The broker-dealer has to decide a price $0 \leq P_j \leq b_j, \forall j \in S$, in a set $\mathbb{P}$.

$$\Downarrow$$

Product of continuous variables

(given by $P_j x_j$ and by duality)

Continuous Pricing Portfolio. *Models*

$$\max_{p,(x^i,y^i,\eta^i,d^i)_{i\in M}} \quad \sum_{i\in M}\sum_{j\in S} p_j x_j^i \qquad \text{(BoT-BILEVEL)}$$

$$\text{s.t.} \quad p \in P \qquad (\mathrm{B^P.a})$$

$$\forall i \in M,\ x^i \in \arg\min_{x^i,y^i,\eta^i,d^i} -\eta^i + \frac{1}{\alpha^i}\sum_{t\in T}\pi_t d_t^i \qquad (\mathrm{I^P})$$

$$\text{s.t.} \quad y_t^i = \sum_{j\in S}(r_{jt}x_j^i - p_j x_j^i), \quad t \in T \qquad (\mathrm{I_0^P.a})$$

$$\sum_{t\in T}\pi_t y_t^i \geq E_{\min}^i \qquad (\mathrm{I_0^P.b})$$

$$\sum_{j\in S} x_j^i = 1 \qquad (\mathrm{I_0^P.c})$$

$$x_j^i \geq 0, \quad j \in S \qquad (\mathrm{I_0^P.d})$$

$$d_t^i \geq \eta^i - y_t^i, \quad t \in T \qquad (\mathrm{CVaR^P.a})$$

$$d_t^i \geq 0, \quad t \in T. \qquad (\mathrm{CVaR^P.b})$$

Continuous Pricing Portfolio. *Case study*

- Dow Jones Index
- **Risk profiles for the investors**: $\alpha = 0.05, 0.25, 0.5, 0.99$
- **Minimum expected return**: 32 problems for different values of $\mu_0 \in [-0.1, 0.72]$ (the highest expected return of any security).

Continuous Pricing Portfolio. *Case study*

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- **Minimum expected return**: 32 problems for different values of $\mu_0 \in [-0.1, 0.72]$ (the highest expected return of any security).

$$\mathbb{P} := \left\{ \sum_{j \in S} P_j \leq 0.3, 0 \leq P_j \leq 0.1 \ \forall j \in S \right\}$$

Continuous Pricing Portfolio. *Case study*

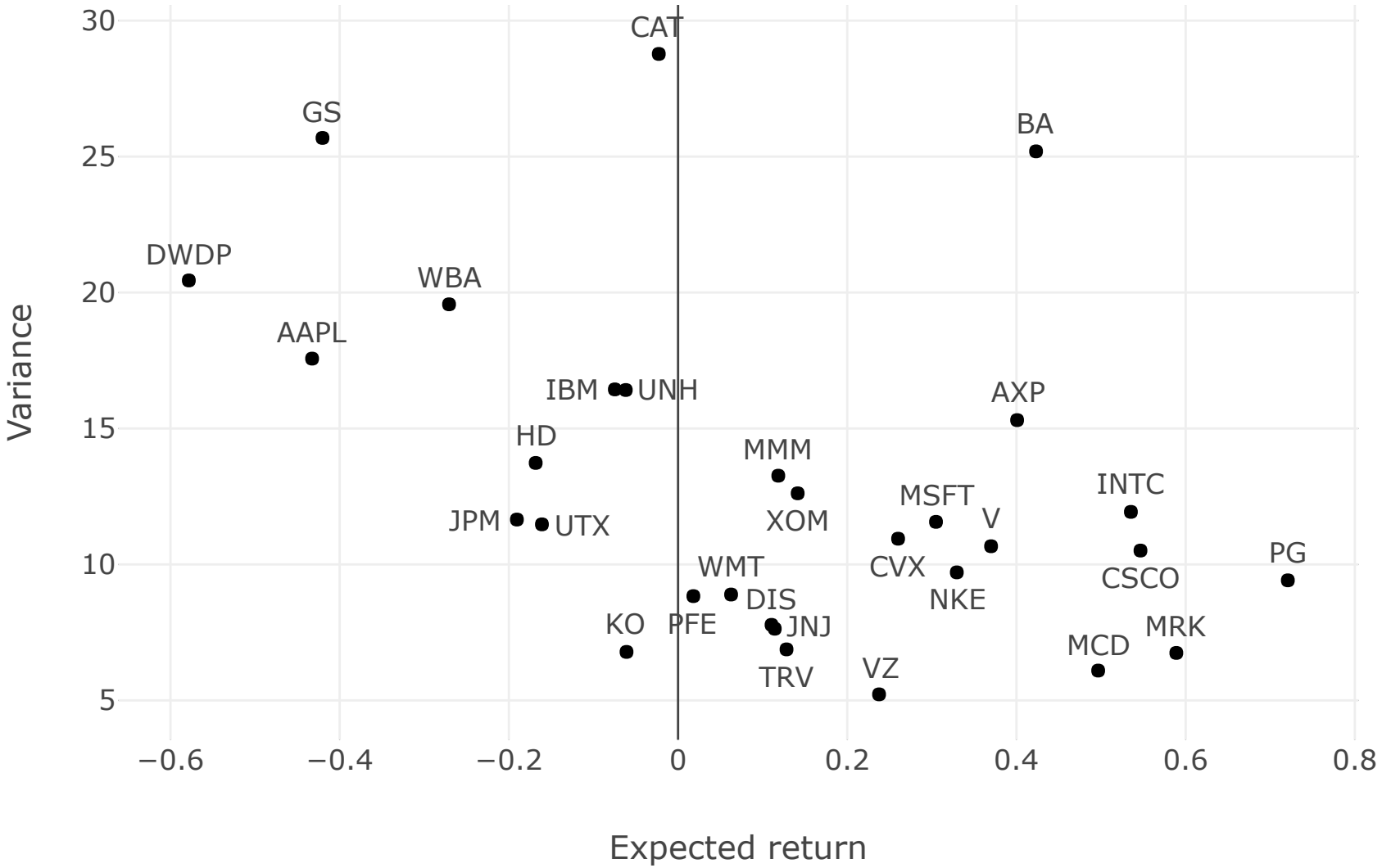


Figure: Expected return and variance for the Dow Jones securities.

Continuous Pricing Portfolio. *Case study*

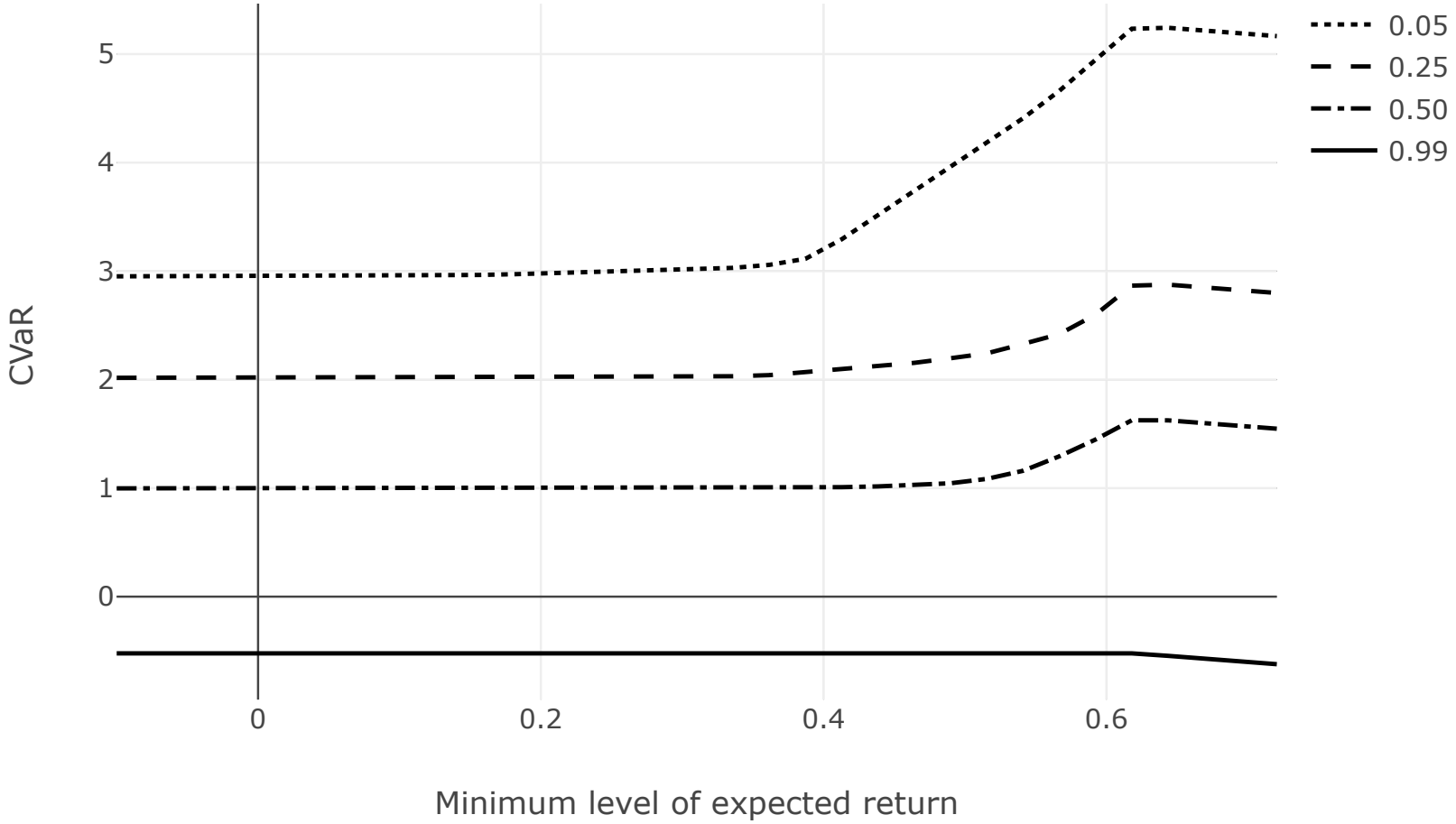


Figure: Objective function for each risk profile and each value of μ_0 for Broker-leader model.

Continuous Pricing Portfolio. *Case study*

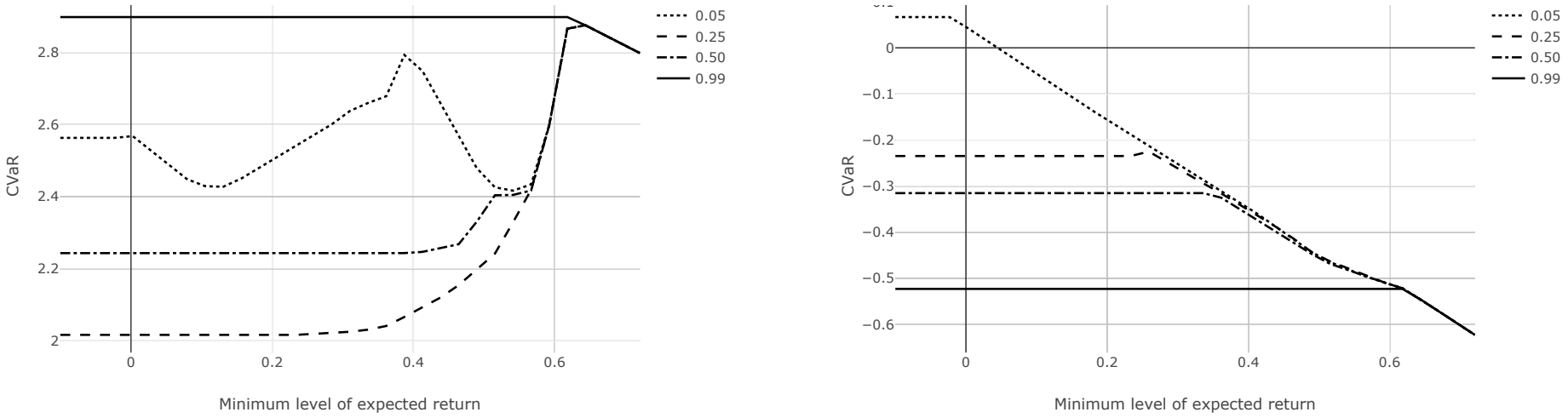


Figure: Comparison of CVaR levels ($\text{CVaR}_{0.25}$ left, $\text{CVaR}_{0.99}$ right) at the optimal solutions of the different risk profiles for Broker-leader model.

Continuous Pricing Portfolio. *Case study*

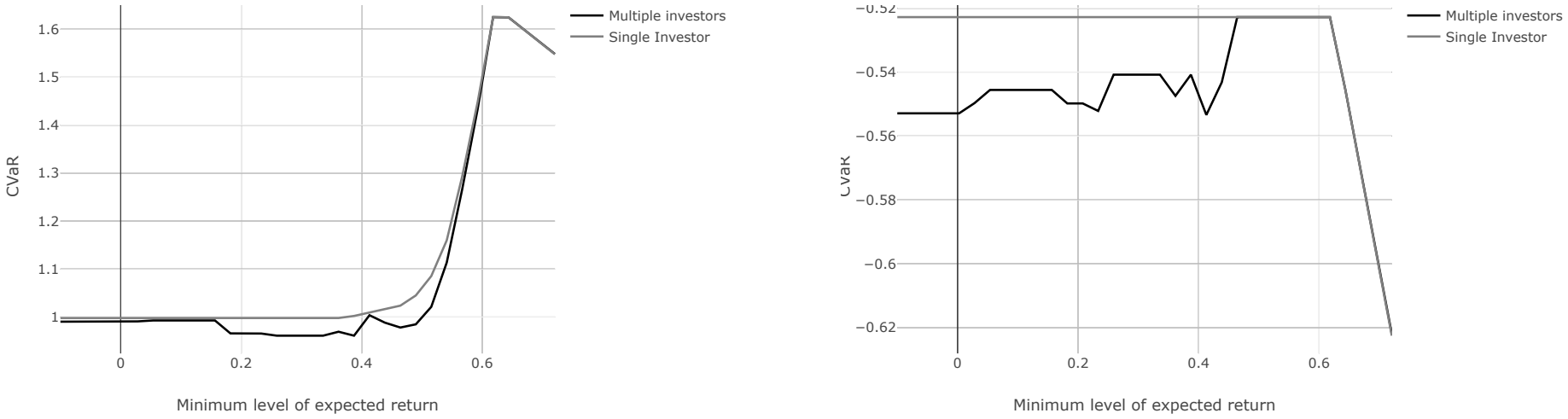


Figure: Comparison of CVaR ($\mathrm{CVaR}_{0.50}$ left, $\mathrm{CVaR}_{0.99}$ right)for one vs several investors for Broker-leader model.

Continuous Pricing Portfolio. *Case study*

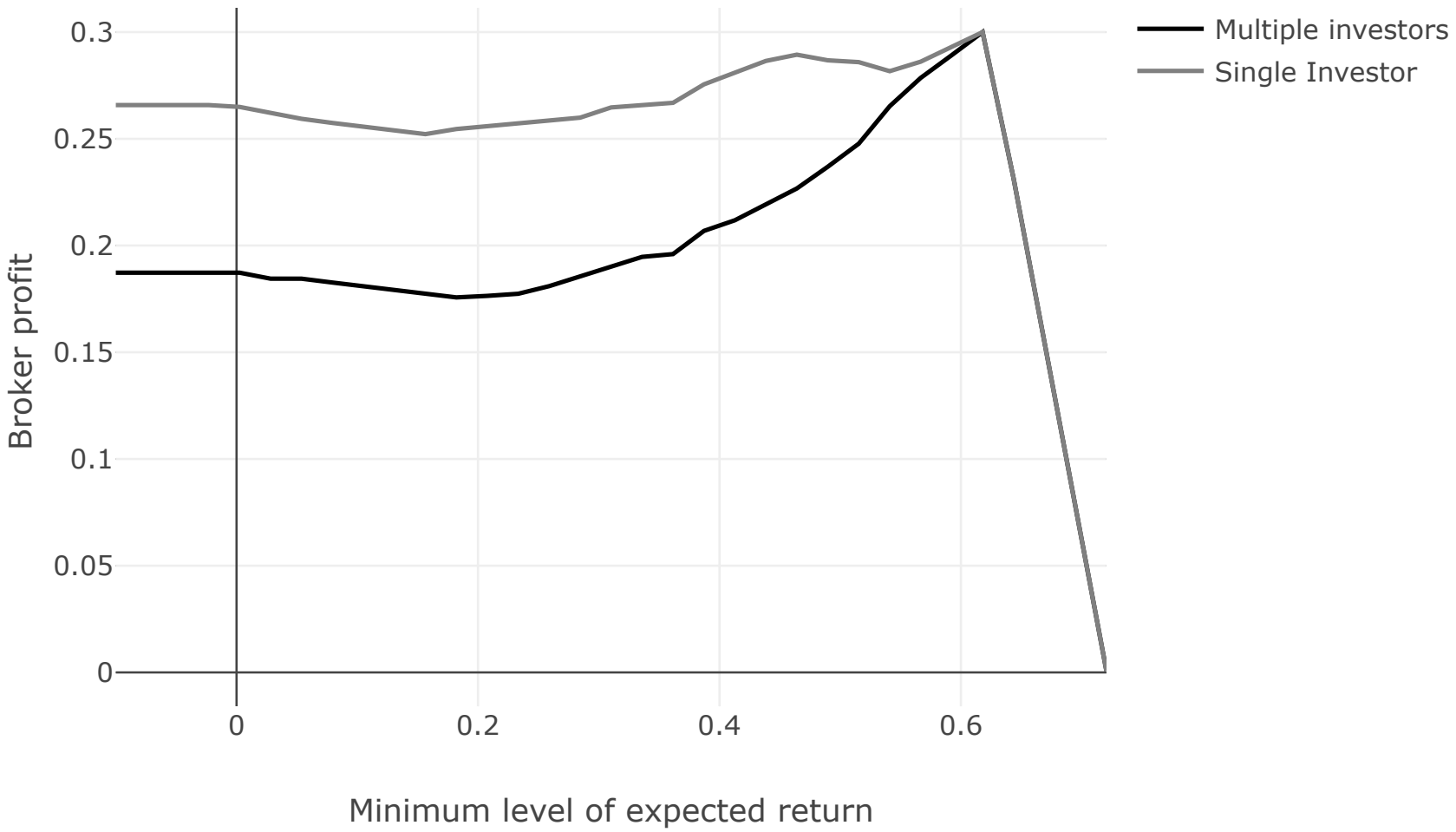
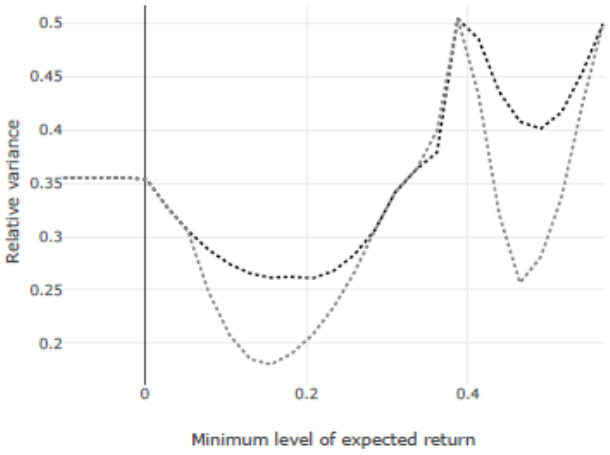
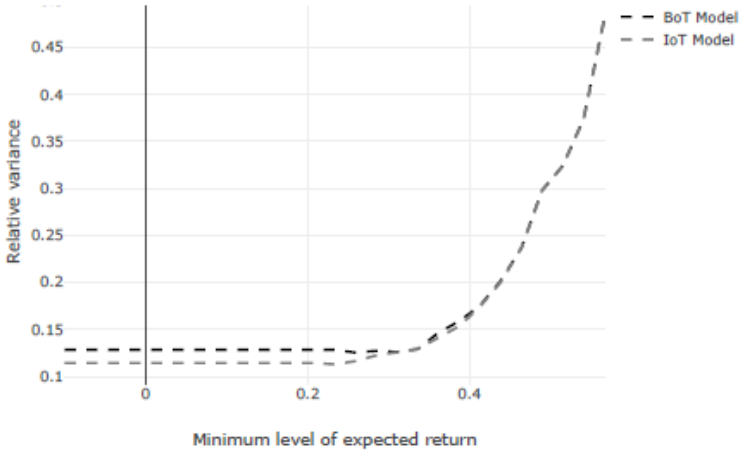


Figure: Profit of the broker for the Broker-leader model with one follower and multiple followers.

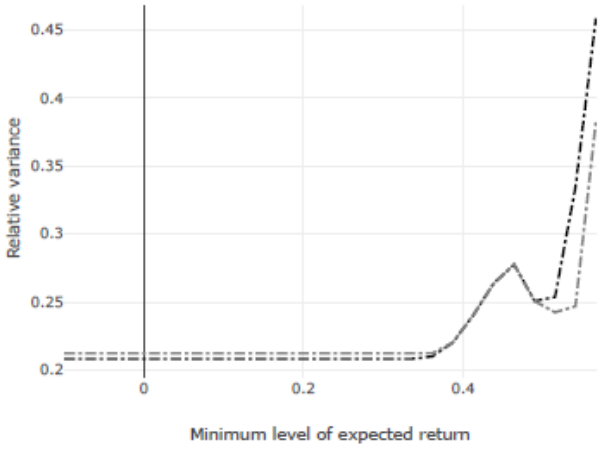
Continuous Pricing Portfolio. *Case study*



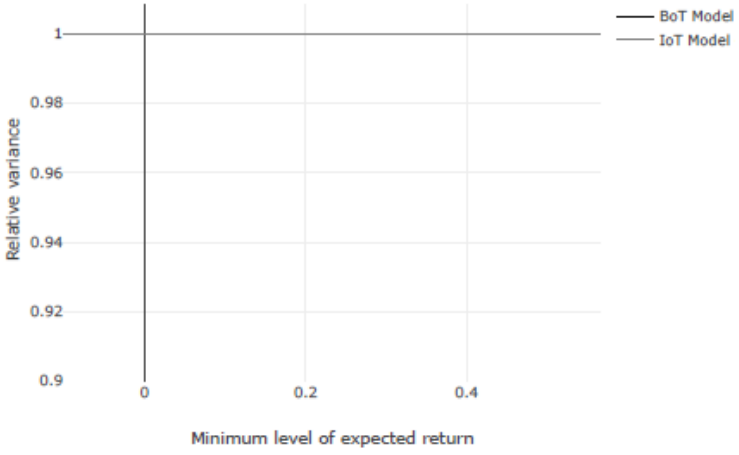
(a) $\mathrm{CVaR}_{0.05}$ risk profiles.



(b) $\mathrm{CVaR}_{0.25}$ risk profiles.

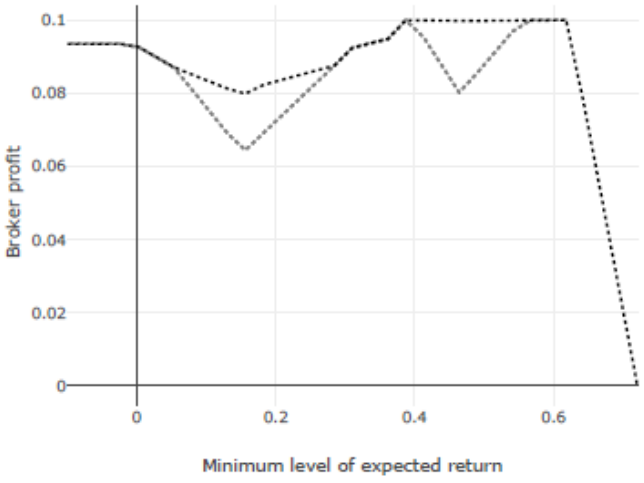


(c) $\mathrm{CVaR}_{0.50}$ risk profiles.

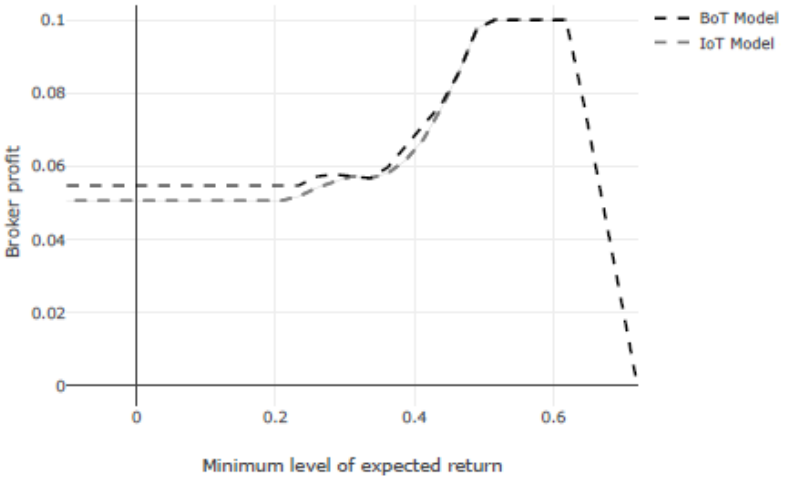


(d) $\mathrm{CVaR}_{0.99}$ risk profiles.

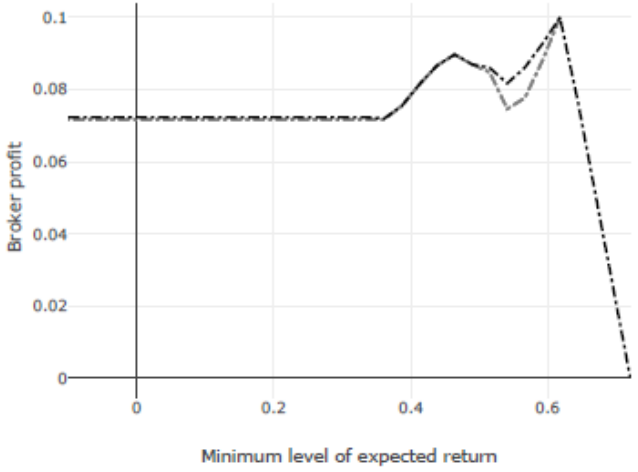
Continuous Pricing Portfolio. *Case study*



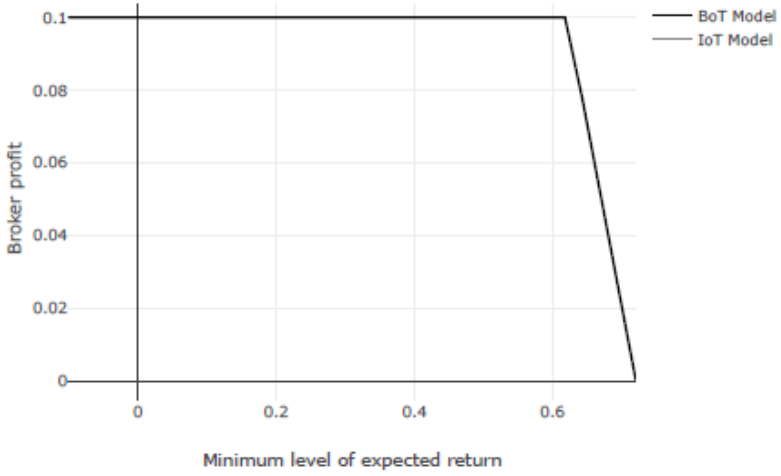
(a) Profit from $\mathrm{CVaR}_{0.05}$ risk profiles.



(b) Profit from $\mathrm{CVaR}_{0.25}$ risk profiles.



(c) Profit from $\mathrm{CVaR}_{0.50}$ risk profiles.



(d) Profit from $\mathrm{CVaR}_{0.99}$ risk profiles.

Outline

Ordered Pricing Portfolio.

S. Benati, M. Leal and J. Puerto
"Bilevel portfolio selection with ordered pricing models".
Work in progress.

Ordered Pricing Portfolio. *Contributions*

Contributions:

- We study PRICES POLICIES
- Broker optimizes a utility function given as an ordered weighted average of the investment.
- The monetary value charged to the investor and the broker's utility function may be different.
- Comparison between discrete and continuous sets of prices.

Broker problem:

$$\begin{aligned} \max \quad & \sum_{i \in S} \lambda_i (px)_{(i)} \\ \text{s.t.} & (px)_{(i)} \geq (px)_{(i+1)} \quad \forall i \in S, \\ & P \in \mathbb{P}. \end{aligned} \tag{$\mathbf{BP}^0$}$$

Investor problem:

$$
\begin{aligned}
\max \quad & \eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t \qquad (\mathbf{IP}^0)\\
\text{s.t.} & \sum_{j\in S}^{n} x_j = 1,\\
& d_t \geq \eta - y_t, \quad t = 1, ..., T,\\
& y_t = \sum_{j=1}^{n} r_{jt}x_j - \sum_{i\in S}\lambda_i'(px)_{(i)}, \quad t = 1, ..., T,\\
& \sum_{t=1}^{T}\pi_t y_t \geq \mu_0\\
& x_j \geq 0, \quad j \in S,\\
& d_t \geq 0, \quad t = 1, ..., T.
\end{aligned}
$$

Broker problem:

$$\begin{aligned} \max \quad & \sum_{i \in S} \lambda_i (px)_{(i)} & (\mathbf{BP}^0) \\ \text{s.t.} & (px)_{(i)} \geq (px)_{(i+1)} & \forall i \in S, \\ & P \in \mathbb{P}. \end{aligned}$$

Broker problem:

$$\begin{aligned} \max \quad & \sum_{i \in S} \lambda_i (px)_{(i)} \\ \text{s.t.} & (px)_{(i)} \geq (px)_{(i+1)} \quad \forall i \in S, \\ & P \in \mathbb{P}. \end{aligned} \qquad (\mathbf{BP}^0)$$

$$\begin{aligned} \max \quad & \sum_{i,j \in S} \lambda_i p_j x_j z_{ij} \\ \text{s.t.} & \sum_{i \in S} z_{ij} \leq 1, \quad j \in S, \\ & \sum_{j \in S} z_{ij} \leq 1, \quad i \in S, \\ & 0 \leq z_{ij} \leq 1, \quad i, j \in S, \\ & P \in \mathbb{P}. \end{aligned}$$

Theorem

Problem $(\mathbf{IP}^0)$ *can be rewritten as the following LP:*

$$
\begin{aligned}
\max \quad & \eta - \frac{1}{\alpha}\sum_{t=1}^{T}\pi_t d_t \\
s.t. & \sum_{j\in S} x_j = 1, \\
& d_t \geq \eta - y_t, \quad t = 1, ..., T, \\
& y_t = \sum_{j=1}^{n} r_{jt}x_j - \left(\sum_{j\in S} u_j + \sum_{i\in S} v_i\right), \quad t = 1, ..., T, \\
& \sum_{t=1}^{T} p_t y_t \geq \mu_0, \\
& u_j + v_i \geq \lambda'_i p_j x_j, \quad i, j \in S, \\
& u_j \geq 0, \quad j \in S, \\
& v_i \geq 0, \quad i \in S, \\
& x_j \geq 0, \quad j \in S, \\
& d_t \geq 0, \quad t = 1, ..., T.
\end{aligned}
$$

Ordered Pricing Portfolio. *Numerical study*

- Dow Jones Index
- **Risk profiles for the investors**: $\alpha = 0.01$
- **Minimum expected return**: 20 problems for different values of $\mu_0 \in [-0.1, 0.72]$ (the highest expected return of any security).
- $$\mathbb{P} := \left\{ \sum_{j \in S} P_j \leq 0.3, 0 \leq P_j \leq 0.1 \forall j \in S \right\}$$

Ordered Pricing Portfolio. *Numerical study*

- Compare charging strategies. Considering the following vectors for λs and λ's:
  - $(1, 0, \ldots, 0)$. *Best security.*
  - $(1, 1, \ldots, 1)$. *All securities.*
  - $(1, 1, 1, 0, \ldots, 0)$. *3 best securities.*
  - $(1, 1, 1, 1, 1, 0, \ldots, 0)$. *5 best securities.*
  - $(1, 0.5, 0.25, \ldots, 0)$. $1 \times 1st + 0.5 \times 2nd + \ldots$.
- Compare discrete and continuous families of prices.

Ordered Pricing Portfolio. *Numerical study*

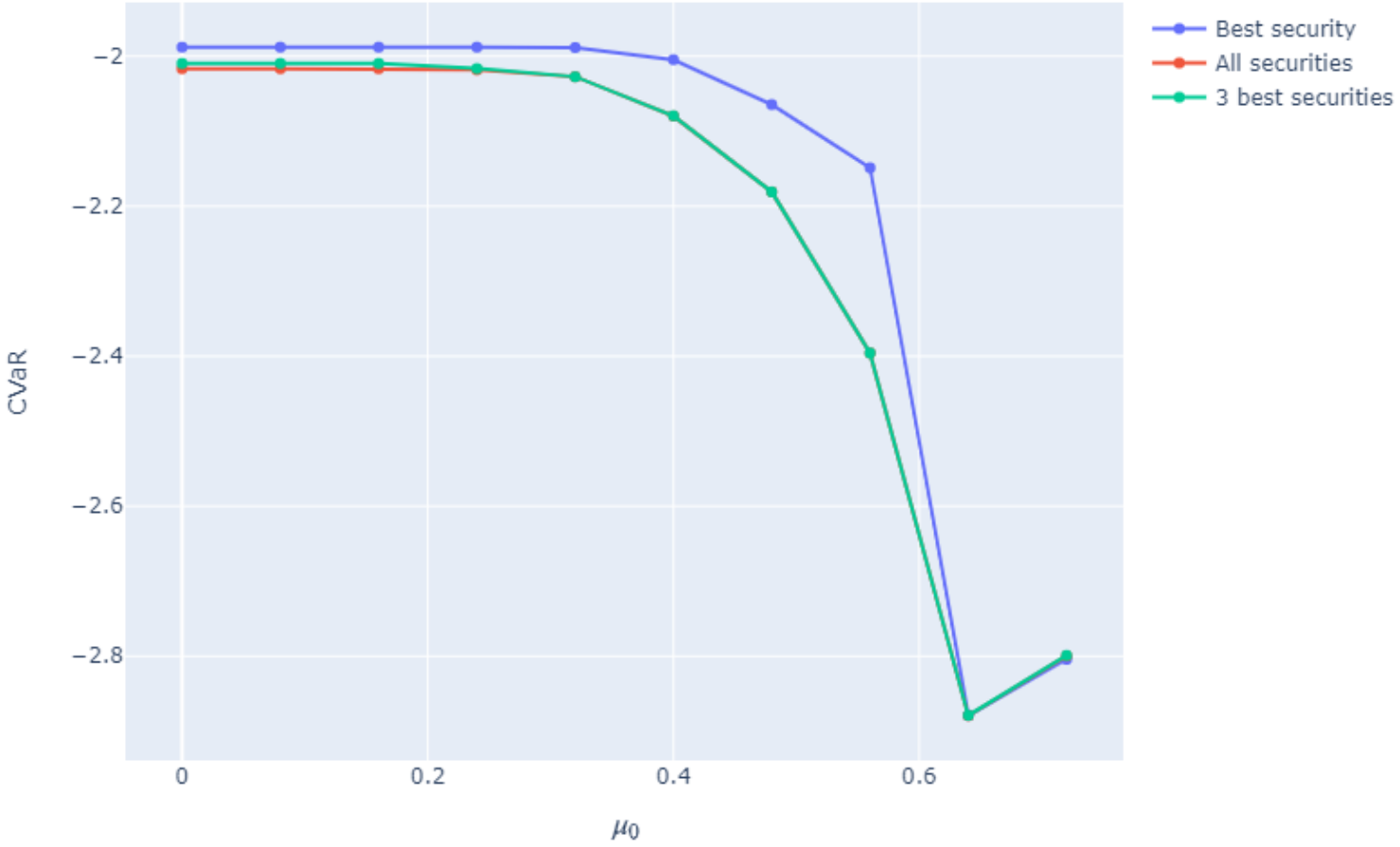


Figure: CVaR for $\mu_0 \in [-0.1, 0.72]$, and different λ

Ordered Pricing Portfolio. *Numerical study*

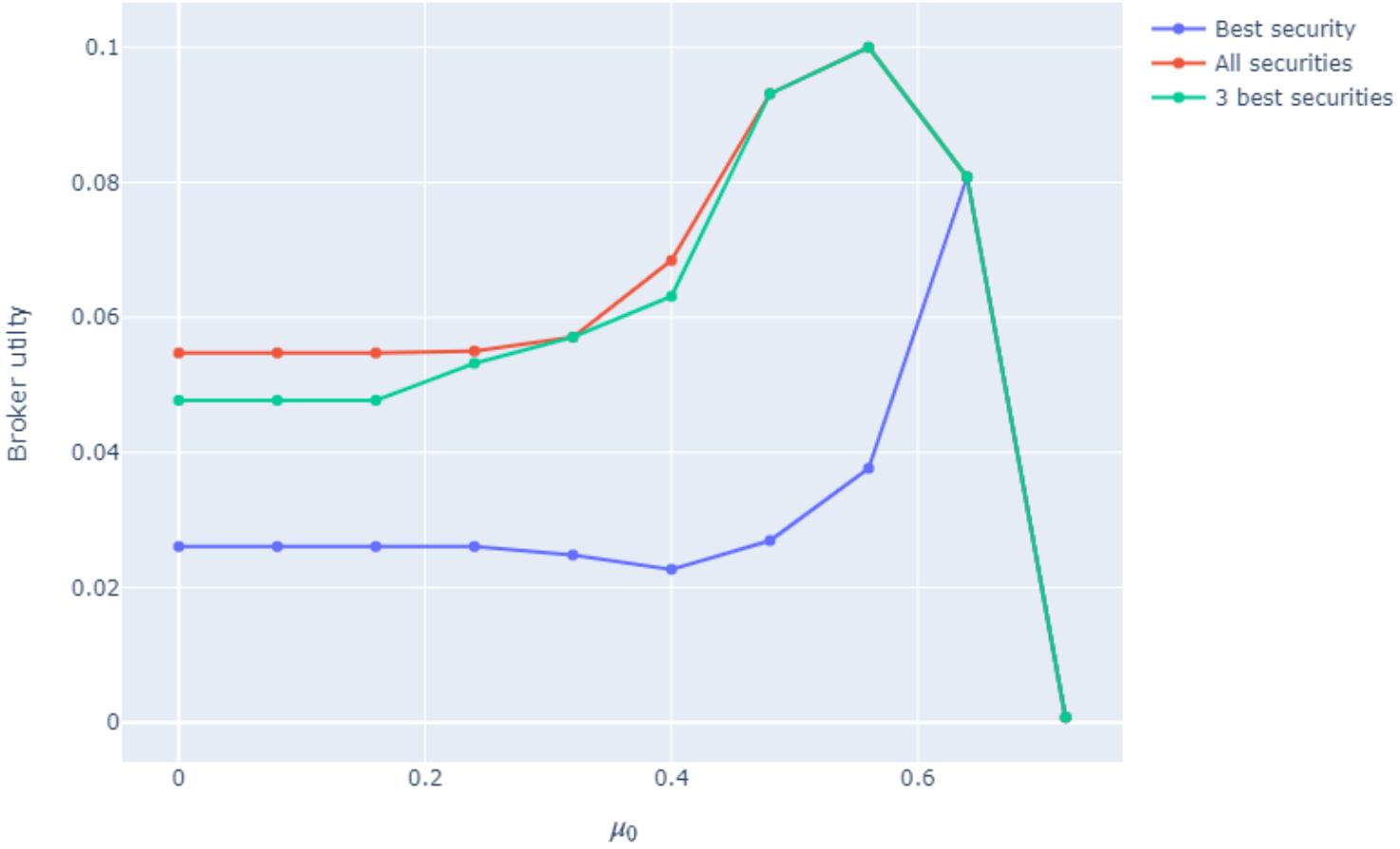


Figure: Broker profit for $\mu_0 \in [-0.1, 0.72]$, and different λ.

Hierarchies, prices and portfolios

Marina Leal Palazón (m.leal@umh.es)

Advances on data analysis, logistics and transportation problems on complex networks