From the Discrete Ordered Median Problem to other combinatorial optimization problems

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Advances on logistics and transportation problems on complex networks: Evaluation and conclusions.
June 23–25, 2021.

1. MILP for DOMP
   - Problem definition and notation
   - Motivations
   - Formulations
2. A Branch-and-Cut-and-Price procedure for DOMP
   - Set partitioning formulation for DOMP
   - Master Problem
   - Restricted relaxed MP and Pricing problem
   - Branching
   - Valid inequalities
3. Convergence of column generation

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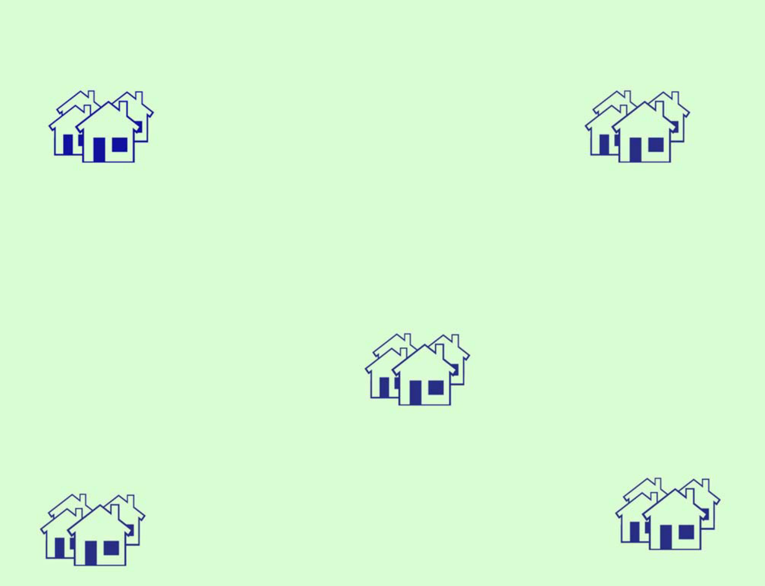

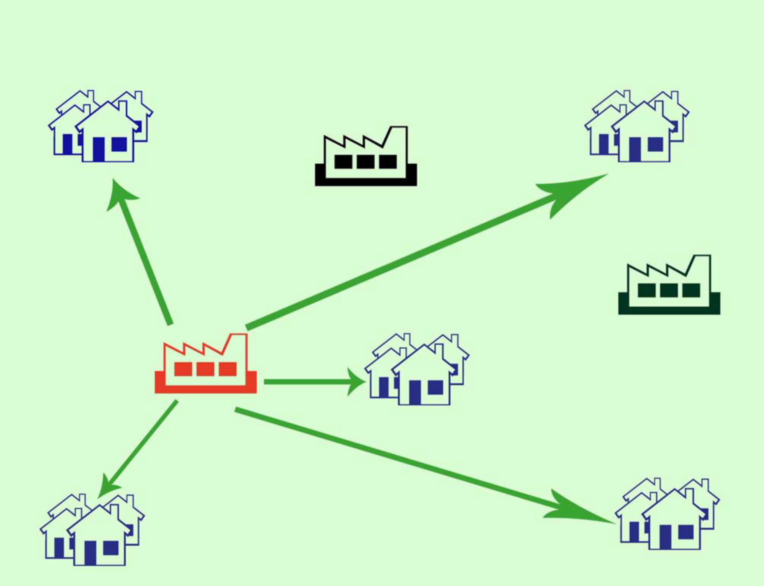

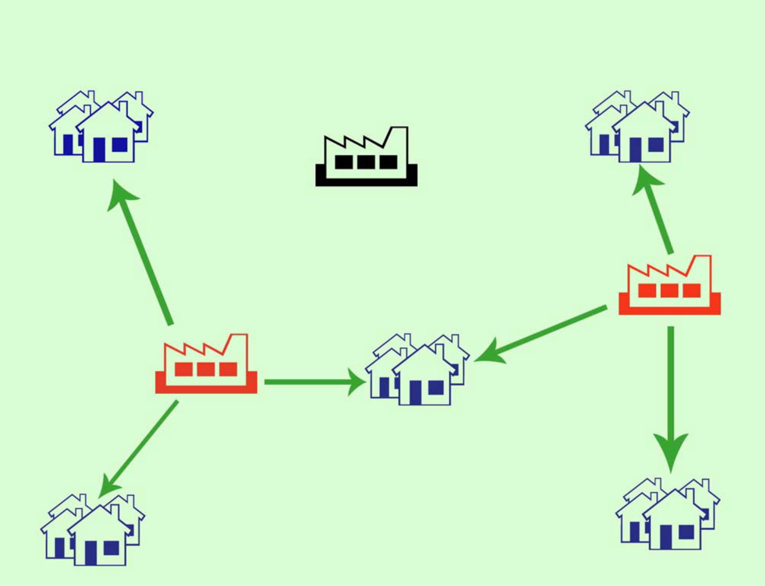

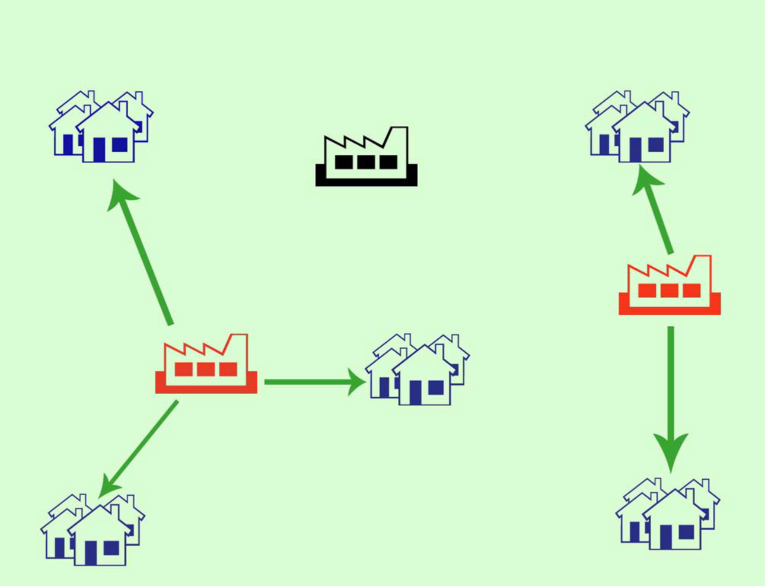

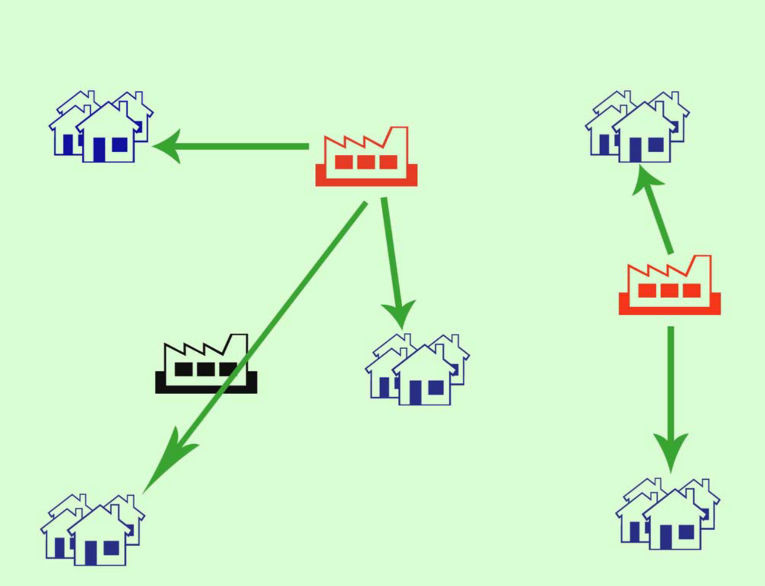

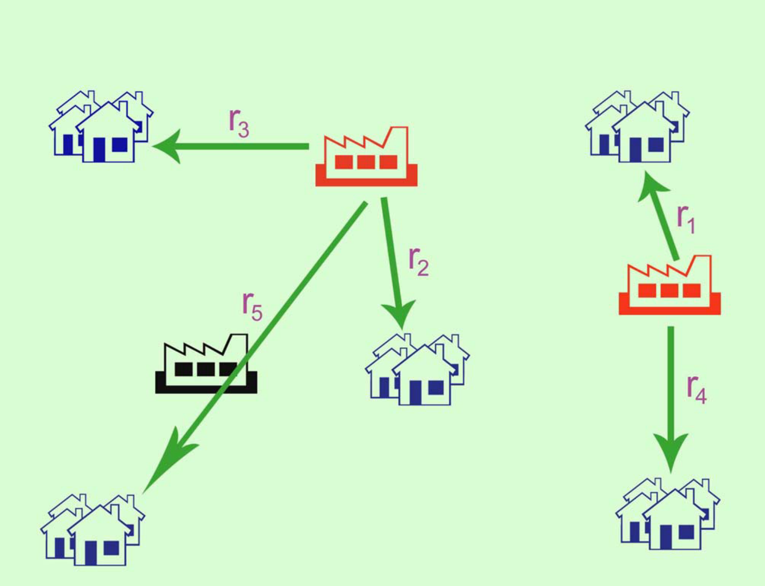
r_3
r_1
r_2
r_5
r_4

- Let I be a set of clients. $|I| = n$.
- Let J be a set of facilities. $|J| = p$.
- Let C be the cost matrix.
  c_{ij} is the associated cost if client i is assigned to facility j.
- Let λ be the weighted ordered vector.

feasible solution

- $J \subseteq I$
- $|J| = p$

$$c_i(J) = \min_{j \in J} c_{ij}.$$

$$c_{\leq}^{1}(J) \leq \cdots \leq c_{\leq}^{n}(J)$$

$$\min_{J} \sum_{k=1}^{n} \lambda_k c_{\leq}^{k}(J)$$

λ-vector	Name
$(1, 1, \dots, 1, 1)$	p-median
$(0, 0, \dots, 0, 1)$	p-center
$(0, 0, \dots, 0, 0, \underbrace{1, 1, \dots, 1, 1}_{k})$	k-centrum
$(\underbrace{0, 0, \dots, 0, 0}_{k_1}, 1, 1, \dots, 1, 1, \underbrace{0, 0, \dots, 0, 0}_{k_2})$	$(k_1 + k_2)$-trimmed mean
$(0, 1, 0, 1, 0, 1, 0, 1, \dots)$	–
$(\dots, 0, 0, 1, 0, 0, 1)$	–
$(\alpha, \alpha, \dots, \alpha, \alpha, 1)$	Centdian
$\vdots$	

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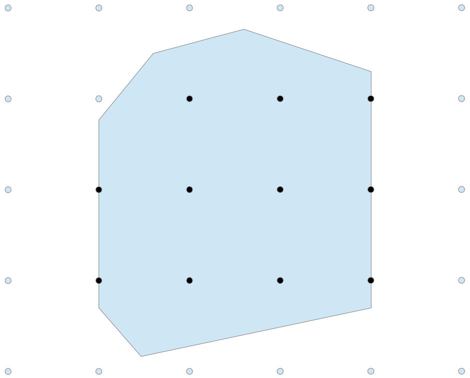
- Problem is NP-hard.
- Tight formulations for combinatorial problems.



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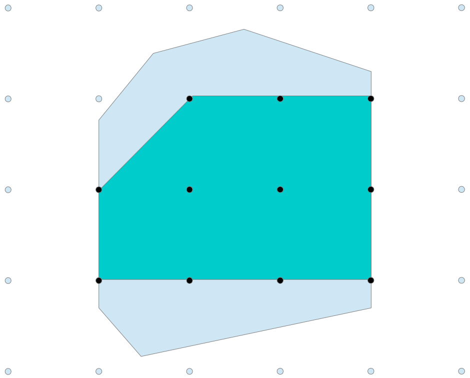

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3. Convergence of column generation

$DOMP_1$ formulation (N. Boland, P. Domínguez-Marín, S.Nickel and J.Puerto, 2006)

Three-index formulations

$$x_{ij}^k = \begin{cases} 1 & \text{if client } i \text{ is served by facility } j \text{ and cost } c_k(J) = c_{ij} \text{ is the } k\text{-th smallest in the ordered sequence } c_{\leq}(J) \\ 0 & \text{otherwise} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{if } j \notin J \end{cases}$$

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$$y_j = \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{if } j \notin J \end{cases}$$

$$\begin{aligned}
\min \quad & \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \lambda^k c_{ij} x_{ij}^k \\
s.t. \quad & \sum_{j=1}^n \sum_{k=1}^n x_{ij}^k = 1 && \forall i \\
& \sum_{i=1}^n \sum_{j=1}^n x_{ij}^k = 1 && \forall k \\
& \sum_{k=1}^n x_{ij}^k \leq y_j && \forall i,j \\
& \sum_{j=1}^n y_j = p \\
& \sum_i^n \sum_j^n c_{ij} x_{ij}^{k-1} \leq \sum_i^n \sum_j^n c_{ij} x_{ij}^k && k = 2, \ldots, n \\
& x_{ij}^k, y_j \in \{0,1\} && \forall i,j,k
\end{aligned}$$

Scheme

Example

$$C = \begin{pmatrix} 0 & 2 & 7 & 4 \\ 1 & 0 & 5 & 5 \\ 3 & 6 & 0 & 2 \\ 9 & 4 & 1 & 0 \end{pmatrix}$$

	11	22	33	44	21	43	12	34	31	14	42	23	24	32	13	41
$k=4$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$k=3$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$k=2$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$k=1$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○

Scheme

Example

$J = \{1, 3\}$

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$k = 3$	○	○	○	○	●	○	○	○	○	○	○	○	○	○	○	○
$k = 2$	○	○	●	○	○	○	○	○	○	○	○	○	○	○	○	○
$k = 1$	●	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○

	11	22	33	44	21	43	12	34	31	14	42	23	24	32	13	41
	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
k	●	●	●	●	●	●	○	○	○	○	○	○	○	○	○	○
$k-1$	○	○	○	○	○	●	●	●	●	●	●	●	●	●	●	●
	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○

	11	22	33	44	21	43	12	34	31	14	42	23	24	32	13	41
	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
k	○	○	○	○	●	○	○	○	○	○	○	○	○	○	○	○
$k-1$	○	○	○	○	○	○	○	○	○	○	●	○	○	○	○	○
	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○

$DOMP_3$ formulation (*strong order constraints*)

Three-index formulations

$$x_{ij}^k = \begin{cases} 1 & \text{if client } i \text{ is served by facility } j \text{ and cost } c_k(J) = c_{ij} \text{ is the } k\text{-th smallest in the ordered sequence } c_{\leq}(J) \\ 0 & \text{otherwise} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{if } j \notin J \end{cases}$$

$$\begin{array}{lll}
\min & \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \lambda^k c_{ij} x_{ij}^k & \\
s.t. & \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{n} x_{ij}^k = 1 & \forall i \\
& \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}^k = 1 & \forall k \\
& \displaystyle\sum_{k=1}^{n} x_{ij}^k \leq y_j & \forall i, j \\
& \displaystyle\sum_{j=1}^{n} y_j = p & \\
& \displaystyle\sum_{\hat{\imath}\hat{\jmath} \succeq ij} x_{\hat{\imath}\hat{\jmath}}^{k-1} + \sum_{\hat{\imath}\hat{\jmath} \preceq ij} x_{\hat{\imath}\hat{\jmath}}^{k} \leq 1 & \forall i, j; k = 2, \dots, n \\
& x_{ij}^k, y_j \in \{0,1\} & \forall i, j, k
\end{array}$$

$DOMP_4$ formulation (*weak order constraints*)

Three-index formulations

$$x_{ij}^k = \begin{cases} 1 & \text{if client } i \text{ is served by facility } j \text{ and cost } c_k(J) = C_{ij} \text{ is the } k\text{-th smallest in the ordered sequence } c_{\leq}(J) \\ 0 & \text{otherwise} \end{cases}$$

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& \sum_{k=1}^{n} x_{ij}^k \leq y_j && \forall i, j \\
& \sum_{j=1}^{n} y_j = p \\
& \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{\substack{i'=1 \\ :i'j'}}^{n} \sum_{\substack{j'=1 \\ \preceq ij}}^{n} x_{i'j'}^k + \sum_{\substack{i'=1 \\ :i'j'}}^{n} \sum_{\substack{j'=1 \\ \succeq ij}}^{n} x_{i'j'}^{k-1} \right) \leq n^2 && k = 2, \dots, n \\
& x_{ij}^k, y_j \in \{0, 1\} && \forall i, j, k
\end{aligned}$$

Remarks

Achievements

- ✓ We have developed several tighter formulations.
- ✓ We have found interesting valid inequalities.

M. Labbé, D. Ponce, and J.Puerto
A comparative study of formulations and solution methods for the discrete ordered p-median problem.
Computers & Operations Research, 78 (2017) 230–242

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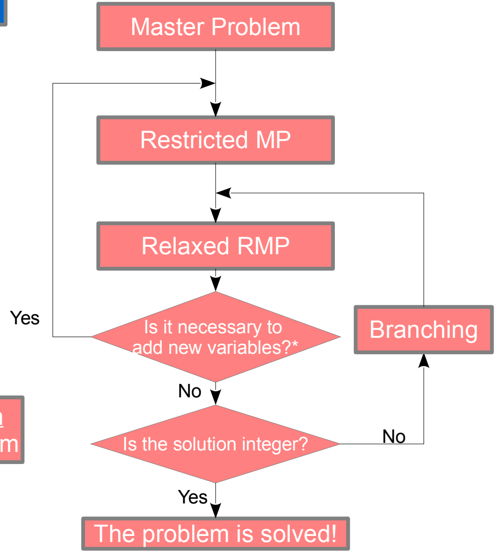
Branch & Price



*Subproblem
Pricing Problem

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$$S = \left\{ \begin{array}{ll} (i,k): & \text{costumer } i\text{'s cost is ranked at position } k \text{ and} \\ & \text{all costumers allocated to the same facility} \end{array} \right\}$$

$$y_S^j = \begin{cases} 1 & \text{if set } S \text{ is in the solution and it is allocated} \\ & \text{to facility } j \text{ that must be open.} \\ 0 & \text{otherwise.} \end{cases}$$

$$x_{ij}^k = \sum_{S \ni (i,k)} y_S^j$$

$$c_S^j = \sum_{(i,k) \in S} \lambda^k c_{ij}$$

Example

$$C = \begin{pmatrix} 1 & 5 & 9 \\ 4 & 2 & 7 \\ 6 & 8 & 3 \end{pmatrix} \qquad \lambda = (2\ 1\ 0.5)$$

$$\begin{array}{lll}
S_1 = \{(1,1)\} & S_{12} = \{(1,1),(3,2)\} & S_{23} = \{(2,1),(3,3)\} \\
S_2 = \{(1,2)\} & S_{13} = \{(1,1),(3,3)\} & S_{24} = \{(2,2),(3,1)\} \\
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S_6 = \{(2,3)\} & S_{17} = \{(1,2),(3,3)\} & S_{28} = \{(1,1),(2,2),(3,3)\} \\
S_7 = \{(3,1)\} & S_{18} = \{(1,3),(2,1)\} & S_{29} = \{(1,1),(2,3),(3,2)\} \\
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\end{array}$$

Associated cost to variable y_{18}^3: $c_{18}^3 = \lambda^3 c_{13} + \lambda^1 c_{23} = 9 \cdot 0.5 + 7 \cdot 2 = 18.5$

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\begin{array}{llll}
\min & \displaystyle\sum_{j=1}^{n}\sum_{S} c_S^j y_S^j & & \\
s.t. & \displaystyle\sum_{j=1}^{n}\sum_{S\ni(i,\cdot)} y_S^j & = 1 & \forall\, i \\
& \displaystyle\sum_{j=1}^{n}\sum_{S\ni(\cdot,k)} y_S^j & = 1 & \forall\, k \\
& \displaystyle\sum_{S} y_S^j & \leq 1 & \forall\, j \\
& \displaystyle\sum_{j=1}^{n}\sum_{S} y_S^j & \leq p & \\
& \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{\substack{S\ni(\hat{\imath},k)\\ :C_{\hat{\imath}\hat{\jmath}}\leq C_{ij}}} y_S^{\hat{\jmath}} + \sum_{\substack{S\ni(\hat{\imath},k-1)\\ :C_{\hat{\imath}\hat{\jmath}}\geq C_{ij}}} y_S^{\hat{\jmath}}\right) & \leq n^2 & k=2,\ldots,n \\
& y_S^j & \in\{0,1\} & \forall\, S,j.
\end{array}
$$

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3 Convergence of column generation

Relaxed Restricted Master Problem

$$
\begin{array}{llll}
\min & \displaystyle\sum_{j=1}^{n}\sum_{S} c_S^j y_S^j & & \\
s.t. & \displaystyle\sum_{j=1}^{n}\sum_{S\ni(i,\cdot)} y_S^j & = 1 & \forall\, i \\
 & \displaystyle\sum_{j=1}^{n}\sum_{S\ni(\cdot,k)} y_S^j & = 1 & \forall\, k \\
 & \displaystyle -\sum_{S} y_S^j & \geq -1 & \forall\, j \\
 & \displaystyle -\sum_{j=1}^{n}\sum_{S} y_S^j & \geq -p & \\
 & \displaystyle -\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{\substack{S\ni(\hat{\imath},k)\\ :C_{\hat{\imath}\hat{\jmath}}\leq C_{ij}}} y_S^{\hat{\jmath}} + \sum_{\substack{S\ni(\hat{\imath},k-1)\\ :C_{\hat{\imath}\hat{\jmath}}\geq C_{ij}}} y_S^{\hat{\jmath}} \right) & \geq -n^2 & k=2,\dots,n \\
 & y_S^j & \geq 0 & \forall\, S, j
\end{array}
$$

Relaxed Restricted Master Problem

$$
\begin{array}{llllll}
\min & \displaystyle\sum_{j=1}^{n}\sum_{S} c_S^j y_S^j & & & & \textbf{Dual Multipliers} \\
s.t. & \displaystyle\sum_{j=1}^{n}\sum_{S\ni(i,\cdot)} y_S^j & = 1 & \forall\, i & & \alpha_i \\
 & \displaystyle\sum_{j=1}^{n}\sum_{S\ni(\cdot,k)} y_S^j & = 1 & \forall\, k & & \beta_k \\
 & \displaystyle -\sum_{S} y_S^j & \geq -1 & \forall\, j & & \gamma_j \geq 0 \\
 & \displaystyle -\sum_{j=1}^{n}\sum_{S} y_S^j & \geq -p & & & \delta \geq 0 \\
 & \displaystyle -\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{\substack{S\ni(\hat{\imath},k)\\ :C_{\hat{\imath}\hat{\jmath}}\leq C_{ij}}} y_S^{\hat{\jmath}} + \sum_{\substack{S\ni(\hat{\imath},k-1)\\ :C_{\hat{\imath}\hat{\jmath}}\geq C_{ij}}} y_S^{\hat{\jmath}}\right) & \geq -n^2 & k=2,\ldots,n & & \epsilon_k \geq 0 \\
 & y_S^j & \geq 0 & \forall\, S, j & &
\end{array}
$$

Dual Problem

$$
\begin{aligned}
\textbf{(DP)}\ \max \quad & \sum_{i=1}^{n} \alpha_i + \sum_{k=1}^{n} \beta_k - \sum_{j=1}^{n} \gamma_j - p\delta - \sum_{k=2}^{n} n^2 \epsilon_k \\
s.t. \quad & \sum_{\substack{i=1 \\ :(i,\cdot)\in S}}^{n} \alpha_i + \sum_{\substack{k=1 \\ :(\cdot,k)\in S}}^{n} \beta_k - \gamma_j - \delta - \sum_{k=2}^{n}\sum_{\hat{\imath}=1}^{n}\sum_{\hat{\jmath}=1}^{n} \left(\sum_{\substack{(i,k)\in S \\ :C_{\hat{\imath}\hat{\jmath}} \geq C_{ij}}} \epsilon_k + \sum_{\substack{(i,k-1)\in S \\ :C_{\hat{\imath}\hat{\jmath}} \leq C_{ij}}} \epsilon_k \right) \leq c_S^j \quad \forall\, j, S \\
& \gamma_j, \delta, \epsilon_k \geq 0 \quad \forall\, i, j, k.
\end{aligned}
$$

Dual Problem

$$
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\textbf{(DP)}\ \max \quad & \sum_{i=1}^{n} \alpha_i + \sum_{k=1}^{n} \beta_k - \sum_{j=1}^{n} \gamma_j - p\delta - \sum_{k=2}^{n} n^2 \epsilon_k \\
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& \gamma_j, \delta, \epsilon_k \geq 0 \quad \forall\, i, j, k.
\end{aligned}
$$

$$
\bar{c}_S^j = c_S^j + \gamma_j^* + \delta^* + \sum_{k=2}^{n}\sum_{\hat{\imath}=1}^{n}\sum_{\hat{\jmath}=1}^{n} \left(\sum_{\substack{(i,k)\in S\\ :C_{\hat{\imath}\hat{\jmath}}\geq C_{ij}}} \epsilon_k^* + \sum_{\substack{(i,k-1)\in S\\ :C_{\hat{\imath}\hat{\jmath}}\leq C_{ij}}} \epsilon_k^* \right) - \sum_{\substack{i=1\\ :(i,\cdot)\in S}}^{n} \alpha_i^* - \sum_{\substack{k=1\\ :(\cdot,k)\in S}}^{n} \beta_k^* .
$$

Pricing subproblem

$$\overline{c}_S^j = c_S^j + \gamma_j^* + \delta^* + \sum_{k=2}^{n} \sum_{\hat{\imath}=1}^{n} \sum_{\hat{\jmath}=1}^{n} \left(\sum_{\substack{(i,k)\in S \\ : C_{\hat{\imath}\hat{\jmath}} \geq C_{ij}}} \epsilon_k^* + \sum_{\substack{(i,k-1)\in S \\ : C_{\hat{\imath}\hat{\jmath}} \leq C_{ij}}} \epsilon_k^* \right) - \sum_{\substack{i=1 \\ :(i,\cdot)\in S}}^{n} \alpha_i^* - \sum_{\substack{k=1 \\ :(\cdot,k)\in S}}^{n} \beta_k^* .$$

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$$\overline{c}_S^j = c_S^j + \gamma_j^* + \delta^* + \sum_{k=2}^{n}\sum_{\hat{\imath}=1}^{n}\sum_{\hat{\jmath}=1}^{n}\left(\sum_{\substack{(i,k)\in S\\ :C_{\hat{\imath}\hat{\jmath}}\geq C_{ij}}} \epsilon_k^* + \sum_{\substack{(i,k-1)\in S\\ :C_{\hat{\imath}\hat{\jmath}}\leq C_{ij}}} \epsilon_k^*\right) - \sum_{\substack{i=1\\ :(i,\cdot)\in S}}^{n} \alpha_i^* - \sum_{\substack{k=1\\ :(\cdot,k)\in S}}^{n} \beta_k^*.$$

$$\overline{c}_S^j = \sum_{(i,k)\in S} d_{ij}^k + \gamma_j + \delta.$$

$$D_j = \begin{pmatrix} d_{i_1j}^1 & d_{i_1j}^2 & \cdots & d_{i_1j}^n \\ d_{i_2j}^1 & & & \\ \vdots & & \ddots & \\ d_{i_nj}^1 & & & d_{i_nj}^n \end{pmatrix}$$

where $C_{i_1j} \leq C_{i_2j} \leq \cdots \leq C_{i_nj}$.

Pricing subproblem

$$\overline{c}_S^j = c_S^j + \gamma_j^* + \delta^* + \sum_{k=2}^{n}\sum_{\hat{\imath}=1}^{n}\sum_{\hat{\jmath}=1}^{n}\left(\sum_{\substack{(i,k)\in S\\ :C_{\hat{\imath}\hat{\jmath}}\geq C_{ij}}} \epsilon_k^* + \sum_{\substack{(i,k-1)\in S\\ :C_{\hat{\imath}\hat{\jmath}}\leq C_{ij}}} \epsilon_k^* \right) - \sum_{\substack{i=1\\ :(i,\cdot)\in S}}^{n} \alpha_i^* - \sum_{\substack{k=1\\ :(\cdot,k)\in S}}^{n} \beta_k^*.$$

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where $C_{i_1j} \leq C_{i_2j} \leq \cdots \leq C_{i_nj}$. Dynamic programming: $O(n^3)$!!!

Lower bound

$$z_{ReLRMP} + p \cdot \min_{j \in I, S \in \mathcal{S}(j)} \overline{c}_S^j \leq z_{LRMP} \leq z_{ReLRMP}, \tag{1}$$

$$z_{ReLRMP} + \sum_{j \in I} \min_{S \in \mathcal{S}(j)} \overline{c}_S^j \leq z_{LRMP} \leq z_{ReLRMP}, \tag{2}$$

where z_{ReLRMP} and z_{LRMP} denote the optimal value of *ReLRMP* and *LRMP* respectively.

1 MILP for DOMP
- Problem definition and notation
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- Formulations

2 A Branch-and-Cut-and-Price procedure for DOMP
- Set partitioning formulation for DOMP
- Master Problem
- Restricted relaxed MP and Pricing problem
- Branching
- Valid inequalities

3 Convergence of column generation

Branching on original variables $x_{ij}^k = \sum_{S \ni (i,k)} y_S^j$

Proposition

If $x_{ij}^k \in \{0,1\}$ *for* $i, j, k = 1, \dots, n$, *then* $y_S^j \in \{0,1\}$.

Branching on original variables $x_{ij}^k = \sum_{S \ni (i,k)} y_S^j$

Proposition

If $x_{ij}^k \in \{0,1\}$ *for* $i,j,k = 1,\ldots,n$, *then* $y_S^j \in \{0,1\}$.

$$0 < \sum_{S \ni (i,k)} y_S^j < 1 \text{ for some } i,j,k = 1,\ldots,n?$$

Branching on original variables $x_{ij}^k = \sum_{S \ni (i,k)} y_S^j$

Proposition

If $x_{ij}^k \in \{0,1\}$ for $i,j,k = 1,\ldots,n$, then $y_S^j \in \{0,1\}$.

$$0 < \sum_{S \ni (i,k)} y_S^j < 1 \text{ for some } i,j,k = 1,\ldots,n?$$

ZERO-branch

$$\sum_{S \ni (i,k)} y_S^j = 0$$

ONE-branch

$$\sum_{S \ni (i,k)} y_S^j = 1$$

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3. Convergence of column generation

Order constraints

$$\sum_{\substack{S \ni (\hat{\imath}, k) \\ : C_{\hat{\imath}\hat{\jmath}} \leq C_{ij}}} y_S^{\hat{\jmath}} + \sum_{\substack{S \ni (\hat{\imath}, k-1) \\ : C_{\hat{\imath}\hat{\jmath}} \geq C_{ij}}} y_S^{\hat{\jmath}} \leq 1, \ \ i, j = 1, \ldots, n, k = 2, \ldots, n. \quad (\zeta_{ij}^k)$$

Order constraints

$$\sum_{\substack{S \ni (\hat{\imath},k) \\ : C_{\hat{\imath}\hat{\jmath}} \leq C_{ij}}} y_S^{\hat{\jmath}} + \sum_{\substack{S \ni (\hat{\imath},k-1) \\ : C_{\hat{\imath}\hat{\jmath}} \geq C_{ij}}} y_S^{\hat{\jmath}} \leq 1, \;\; i,j = 1,\dots,n, k = 2,\dots,n. \quad (\zeta_{ij}^k)$$

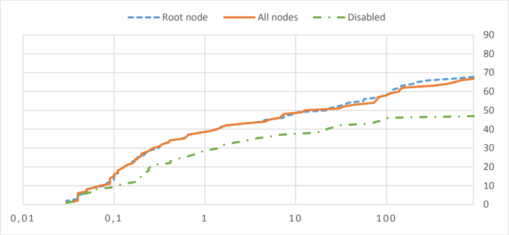

Figure: Number of solved problems per time using different cut strategies.

Remarks

Achievements

- ✓ Set partitioning formulation with sets of pairs.
- ✓ Cutting planes added to the model.

S. Deleplanque, M. Labbé, D. Ponce, and J.Puerto
A Branch-Price-and-Cut Procedure for the Discrete Ordered Median Problem.
INFORMS Journal on Computing, 32 (2020) 582–599

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3. Convergence of column generation

Du Merle, Villeneuve, Desrosiers, and Hansen (1999)

The classical column generation algorithm performs poorly, as degeneracy occurs at two levels: when solving the current linear program and also during several successive major iterations for which the added columns do not suffice to modify the objective function value.

Pessoa, Uchoa, Poggi, and Rodrigues (2010)

Du Merle et al. (1999) proposed a dual stabilization technique to alleviate the convergence difficulties in column generation, based on a simple observation: the columns that will be part of the final solution are only generated in the last iterations, when the dual variables are already close to their optimal values.

Algorithm 1 Stabilization in *ReLRMP*.

$\Delta = \Delta_{init}$; $\overline{\pi} = 0$; $LB(\overline{\pi}) = 0$; $GAP = 1$;
while $GAP > \epsilon$ **do**
  Solve ReLRMP, obtaining z_{ReLRMP} and π_{ReLRMP}; $\pi_{st} = \Delta\pi_{ReLRMP} + (1-\Delta)\overline{\pi}$;
  for $j = 1, \ldots, n$ **do**
    Solve the pricing using π_{st}, obtaining S;
    if $\overline{c}^j_S(\pi_{ReLRMP}) < 0$ **then** Add variable y^j_S; **end if**
  end for
  $LB(\pi_{st}) = z(\pi^t_{st}) + \sum_{S,j:y^j_S \, added} \overline{c}^j_S(\pi_{st})$;
  if At least one variable was added **then**
    if $LB(\pi_{st}) > LB(\overline{\pi})$ **then**
      $\overline{\pi} = \pi_{st}$; $LB(\overline{\pi}) = LB(\pi_{st})$;
    end if
  else
    $\overline{\pi} = \pi_{st}$; $LB(\overline{\pi}) = LB(\pi_{st})$;
  end if
  $GAP = \frac{z_{ReLRMP} - LB(\overline{\pi})}{z_{ReLRMP}}$;
  if $GAP < 1 - \Delta$ **then** $\Delta = 1 - GAP$; **end if**
end while

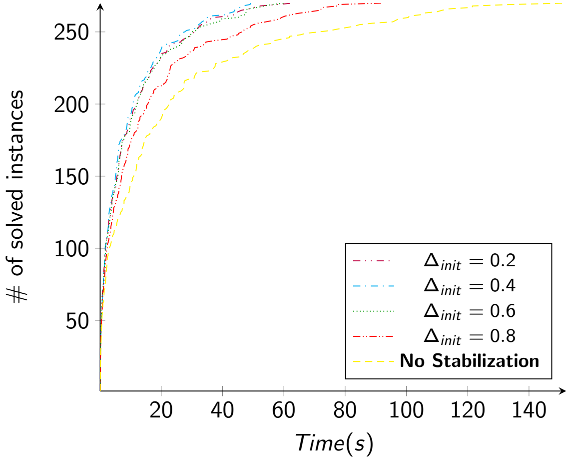
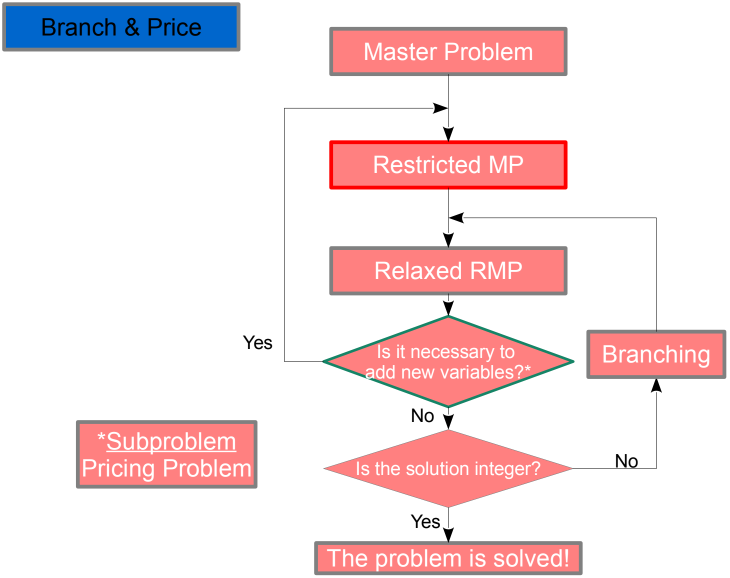
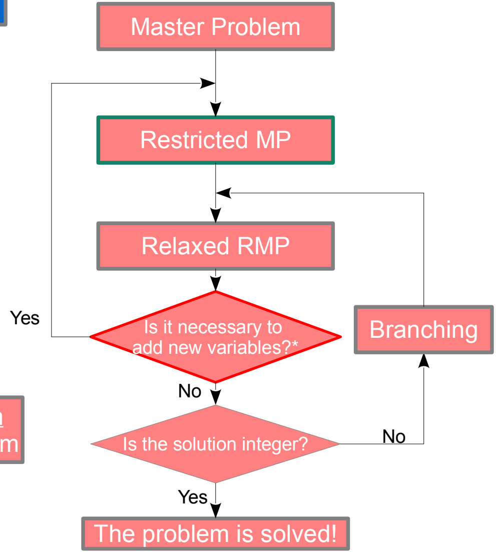


Figure: Performance profile graph with different combination of Δ_{init}, #solved instances / n.


Branch & Price
Master Problem
Restricted MP
Relaxed RMP
Yes
Is it necessary to add new variables?*
Branching
No
*Subproblem
Pricing Problem
Is the solution integer?
No
Yes
The problem is solved!

Branch & Price



*Subproblem
Pricing Problem

V. Blanco, A. Japón, D. Ponce, and J. Puerto
On the multisource hyperplanes location problem to fitting set of points.
Computers and Operations Research, 128 (2021) 105124

S. Benati, D. Ponce, J. Puerto, and A. M. Rodríguez-Chía
A branch-and-price procedure for clustering data that are graph connected.
European Journal of Operational Research, (2021)

V. Blanco, R. Gázquez, D. Ponce, and J. Puerto
A Branch-and-Price approach for the Continuous Multifacility Monotone Ordered Median Problem.
Almost submitted

n	Heurvar	Iterations		Vars	Time
		Exact	Total		
20	FALSE	13	13	2189	64.92
	TRUE	4	23	2219	18.02
30	FALSE	15	15	2827	1034.97
	TRUE	3	60	2856	191.84
40	FALSE	50	50	4713	9086.33
	TRUE	13	136	4511	2229.21

Table: Average number of pricer iterations, variables and time using the combined heuristic and exact pricers or only using the exact pricer

MUCHAS GRACIAS POR SU ATENCIÓN