On k-sum optimization

J. Puerto, A. Tamir and A.M. Rodríguez-Chía

Advances on logistics and transportation problems on complex networks: Evaluation and conclusions

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Motivación:

1. Arie Tamir. Department of Statistics and Operations Research. School of Mathematical Sciences. Tel Aviv University.

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2. J. Puerto, A.M. Rodríguez-Chía, A. Tamir. **Revisiting k-sum Optimization.** *Mathematical Programming*, 165(2):579-604, 2017.

1. Introduction
   - k-sum optimization
2. Linear k-sum optimization
   - Consequences
3. k-sum integer optimization
   - Consequences
4. k-sum combinatorial optimization problem
   - Consequences

The raw problem:

Let E be a finite set of elements, where each $e \in E$ is associated with a pair of real weights (c_e, d_e), where $d_e \geq 0$. Let S be a collection of subsets of E.

- The MINSUM problem is to find a subset $X \in S$ of minimum total weight, $c(X) + d(X) = \sum_{e \in X}(c_e + d_e)$.

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Examples

assignment, shortest paths, matching, ...

Background:

- The name: Gupta and Punnen 1990.
- k-centrum problem on networks (Slater 1978, 1981).
- Partial sum problems (Gupta and Punnen ORL 1990, Punnen JORS 1992, Punnen and Aneja ORL 1996).
- k-centrum multifacility location (Tamir, DAM 2001; Tamir, Puerto, Perez, DAM 2002; Kalcsics, Nickel, Puerto , Networks 2003)
- Continuous k-centrum (Ogryczak, Tamir IFL 2003)
- Robust optimization (Bertsimas and Sim, Math. Prog. 2003)
- Locating k-centrum subtrees (strategical and tactical) (Puerto and Tamir, Math. Prog. 2005)
- The k-sum Shortest Path Problem, (Garfinkel, Fernandez, Lowe TOP, 2006)

Goal:

1. To develop a new methodology applicable to the optimization of k-sum objective functions in great generality,
2. To obtain new algorithms and complexity results for a number of problems, improving or getting similar bounds, but using the same approach in all cases.

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General Idea:

Our methodology consists of solving a k-sum optimization problem by solving a polynomial number of minisum problems in the same or slightly modified feasible region.

Achievements in this paper...

Problem	Best known complexity	Our complexity
k-sum minimum cost network flow problem	Approximate alg., Bertsimas & Sim 2003	Strongly polynomial
k-centrum path problem on trees	Unknown	$O(n^2 \log n)$
Continuous tactical k-centrum subree problem on trees	$O(n^3 + n^{2,5} l)$,Puerto, Tamir 2005	$O(n \log n)$
Continuous tactical k-centrum path problem on trees	Unkonwn	$O(n(n\alpha(n) \log n)^2)$
Continuous strategic k-centrum subtree problem on trees	$O(kn^7)$, Puerto, Tamir 2005	$O(n \log n)$
Single facility k-centrum problem:		
Undirected general networks	$O(nm \log n)$, Kalcsics et al. 2002	$O(mn^2 \log n)$
Continuous ℓ_1-norm	$O(n)$, Tamir 2003	$O(n \log n)$
k-sum Chinese Postman Problem	Unknown	Strongly polyn.
The k-centrum p-facility problem on trees	$O(pk^2 n^2)$, Kalcsics 2011	$O(pn^4)$
The k-centrum p-facility problem on paths	Unknown	$O(pn^3)$
The discrete tactical k-centrum path problem on trees	Unknown	$O(n^3 \log n)$
The discrete strategic k-centrum subtree problem on trees	$O(kn^3)$, Puerto & Tamir 2005	$O(n^3)$
The k-sum shortest path problem	$O(n^2 m^2)$, Garfinkel et al. 2006	$O(m^2 + mn \log n)$
The continuous multifacility OMP $\lambda = (a, \overset{s}{\ldots}, a, b, \overset{n-s}{\ldots}, b)$	$O(pn^9 s^2)$, Kalcsics et al 2003	$O(pn^8 \log^4 n)$
The convex continuous OMP	Unknown	Polynomial

FORMULATION k-SUM/k-CENTRUM:

$$Z_X^* := \min_{x \in X}(cx + \max\{\sum_{j \in S_k} d_j x_j : S_k \subseteq \{1, ..., n\}, |S_k| = k\}),$$

where $X = \{x : Ax = b, x \in \mathcal{X}\}$, $\mathcal{X} = \mathbb{R}^n_+, \mathbb{N}^n, \{0,1\}^n$.

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The inner maximization for
a fixed $x \in X$ is $(d \geq 0)$:

$$\begin{aligned}
\max \quad & \sum_{j=1}^{n} d_j x_j v_j \\
s.t. \quad & \sum_{j=1}^{n} v_j = k \\
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$$\begin{aligned} \min_{(r,p)} \quad & kr + \sum_{j=1}^{n} p_j, \\ s.t. \quad & p_j \geq d_j x_j - r, j = 1, ..., n, \\ & p_j \geq 0, j = 1, ..., n, \end{aligned}$$

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The problem above is:

$$\begin{aligned} Z_X^* = & \quad \min_{r\geq 0} Z_X(r), \\ Z_X(r) = & \quad kr + \min_{(x,p)}(cx + \sum_{j=1}^{n} p_j), \\ s.t. & \quad p_j \geq d_j x_j - r, j = 1, ..., n, \\ & \quad p_j \geq 0, j = 1, ..., n, \end{aligned}$$

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$$Z_X(r) = kr + \min_{x\in X}\left(cx + \sum_{j=1}^{n} \max\{d_j x_j - r, 0\}\right)$$

I. X a polytope in $\mathbb{R}^n$

$$Z_X(r) = kr + \min_{x \in X} \left(cx + \sum_{j=1}^{n} \max\{d_j x_j - r, 0\} \right)$$

$$\begin{aligned} \min_{p,x} \quad & \left\{ cx + \sum_{j=1}^{n} p_j \right\} \\ s.t. \quad & p_j - d_j x_j \geq -r,\ j = 1, \ldots, n, \\ & \sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = 1, \ldots, m, \\ & x_j, p_j \geq 0, \quad j = 1, \ldots, n. \end{aligned}$$

$$\begin{aligned} \max_{\alpha,\beta} \quad & \left\{ -r \sum_{j=1}^{n} \alpha_j + \sum_{i=1}^{m} b_i \beta_i \right\} \\ s.t. \quad & \alpha_j \geq 1, \quad j = 1, \ldots, n, \\ & -\alpha_j d_j + \sum_{i=1}^{m} a_{ij} \beta_i \geq c_j,\ \forall j. \end{aligned}$$

I. X a polytope in $\mathbb{R}^n$

Let $X_L := \{x : Ax = b, x \geq 0\}$ be the region X for this particular case

Theorem

1. *$Z_{X_L}(r)$ is a piecewise linear convex function.*
   Use duality from the previous reformulation!
2. *Suppose that there is a combinatorial algorithm of $O(T(n,m))$ complexity to compute $Z_{X_L}(r)$ for any given r. Then, $Z^*_{X_L}$ can be computed in $O((T(n,m))^2)$ time. Moreover, if $T(n,m) = O(n)$ then $Z^*_{X_L}$ can be computed in $O(n \log n)$ time.*
   Use Megiddo's parametric approach on $Z_{X_L}(r)$.!

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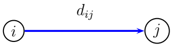
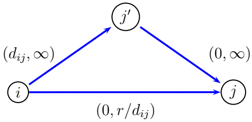
CONSEQUENCES:

I.1 Robust minimum cost network flow problem in Bertismas and Sim (2003). (Only approximately solved!)

$$Z_X(r) = kr + \min_{x \in X} \left(cx + \sum_{i,j=1}^{n} \max\{d_{ij}x_{ij} - r, 0\} \right)$$

$$Z_X(r) = kr + \min_{x \in X} \left(cx + \sum_{i,j=1}^{n} \max\{d_{ij}(x_{ij} - r/d_{ij}), 0\} \right)$$



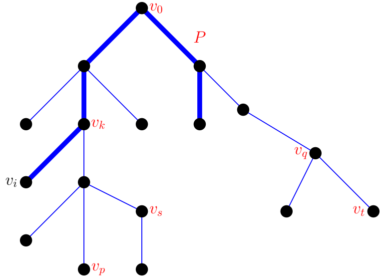


Theorem

k-sum flow is strongly polynomial solvable.

$T(n, m) = O((m \log n)(m + n \log n))$.

I.2 The k-centrum path problem on trees.



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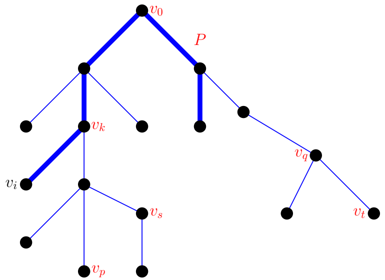
A path containing v_0 minimizing the weighted sum of distances can be found in $O(n)$ time, Averbakh and Berman (1999).



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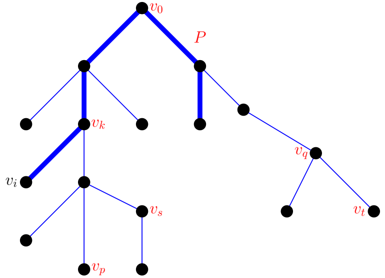

Using the reformulation

$$\min \sum_{k=1}^{n-1} w_k \sum_{j: e_j \in P[v_k, v_0)} \ell_j (1 - x_j)$$

$$s.t. \quad \sum_{k \in ES(e_i)} x_k \leq x_i, \quad \forall i = 1, \ldots, n-1$$

$$0 \leq x_j \leq 1, \quad \forall j = 1, \ldots, n-1.$$

Theorem

The k-centrum path problem on a tree can be solved in $O(n^2 \log n)$ time.

I.3 The continuous tactical k-centrum subtree/path problem on trees

Median version of the problem:

$$\begin{array}{ll} \min_{Y \subseteq A(T)} & \sum_{i=1}^{n} w_i d(v_i, Y) \\ s.t. & L(Y) \leq L. \end{array}$$

Median subtree version: $O(n)$, Tamir 1998.

k-centrum subtree: Puerto and Tamir (2005): $O(n^3 + n^{2,5} I))$ where I is the total number of bits needed to represent the input. Nestedness property.

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Path Median: $O(n\alpha(n)\log n)$, Alstrup et al. 1997.

k-centrum subtree: Puerto and Tamir (2005): $O(n^3 + n^{2,5}I))$ where I is the total number of bits needed to represent the input. Nestedness property.

Theorem

1. *The continuous tactical k-centrum subtree problem on trees can be solved in $O(n \log n)$ time.*
2. *The continuous tactical k-centrum path problem on trees can be solved in $O(n(n\alpha(n)\log n)^2)$ time, where $\alpha(n)$ is the inverse of the Ackermann function.)*

I.4 The continuous strategic k-centrum subtree problem on trees

Median version of the problem:

$$\min_{Y \subseteq A(T)} \sum_{i=1}^{n} w_i d(v_i, Y) + \delta L(Y), \text{ with } \delta \in \mathbb{R}.$$

Median subtree: O(n), Kim et al. (1996).
k centrum subtree: $O(kn^7)$, Puerto y Tamir (2011). (Nestedness property).

Theorem

The continuous strategic k-centrum subtree problem on trees is solvable in $O(n \log n)$ time.

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II. k-sum integer optimization

$$X \longrightarrow X_I = \{x \in \mathbb{R}^n : Ax = b, x_j \in \{0, 1, 2, ...\}, j = 1, ..., n\}.$$

Some negative results

Unlike the linear case, even for the binary case, the function $Z_{X_I}(r)$ is not generally convex, and is not generally unimodal.

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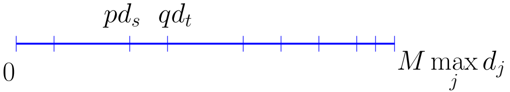
Positive results

If all the integer variables are bounded by $M = M(n, m)$, where $M(n, m)$ is a polynomial in m, n, the integer model is polynomially solvable.

$$Z_{X_I}(r) = kr + \min_{x \in X_I} \left(cx + \sum_{j=1}^{n} \max\{d_j x_j - r, 0\} \right).$$

$\mathcal{I} = [pd_s, qd_t]$ con $p, q \in \{0, \ldots, M\}$ y $s, t \in \{1, \ldots, n\}$.



$$Z_{X_I}(r) = kr + \min_{x \in X_I} \left(cx + \sum_{\substack{j=1 \\ x_j > h_j}}^{n} (d_j x_j - r) \right)$$ is concave for $r \in \mathcal{I}$.

Hence, we may conclude that without loss of generality $r^* \in \{pd_s, qd_t\}$.

Theorem

Consider the k-sum integer optimization problem $Z^*_{X_I}$, and assume that the matrix A is totally unimodular. Suppose further that all integer variables are bounded by some polynomial $M(n, m)$. Then, $Z^*_{X_I}$ can be computed in strongly polynomial time.

Proof. $Z^*_{X_I}$ can be computed by evaluating $Z_{X_I}(r)$ for $O(nM(n, m))$ values of the parameter r. For a fixed value of r, solve:

$$\begin{array}{ll} \min & cx + \sum_{j=1}^{n} \max\{d_j x_j - r, 0\}, \\ s.t. & x \in X_I. \end{array}$$

The above can be solved in strongly polynomial time by substituting $x_j = u_j + v_j + z_j$, $j = 1, \ldots, n$, and solving the respective integer program, defined by a totally unimodular system,

$$\begin{aligned}
\min \quad & c(u+v+z) + \sum_{j=1}^{n} (d_j(\lceil r/d_j \rceil - r/d_j) v_j + d_j z_j), \\
s.t. \quad & A(u+v+z) = b, \\
& u_j \in \{0, 1, ..., \lfloor r/d_j \rfloor\}, \quad j = 1, ..., n, \\
& v_j \in \{0, 1\}, \quad j = 1, ..., n, \\
& z_j \in \{0, 1, 2, ...\}, \quad j = 1, ..., n.
\end{aligned}$$

Since A is totally unimodular this problem is an LP with $\{0, \pm 1\}$-matrix and therefore, by Tardos (1985), it is solvable by a strongly polynomial algorithm.

Theorem

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Conjecture Closed:

This result gives a positive answer to a conjecture in Punnen (1992), since it proves that k-sum optimization problem is polynomially solvable assuming that the constraint matrix is totally unimodular and the variables are bounded.

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Applications

The k-sum Chinese Postman Problem defined on undirected connected graphs and on strongly connected directed graphs is solvable in strongly polynomial time.

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III. Combinatorial case: $\mathcal{X} = \{0,1\}^n$.

Therefore, given a finite set of elements E, where each $e \in E$ is associated with a pair of real weights (c_e, d_e) and X_C be a collection of subsets of E; MINSUM problem is to find a subset $x \in X_C$ of minimum total weight, $c(x) + d(x) = \sum_{e \in x}(c_e + d_e)$.

k-sum optimization problem with respect to the d weights

Find a subset $S \in \mathcal{S}$ minimizing the sum of $c(S)$ and the sum of the k-largest elements in the set $\{d_e : e \in S\}$.

Theorem (Punnen & Aneja (1996), Bertsimas & Sim (2005))

Suppose that for each real r the MINSUM problem with respect to the weights $(c_e, \max(0, d_e - r))$, $e \in E$, is solvable in $T(m)$ time, where $m = |E|$. Then, the k-centrum problem with respect to the d weights can be solved in $O(m' T(m))$ time, where m' is the number of distinct elements in the set $\{d_e : e \in E\}$.

Remark

The supposition that $d_e \geq 0$, for each $e \in E$, which is made in the papers by Punnen & Aneja is used extensively in the proofs. Based on this nonnegativity supposition, they can relax the formulation and introduce the constraint that at most k elements are selected, i.e., $\sum_{e \in E} u_e \leq k$.

From the proof of the above result we note that it actually holds also for some specific linear functions as stated in the next theorem.

Consider the case of arbitrary $\{d_e\}$. For the general case we need to impose the constraint $\sum_{e \in E} u_e = k$. We will then obtain that the parameter r is unrestricted in sign and we will get the following result for general $\{d_e\}$:

Theorem

Suppose that for any real r the MINSUM problem with respect to the weights $(c_e, \max(0, d_e - r))$, $e \in E$, is solvable in $T(m)$ time, where $m = |E|$. Then, the k-centrum problem with respect to the d weights can be solved in $O(m'T(m) + T'(m))$ time, where m' is the number of distinct elements in the set $\{d_e : e \in E\}$, and $T'(m)$ is the time to solve the original MINISUM problem with respect to the weights (c_e, d_e), $e \in E$.

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III.1 The k-centrum p-facility problem on trees and paths

$X_{med(p)}$ the lattice points defined by p-median polytope.
The p-median version: solvable in polynomial time provided that c_{ij} are distances induced by the metric of shortest paths on a tree Hassin and Tamir (2002). (It is NP-hard for a general linear objective function.)
k-sum: requires to solve $O(G)$ problems of the form:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{n} \max\{d_{ij} - d_{(\ell)}, 0\} x_{ij} \\ s.t. \quad & x \in X_{med(p)} \end{aligned}$$

p-median on a path: $O(pn)$, Hassin and Tamir (1991).
p-median on a tree: $O(pn^2)$, Tamir (1996).

Teorema

The k-centrum p-facility on a path is solvable in $O(pn^3)$ and on trees is solvable in $O(pn^4)$.

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Teorema

The k-centrum p-facility on a path is solvable in $O(pn^3)$ and on trees is solvable in $O(pn^4)$.

This improves upon the $O(\min(k,p)kpn^5)$ bound in Tamir (2000) and equals the complexity reported in Kalcsics (2011).

III.2 The discrete tactical k-centrum path/subtree problem on trees

Discrete median subtree: NP-hard, Hakimi et al. (1993).

Discrete median path: $O(n \log n)$, Alstrup et al 1997).

Theorem

The k-centrum version of this model can be solve in $O(n^3 \log n)$ time.

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III.3. The discrete strategic k-centrum subtree problem on trees

Median subtree: $O(n)$, Kim et al. (1996).

K-centrum subtree: $O(kn^3)$ (Puerto & Tamir 2005).

Teorema

The complexity of this problem is $O(n^3)$ time.

III.4 The k-centrum shortest path problem

K-centrum can be solved in $O(n^2 m^2)$ time provided that any simple $s-t$-path there are at least k arcs, otherwise this problem is NP-hard, see Garfinkel, Fernández, Lowe (2006).

Theorem

We improve the bound to $O(m^2 + mn \log n)$ time.

Corollary:

The k-centrum minimum weight matching problem is also solvable in polynomial time applying the above theorem.