Coordenadas de Kunz

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(DEMACON2)

Computational tools

- GAP from www.gap-system.org, and the following packages
  - numericalsgps by M. Delgado, PAGS and J. J. Morais. The idea is to offer methods depending on which of the following packages are installed in the user's machine
  - 4ti2Interface by S. Gutsche
  - NormalizInterface by S. Gutsche, M. Horn, C. Söger (under development)
  - SingularInterface by M. Barakat, M. Horn, F. Lübeck, O. Motsak, M. Neunhoeffer, H. Shoenemann (under development)
  - Singular by M. Costantini, W. de Graaf
  - gap4ti2 by A. Sánchez-R.-Navarro (under development)

Atoms in a block monoid

$G \cong \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$
Let $g_1, \ldots, g_n \in G$. The set of zerosum sequences corresponds to the set of nonnegative integer solutions of

$$\begin{cases} g_{11}x_1 + \cdots + g_{n_1}x_n \equiv 0 \bmod d_1 \\ \cdots \\ g_{1r}x_1 + \cdots + g_{n_r}x_n \equiv 0 \bmod d_r \end{cases}$$

So we can use `Normaliz` with the option "congruences"

Atoms in a block monoid, example

Let us compute the atoms of $\mathcal{B}(C_2^3)$

```
gap> a:=AffineSemigroup("equations",
[TransposedMat([[1,0,0],[0,1,0],[1,1,0],[0,0,1],[1,0,1],[0,1,1],[1,1,1]]),
[2,2,2])];
<Affine semigroup>
gap> at:=GeneratorsOfAffineSemigroup(a);
[ [ 0, 0, 0, 0, 0, 0, 2 ], [ 0, 0, 0, 0, 2, 0, 0 ], [ 0, 0, 0, 0, 0, 2, 0 ],
  [ 0, 0, 0, 2, 0, 0, 0 ], [ 0, 0, 2, 0, 0, 0, 0 ], [ 0, 2, 0, 0, 0, 0, 0 ],
  [ 2, 0, 0, 0, 0, 0, 0 ], [ 0, 0, 1, 0, 1, 1, 0 ], [ 0, 0, 1, 1, 0, 0, 1 ],
  [ 0, 1, 0, 0, 1, 0, 1 ], [ 0, 1, 0, 1, 0, 1, 0 ], [ 1, 0, 0, 0, 0, 1, 1 ],
  [ 1, 0, 0, 1, 1, 0, 0 ], [ 1, 1, 1, 0, 0, 0, 0 ], [ 0, 0, 0, 1, 1, 1, 1 ],
  [ 0, 1, 1, 0, 0, 1, 1 ], [ 0, 1, 1, 1, 1, 0, 0 ], [ 1, 0, 1, 0, 1, 0, 1 ],
  [ 1, 0, 1, 1, 0, 1, 0 ], [ 1, 1, 0, 0, 1, 1, 0 ], [ 1, 1, 0, 1, 0, 0, 1 ] ]
```

So, from this point on, we “live” inside $\mathbb{N}^7$

Factorizations

We look for the factorizations of an element b in elements of atoms $\mathcal{A}$

In our setting b and the elements in $\mathcal{A}$ are in $\mathbb{N}^k$ for some k, so we have to solve the system

$$Ax = b$$

where A has the elements of $\mathcal{A}$ as columns

Hence we can use for this `Normaliz` with the option "inhom_equations" or `4ti2`

```
gap> FactorizationsVectorWRTList([3,3],[[2,0],[1,1],[0,2]]);
[ [ 0, 3, 0 ], [ 1, 1, 1 ] ]
```

And thus one can easily compute sets of lengths, delta sets, catenarities, tame degrees of an element

Elasticity

We want to compute the maximum of $\frac{\max \mathcal{L}(s)}{\min \mathcal{L}(s)}$ for s ranging in our semigroup

Let A be the matrix containing as columns the atoms of the semigroup, and let G be a Graver basis of $Ax = 0$

The elasticity

The elasticity of the monoid is the maximum of $\frac{|v^+|}{|v^-|}$ where v ranges in G, and $v^+, v^- \in \mathbb{N}^k$ with $v = v^+ - v^-$ and $v^+ \cdot v^- = 0$

Actually A. Philipp proved that one has to look among the circuits

Circuits can be computed as explained by Eisenbud and Sturmfels

The elasticity, an example

We compute next the elasticity of $\mathcal{B}(C_2^2)$

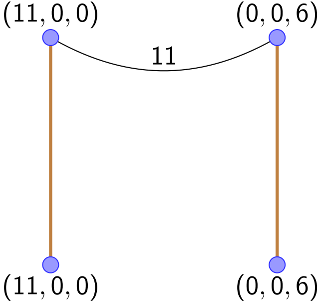
```
gap> AffineSemigroup("equations",
[TransposedMat([[1,0],[0,1],[1,1]]),[2,2]]);;
gap> ElasticityOfAffineSemigroup(last);
3/2
```

The catenary degree of an element with an example

$66 \in S = \langle 6, 9, 11 \rangle$, $c(S) = 4$
The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$Z(66) = \{(0,0,6), (1,3,3), (2,6,0), (4,1,3), (5,4,0), (8,2,0), (11,0,0)\}$$

The distance between $(11, 0, 0)$ and $(0, 0, 6)$ is 11.



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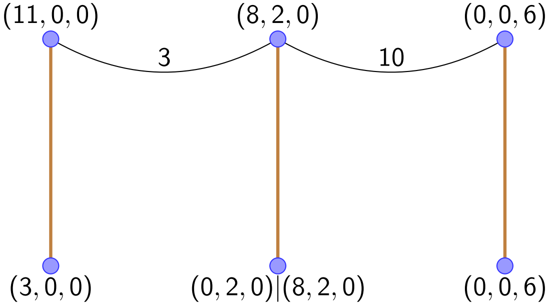

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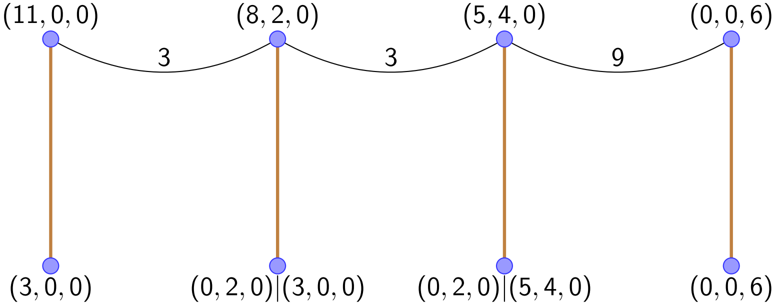

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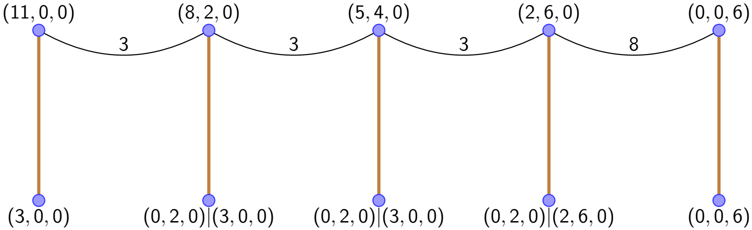

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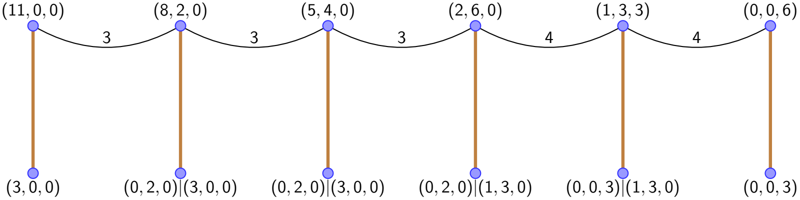
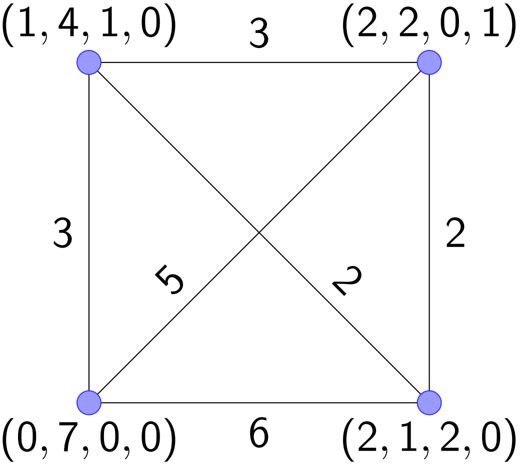


The catenary degree of $77 \in \langle 10, 11, 23, 35 \rangle$



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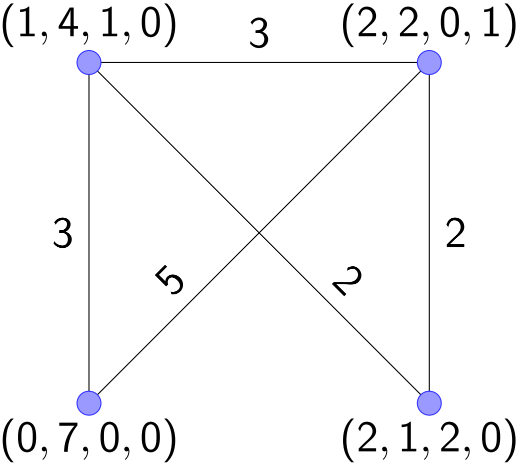

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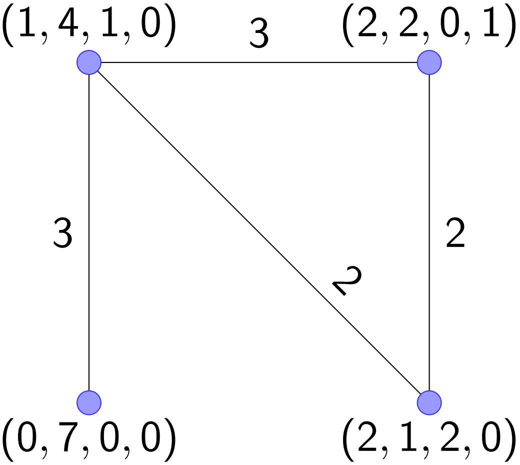

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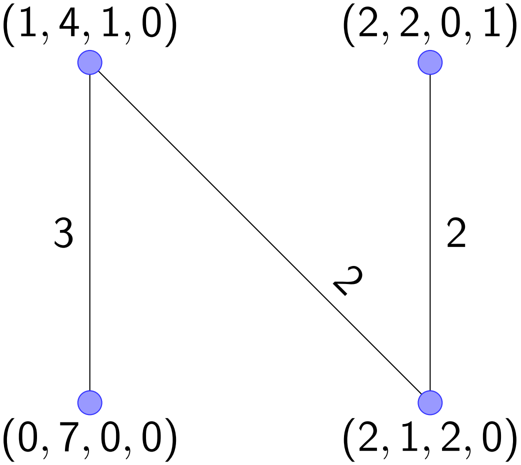


$c(77) = 4$

The catenary degree of a monoid

Let S be an affine semigroup minimally generated by $\mathcal{A}$
The catenary degree of S is defined as

$$\mathsf{c}(S) = \max\{\mathsf{c}(s) \mid s \in S\}$$

For $n \in S$, define the graph G_n as the graph with vertices $a \in \mathcal{A}$ if $n - a \in S$, and edges ab if $n - (a + b) \in S$
Let $\mathrm{Betti}(S)$ be the set of $n \in S$ with G_n nonconnected

Calculating the catenary degree

$$\mathsf{c}(S) = \max\{\mathsf{c}(s) \mid s \in \mathrm{Betti}(S)\}$$

Which graphs are non connected?

- In a numerical semigroup S minimally generated by $\{n_1, \ldots, n_e\}$, if G_n is not connected, then $n = w + n_j$ with $w \in S \setminus \{0\}$, $w - n_1 \notin S$ and $j \in \{2, \ldots, e\}$
- In the affine case we can use Herzog's correspondence and the fact that a minimal presentation for S is constructed from factorizations of elements inf Betti(S)

$$\varphi : \mathbb{N}^e \to S \qquad \psi : K[x_1, \ldots, x_e] \to K[S] = \bigoplus_{s \in S} Kt^s$$
$$e_i \mapsto n_i \qquad\qquad x_i \mapsto t^{n_i}$$

$$(a, b) \in \ker \phi \text{ if and only if } X^a - X^b \in \ker \psi$$

Elimination and nonconnected graphs

G_n is not connected if and only if $n = \varphi(a)$ for some a such that there exist $b \in \mathbb{N}^e$ such that $X^a - X^b$ is in a minimal generating set of $\ker \psi$

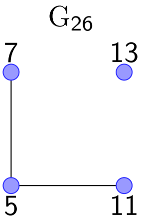
`Singular`+eliminate+minbase

or

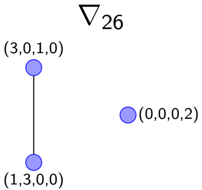
`4ti2`+removing non connected graphs (`4ti2` computes binomial Gröbner basis, and our ideals are binomial)

Some more graphs

$S = \langle 5, 7, 11, 13 \rangle$



∇_n is a (nonoriented) graph with vertices the factorizations of n, and there is an edge if $x \cdot y \neq 0$



Sets of distances

Let as in the previous lectures, $\Delta(s)$ denote set of distances (delta set) of factorizations of s, that is, the differences of two consecutive lengths of factorizations

$$\Delta(S) = \bigcup_{s \in S} \Delta(s)$$

The minimum

The minimum is actually the greatest common divisor of $\Delta(S)$

The maximum

The maximum is achieved in the set $\mathrm{Betti}(S)$

Tame degree with an example

We go back to $66 \in S = \langle 6, 9, 11 \rangle$, $\mathrm{t}(S) = 7$ The factorizations of $66 \in \langle 6, 9, 11 \rangle$ are

$$\mathrm{Z}(66) = \{(0,0,6),(1,3,3),(2,6,0),(4,1,3),(5,4,0),(8,2,0),(11,0,0)\}$$

Besides, 9 *divides* 66

$$(11, 0, 0)$$

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and 11 also *divides* 66

$$\begin{array}{c} (8,2,0) \\ {\scriptstyle 3}\big| \\ (11,0,0) \end{array}$$

Tame degree with an example

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$(8, 2, 0)$

3 |

$(11, 0, 0)$

7 |

$(4, 1, 3)$

Tame degree of the monoid

The tame degree of an affine semigroup is the maximum of the tame degrees of its elements

Calculating the tame degree

$\mathrm{t}(S)$ is the maximum of the $\mathrm{t}(s)$ with $s \in S$ having associated graph G_s not complete

Tame degree of the monoid, practical info

In the numerical semigroup case, G_s not complete means that $s = w + n_j$ where $w \in S \setminus \{0\}$, $w - n_i \notin S$ for some n_i, n_j atoms of the monoid

In the affine case, Apéry sets are not that easy to compute, but one can still use the following fact

Primitive elements and tame degree

Let A be the matrix whose columns are the atoms of the monoid
The tame degree of the monoid is achieved in an element s such that there exists $v = v^+ - v^-$ in a Graver basis of $Ax = 0$ with $\varphi(v^+) = s$

So, we can use once more `4ti2` for the Hilbert basis computations and `Normaliz` or `4ti2` for the factorizations of each candidate

The ω-primality

Let S be an affine semigroup with atoms $\mathcal{A} = \{a_1, \ldots, a_k\}$, and let $s \in S$
The ω-primality of s, $\omega(s)$, is the least integer N such that whenever $(\sum_{i=1}^{k} \lambda_i a_i) - s \in S$, there exists $(\beta_1, \ldots, \beta_k) \leq (\lambda_1, \ldots, \lambda_k)$ such that $(\sum_{i=1}^{k} \beta_i a_i) - s \in S$ and $\sum \beta_i \leq N$

Calculating ω-primality

$\omega(s)$ is the maximum of the lengths of the minimal elements of $\mathrm{Z}(s + S)$

The ω-primality, practical information

We want to calculate $\omega(s)$, and the atoms are $\mathcal{A}$; A is a matrix with columns the elements of $\mathcal{A}$

- For numerical semigroups, one only has to look at factorizations of elements of the form $w + a$ with $w \in S \setminus \{0\}$, $w - s \notin S$ and $a \in \mathcal{A}$
- For affine semigroups, we can compute the minimals of $\mathsf{Z}(s + S)$ or find the solutions to $Ax = s + Ay$, project on x, and take the minimal ones

So we can either use `Singular`+preimage or `Normaliz`+inhom_equations or `4ti2`

```
gap> OmegaPrimalityOfElementInAffineSemigroup(
[1000],[[31],[51],[75],[49]]);
37
```