Physical refraction and Snell Law: a new framework for location models with rapid transit media

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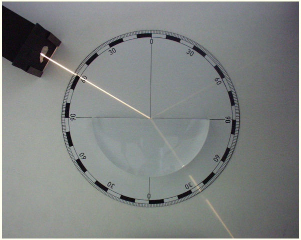
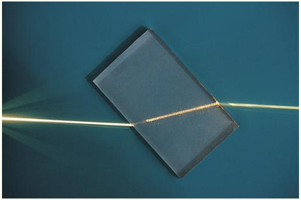
(joint work with V. Blanco and D. Ponce)

Refraction

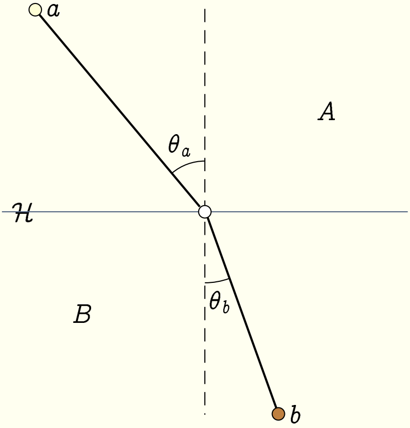
Change in direction of propagation of any wave as a result of its traveling at different speeds at different points along the wave front.





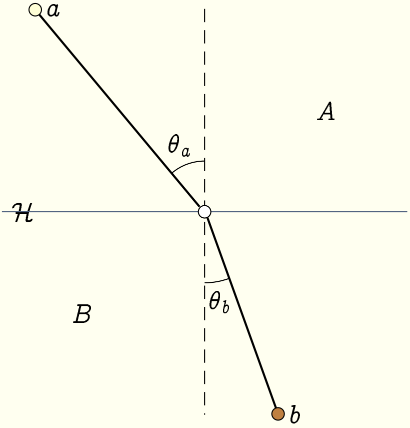
Applications: Transportation Systems connecting urban and rural areas; natural barriers or borders, ...

Euclidean Planar Snell's Law



ω_A, ω_B refraction indices.

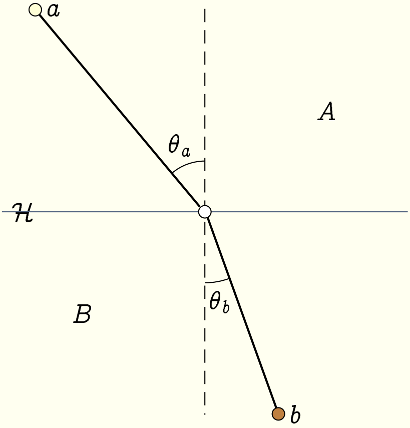
Euclidean Planar Snell's Law



ω_A, ω_B refraction indices.

$$\min_{x \in \mathcal{H}} \quad \omega_A \|a - x\| + \omega_B \|x - b\|$$

Euclidean Planar Snell's Law



ω_A, ω_B refraction indices.

$$\min_{x \in \mathcal{H}} \; \omega_A \|a - x\| + \omega_B \|x - b\|$$

$$\Downarrow$$

$$\omega_A \sin \theta_a = \omega_B \sin \theta_b$$

SP between points separated by a hyperplane

$\mathcal{H} = \{x \in \mathbb{R}^d : \alpha^t x = \beta\}$.

$a \in H_A = \{x \in \mathbb{R}^d : \alpha^t x \leq \beta\}$ ($p_A = \frac{r_A}{s_A}$-norm)

$b \in H_B = \{x \in \mathbb{R}^d : \alpha^t x > \beta\}$ ($p_B = \frac{r_B}{s_B}$-norm)

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$b \in H_B = \{x \in \mathbb{R}^d : \alpha^t x > \beta\}$ ($p_B = \frac{r_B}{s_B}$-norm)

Lemma

If $1 < p_A, p_B < +\infty$, the length $d_{p_A p_B}(a, b)$ of the shortest weighted path between a and b is

$$d_{p_A p_B}(a, b) = \omega_a \|x^* - a\|_{p_A} + \omega_b \|x^* - b\|_{p_B},$$

where $x^* = (x_1^*, \ldots, x_d^*)^t$, $\alpha^t x^* = \beta$ must satisfy the following conditions:

SP between points separated by a hyperplane

1. For all j such that $\alpha_j = 0$:

$$\omega_a \left[\frac{|x_j^* - a_j|}{||x^* - a||_{p_A}} \right]^{p_A - 1} \mathrm{sg}(x_j^* - a_j) + \omega_b \left[\frac{|x_j^* - b_j|}{||x^* - b||_{p_B}} \right]^{p_B - 1} \mathrm{sg}(x_j^* - b_j) = 0.$$

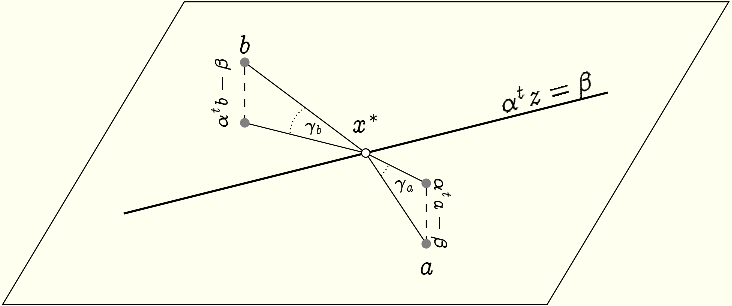
2. For all i, j such that $\alpha_i \alpha_j \neq 0$.

$$\omega_a \left[\frac{|x_i^* - a_i|}{||x^* - a||_{p_A}} \right]^{p_A - 1} \frac{\mathrm{sg}(x_i^* - a_i)}{\alpha_i} + \omega_b \left[\frac{|x_i^* - b_i|}{||x^* - b||_{p_B}} \right]^{p_B - 1} \frac{\mathrm{sg}(x_i^* - b_i)}{\alpha_i} =$$

$$\omega_a \left[\frac{|x_j^* - a_j|}{||x^* - a||_{p_A}} \right]^{p_A - 1} \frac{\mathrm{sg}(x_j^* - a_j)}{\alpha_j} + \omega_b \left[\frac{|x_j^* - b_j|}{||x^* - b||_{p_B}} \right]^{p_B - 1} \frac{\mathrm{sg}(x_j^* - b_j)}{\alpha_j}.$$

Generalized Snell's Law

$$\sin_{p_A} \gamma_{a_j} := \frac{|\alpha_j a_j - \alpha_j x_j^*|}{\|a - x^*\|_{p_A}}, \; j = 1, \ldots, d.$$



Generalized Snell's Law

Corollary (Snell's-like result)

The point x^* in $\mathcal{H}$ must satisfy:

1. *For all j such that $\alpha_j = 0$:*

$$\omega_a \left[\frac{|x_j^* - a_j|}{\|x^* - a\|_{p_A}}\right]^{p_A - 1} \operatorname{sg}(x_j^* - a_j) + \omega_b \left[\frac{|x_j^* - b_j|}{\|x^* - b\|_{p_B}}\right]^{p_B - 1} \operatorname{sg}(x_j^* - b_j) = 0.$$

2. *For all i, j, $\alpha_i \alpha_j \neq 0$.*

$$\omega_a \left[\frac{\sin_{p_A} \gamma_{a_i}}{|\alpha_i|}\right]^{p_A - 1} \frac{\operatorname{sg}(x_i^* - a_i)}{\alpha_i} + \omega_b \left[\frac{\sin_{p_B} \gamma_{b_i}}{|\alpha_i|}\right]^{p_B - 1} \frac{\operatorname{sg}(x_i^* - b_i)}{\alpha_i} =$$

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Generalized Snell's Law

Corollary (Snell's Law)

If $d = 2$, $p_A = p_B = 2$ *the point* x^* *satisfies:*

$$\omega_A \sin\theta_A = \omega_B \sin\theta_B,$$

where θ_A *and* θ_B *are:*

1. *if* $\alpha_1 \leq \alpha_2$, *the angles between the vectors* $a - x^*$ *and* $(-\alpha_2, \alpha_1)^t$, *and* $b - x^*$ *and* $(\alpha_2, -\alpha_1)^t$.
2. *if* $\alpha_1 > \alpha_2$, *the angles between the vectors* $a - x^*$ *and* $(\alpha_2, -\alpha_1)^t$, *and* $b - x^*$ *and* $(-\alpha_2, \alpha_1)^t$.

Location under Refraction

Given $\mathcal{H} = \{x \in \mathbb{R}^d : \alpha^t x = \beta\}$, $A \subseteq \mathrm{H}_A$, $B \subseteq \mathrm{H}_B$:

$$f^* := \inf_{x \in \mathbb{R}^d} \sum_{a \in A} \omega_a \; d_{p_A,p_B}(x,a) + \sum_{b \in B} \omega_b \; d_{p_A,p_B}(x,b) \tag{P}$$

where $d_{p_A,p_B}(x,y)$ is the length of the SP between $x, y \in \mathbb{R}^d$.

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- Parlar, 1994. (Planar and heuristics).
- Carrizosa & Rodríguez-Chía, 1997. (Rapid transit lines induced by networks on the plane).
- Brimberg, Kakhki & Wesolowsky, 2003, 2005 (Planar and ℓ_1-ℓ_2 norms, bounded regions).
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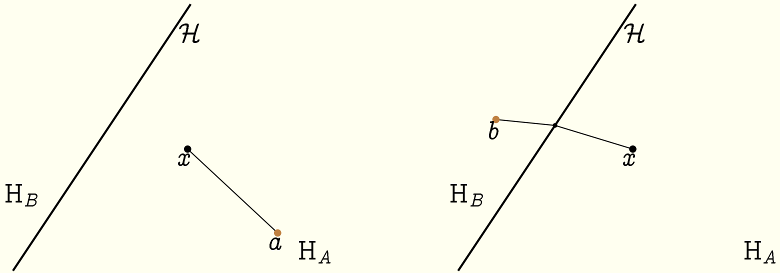
Our Goal: Exact approach for solving (P) for any d and any ℓ_p-norms.

Shortest paths



Shortest paths

For $x \in \mathbb{R}^d$, then we assume that the shortest path length between x and $a \in \mathrm{H}_A$ or $b \in \mathrm{H}_B$ is:

$$d_{p_A,p_B}(x,a) = \begin{cases} \|x-a\|_{p_A} & \text{if } x \in \mathrm{H}_A, \\ \min_{y \in \mathcal{H}} \|y-a\|_{p_A} + \|x-y\|_{p_B} & \text{if } x \in \mathrm{H}_B, \end{cases}$$

and

$$d_{p_A,p_B}(x,b) = \begin{cases} \|x-b\|_{p_B} & \text{if } x \in \mathrm{H}_B, \\ \min_{y \in \mathcal{H}} \|y-b\|_{p_B} + \|x-y\|_{p_A} & \text{if } x \in \mathrm{H}_A. \end{cases}$$

Shortest paths

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and

$$d_{p_A,p_B}(x,b) = \begin{cases} \|x-b\|_{p_B} & \text{if } x \in \mathrm{H}_B, \\ \min\limits_{y\in\mathcal{H}} \|y-b\|_{p_B} + \|x-y\|_{p_A} & \text{if } x \in \mathrm{H}_A. \end{cases}$$

Theorem

Assume that $\min\{|A|,|B|\} > 2$. *If the points in* A *or* B *are not collinear and* $p_A < +\infty$, $p_B > 1$ *then Problem* (P) *always has a unique optimal solution.*

Formulation

$$\min \sum_{a\in A} \omega_a Z_a + \sum_{b\in B} \omega_b Z_b$$

$$\begin{aligned}
\text{s.t.}\quad & z_a - Z_a \le M_a(1-\gamma), && \forall a \in A,\\
& \theta_a + u_a - Z_a \le M_a\,\gamma, && \forall a \in A,\\
& z_b - Z_b \le M_b\,\gamma, && \forall b \in B,\\
& \theta_b + u_b - Z_b \le M_b\,(1-\gamma), && \forall b \in B,\\
& z_a \ge \|x - a\|_{p_A}, && \forall a \in A,\\
& \theta_a \ge \|x - y_a\|_{p_B}, && \forall a \in A,\\
& u_a \ge \|a - y_a\|_{p_A}, && \forall a \in A,\\
& z_b \ge \|x - b\|_{p_B}, && \forall b \in B,\\
& \theta_b \ge \|x - y_b\|_{p_A}, && \forall b \in B,\\
& u_b \ge \|b - y_b\|_{p_B}, && \forall b \in B,\\
& \alpha^t x - \beta \le M(1-\gamma), && \\
& \alpha^t x - \beta \ge -M\gamma, && \\
& \alpha^t y_a = \beta, && \forall a \in A,\\
& \alpha^t y_b = \beta, && \forall b \in B,\\
& Z_a, z_a, \theta_a, u_a \ge 0, && \forall a \in A,\\
& Z_b, z_b, \theta_B, u_B \ge 0, && \forall b \in B,\\
& y_a, y_b \in \mathbb{R}^d, && \forall a \in A, b \in B,\\
& \gamma \in \{0, 1\}.
\end{aligned}$$

Divide et impera

Theorem

Let $x^* \in \mathbb{R}^d$ be the optimal solution of (P). Then, x^* is the solution of one of the following two problems:

$$\min_{x \in \mathrm{H}_A} f * \qquad (\mathrm{P}_A) \qquad\qquad \min_{x \in \mathrm{H}_B} f * \qquad (\mathrm{P}_B)$$

Divide et impera

P_A

$$\min \sum_{a\in A} \omega_a z_a + \sum_{b\in B} \omega_b \theta_b + \sum_{b\in B} \omega_b u_b$$

s.t.

$$
\begin{aligned}
& z_a \geq \|x - a\|_{p_A}, \forall a \in A,\\
& \theta_b \geq \|x - y_b\|_{p_A}, \forall b \in B,\\
& u_b \geq \|b - y_b\|_{p_B}, \forall b \in B,\\
& \alpha^t y_b = \beta, \forall b \in B,\\
& \alpha^t x \leq \beta,\\
& z_a \geq 0, \ \forall a \in A,\\
& \theta_b, u_b \geq 0, \ \forall b \in B,\\
& x, y_b \in \mathbb{R}^d.
\end{aligned}
$$

P_B

$$\min \sum_{b\in B} \omega_a z_b + \sum_{a\in A} \omega_a \theta_a + \sum_{a\in A} \omega_a u_a$$

s.t.

$$
\begin{aligned}
& z_b \geq \|x - b\|_{p_B}, \forall b \in B,\\
& \theta_a \geq \|x - y_a\|_{p_B}, \forall a \in A,\\
& u_a \geq \|a - y_a\|_{p_A}, \forall a \in A,\\
& \alpha^t y_a = \beta, \forall a \in A,\\
& \alpha^t x \geq \beta,\\
& z_b \geq 0, \ \forall b \in B,\\
& \theta_a, u_a \geq 0, \ \forall a \in A,\\
& x, y_a \in \mathbb{R}^d.
\end{aligned}
$$

NLP Formulation

Lemma

$Z \geq \|X - Y\|_p$, for any $p = \frac{r}{s}$ with $r, s \in \mathbb{N} \setminus \{0\}$, $r > s$ and $\gcd(r, s) = 1$, and X, Y variables in $\mathbb{R}^d$, can be equivalently written as the following set of constraints:

$$
\begin{aligned}
Q_k + X_k - Y_k \geq 0,\ k = 1, \ldots, d, \\
Q_k - X_k + Y_k \geq 0,\ k = 1, \ldots, d, \\
Q_k^r \leq \xi_k^s Z^{r-s}, k = 1, \ldots, d, \\
\sum_{k=1}^{d} \xi_k \leq Z, \\
\xi_k \geq 0, k = 1, \ldots, d.
\end{aligned}
$$

NLP Formulation

Theorem

Let $\|\cdot\|_{p_i}$ *be a* ℓ_{p_i}*-norm with* $p_i = \frac{r_i}{s_i} > 1$, $r_i, s_i \in \mathbb{N} \setminus \{0\}$, *and* $\gcd(r_i, s_i) = 1$ *for* $i \in \{A, B\}$. *Then, solving* (P_A) *is equivalent to*

$$\min \sum_{a \in A} \omega_a z_a + \sum_{b \in B} \omega_j \theta_j + \sum_{b \in B} \omega_b u_b$$

$$\begin{aligned}
s.t.\ & x \in \mathrm{H}_A, y \in H, \\
& t_{ak} - x_k + a_k \geq 0, \\
& t_{ak} + x_k - a_k \geq 0, \\
& v_{bk} + x_k - y_{bk} \geq 0, \\
& v_{bk} - x_k + y_{bk} \geq 0, \\
& g_{bk} - y_{bk} + b_k \geq 0, \\
& g_{bk} + y_{bk} - b_k \geq 0, \\
& t_{ak}^{r_A} \leq \xi_{ak}^{s_A} z_a^{r_A - s_A}, \\
& v_{bk}^{r_A} \leq \rho_{bk}^{s_A} \theta_b^{r_A - s_A}, \\
& g_{bk}^{r_B} \leq \psi_{bk}^{s_B} u_b^{r_B - s_B},
\end{aligned}$$

$$\begin{aligned}
& \sum_{k=1}^{d} \xi_{ak} \leq z_a, \\
& \sum_{k=1}^{d} \rho_{bk} \leq \theta_b, \\
& \sum_{k=1}^{d} \psi_{bk} \leq u_b, \\
& \xi_{ak}, t_{ak}, \rho_{bk}, v_{bk}, \psi_{bk}, g_{bk} \geq 0, \\
& z_a, \theta_b, u_b \geq 0, \\
& x, y_b \in \mathbb{R}^d.
\end{aligned}$$

NLP Formulation

Theorem

Let $\|\cdot\|_{p_i}$ *be a* ℓ_{p_i}*-norm with* $p_i = \frac{r_i}{s_i} > 1$, $r_i, s_i \in \mathbb{N} \setminus \{0\}$, *and* $\gcd(r_i, s_i) = 1$ *for* $i \in \{A, B\}$. *Then, solving* (P_A) *is equivalent to*

$$\min \sum_{a \in A} \omega_a z_a + \sum_{b \in B} \omega_j \theta_j + \sum_{b \in B} \omega_b u_b$$

$$\begin{aligned}
s.t.\ \ & x \in \mathrm{H}_A, y \in H, && \sum_{k=1}^{d} \xi_{ak} \leq z_a, \\
& t_{ak} - x_k + a_k \geq 0, && \\
& t_{ak} + x_k - a_k \geq 0, && \sum_{k=1}^{d} \rho_{bk} \leq \theta_b, \\
& v_{bk} + x_k - y_{bk} \geq 0, && \\
& v_{bk} - x_k + y_{bk} \geq 0, && \\
& g_{bk} - y_{bk} + b_k \geq 0, && \sum_{k=1}^{d} \psi_{bk} \leq u_b, \\
& g_{bk} + y_{bk} - b_k \geq 0, && \\
& t_{ak}^{r_A} \leq \xi_{ak}^{s_A} z_a^{r_A - s_A}, && \xi_{ak}, t_{ak}, \rho_{bk}, v_{bk}, \psi_{bk}, g_{bk} \geq 0, \\
& v_{bk}^{r_A} \leq \rho_{bk}^{s_A} \theta_b^{r_A - s_A}, && z_a, \theta_b, u_b \geq 0, \\
& g_{bk}^{r_B} \leq \psi_{bk}^{s_B} u_b^{r_B - s_B}, && x, y_b \in \mathbb{R}^d.
\end{aligned}$$

SOCP Formulation

Lemma (B., Puerto, ElHaj-BenAli, 2014)

Let $\tau = \frac{r}{s}$ *be such that* $r, s \in \mathbb{N} \setminus \{0\}$, $r > s$, $\gcd(r,s) = 1$ *and* $k = \lceil \log_2(r) - 1 \rceil$.
Let x, u *and* t *be non negative and satisfying* $\mathbf{x^r} \leq \mathbf{u^s t^{r-s}}$, *then, there exists* w *such that, such a constraint is equivalent to:*

$$\begin{aligned} \theta_i^2 &\leq A_i B_i, && \forall i = 1, \ldots, m \\ x^2 &\leq A_{m+1} B_{m+1}, \end{aligned}$$

where $A_i, B_i \in \{\theta_{i-1}, u, t, x\}$ *for* $i = 1, \ldots, m$ *and* $m = 1 + 2\#\{i : \alpha_i + \beta_i + \gamma_i \geq 2, 1 \leq i < k-1\} + \#\{i : \alpha_i + \beta_i + \gamma_i \leq 1, 1 \leq i < k-1\} \sim \mathcal{O}(log(r))$ *with:*

$$\begin{aligned} s &= \alpha_{k-1} 2^{k-1} + \alpha_{k-2} 2^{k-2} + \ldots + \alpha_1 2^1 + \alpha_0 2^0, \\ r - s &= \beta_{k-1} 2^{k-1} + \beta_{k-2} 2^{k-2} + \ldots + \beta_1 2^1 + \beta_0 2^0, \\ 2^k - r &= \gamma_{k-1} 2^{k-1} + \gamma_{k-2} 2^{k-2} + \ldots + \gamma_1 2^1 + \gamma_0 2^0, \\ 2^k &= (\alpha_{k-1} + \beta_{k-1} + \gamma_{k-1}) 2^{k-1} + \cdots + (\alpha_0 + \beta_0 + \gamma_0) 2^0, \end{aligned}$$

$\alpha_i,\ \beta_i,\ \gamma_i \in \{0, 1\}$.

SOCP Formulation

$$X^2 \leq YZ \quad \Leftrightarrow \begin{pmatrix} Y+Z & 0 & 2X \\ 0 & Y+Z & Y-Z \\ 2X & Y-Z & Y+Z \end{pmatrix} \succeq 0, \; Y+Z \geq 0.$$

Corollary

Problem (P_A) *can be represented as a semidefinite programming problem with:*

- $|A|(2d+1) + |B|(4d+3) + 1$ *linear constraints, and*
- *at most* $4d(|A| \log r_A + |B| \log r_A + |B| \log r_B)$ *positive semidefinite constraints.*

Constrained Case

Theorem

Let $\mathbf{K} := \{x \in \mathbb{R}^d : g_j(x) \geq 0,\ j = 1, \ldots, l\}$ be a basic closed, compact semialgebraic set with nonempty interior, and consider the restricted problem:

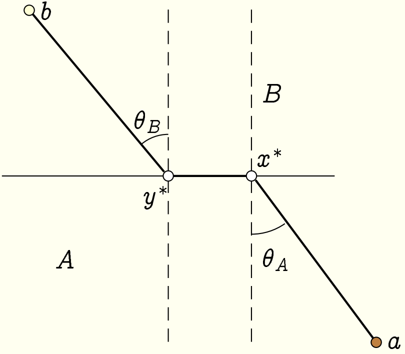
$$\min_{x \in \mathbf{K}} \sum_{a \in A} \omega_a\ d(x, a) + \sum_{b \in B} \omega_b\ d(x, b). \tag{1}$$

Assume that $\mathbf{K}$ satisfies the Archimedean property and further that any of the following conditions hold:

1. *$g_i(x)$ are concave for $i = 1, \ldots, l$ and $-\sum_{i=1}^{l} \mu_i \nabla^2 g_i(x) \succ 0$ for each dual pair (x, μ) of the problem of minimizing any linear functional $c^t x$ on $\mathbf{K}$ (Positive Definite Lagrange Hessian (PDLH)).*
2. *$g_i(x)$ are sos-concave on $\mathbf{K}$ for $i = 1, \ldots, l$ or $g_i(x)$ are concave on $\mathbf{K}$ and strictly concave on the boundary of $\mathbf{K}$ where they vanish, i.e. $\partial\mathbf{K} \cap \partial\{x \in \mathbb{R}^d : g_i(x) = 0\}$, for all $i = 1, \ldots, l$.*
3. *$g_i(x)$ are strictly quasi-concave on $\mathbf{K}$ for $i = 1, \ldots, l$.*

Then, there exists a constructive finite dimension embedding, which only depends on p_A, p_B and g_i, $i = 1, \ldots, l$, such that the solution of (1) can be obtained by solving two semidefinite programming problems.

Hyperplane Endowed with a third norm...



$$d_t(a,b) = \begin{cases} \|a-b\|_{p_i} & \text{if } a,b \in \mathrm{H}_i,\ i \in \{A,B\}, \\ \min\limits_{x,y\in\mathcal{H}} \|x-a\|_{p_A} + \|x-y\|_{p_H} + \|y-b\|_{p_B} & \text{if } a \in \mathrm{H}_A,\ b \in \overline{\mathrm{H}}_B, \end{cases} \tag{DT}$$

Snell's like result

Assume that $\|\cdot\|_{p_A}$, $\|\cdot\|_{p_B}$, $\|\cdot\|_{p_H}$ are ℓ_p-norms with $1 < p < +\infty$. Let $x^*, y^* \in \mathbb{R}^d$, $\alpha^t x^* = \alpha^t y^* = \beta$. Then, x^* and y^* define the shortest weighted path between a and b when traversing the hyperplane is allowed if and only if the following conditions are satisfied:

1. For all j such that $\alpha_j = 0$:

$$\omega_a \left[\frac{|x_j^* - a_j|}{\|x^* - a\|_{p_A}} \right]^{p_A - 1} \operatorname{sg}(x_j^* - a_j) + \omega_H \left[\frac{|x_j^* - y_j^*|}{\|x^* - y^*\|_{p_H}} \right]^{p_H - 1} \operatorname{sg}(x_j^* - y_j^*) = 0,$$

$$\omega_b \left[\frac{|y_j^* - b_j|}{\|y^* - b\|_{p_B}} \right]^{p_B - 1} \operatorname{sg}(y_j^* - b_j) - \omega_H \left[\frac{|x_j^* - y_j^*|}{\|x^* - y^*\|_{p_H}} \right]^{p_H - 1} \operatorname{sg}(x_j^* - y_j^*) = 0.$$

Snell's like result

2. For all i, j, such that $\alpha_i \alpha_j \neq 0$:

$$\omega_a \left[\frac{\sin \gamma_{a_i}}{|\alpha_i|}\right]^{p_A - 1} \frac{\mathrm{sg}(x_i^* - a_i)}{\alpha_i} + \omega_H \left[\frac{|x_i^* - y_i^*|}{\|x^* - y^*\|_{p_H}}\right]^{p_H - 1} \frac{\mathrm{sg}(x_i^* - y_i^*)}{\alpha_i} = \omega_a \left[\frac{\sin \gamma_{a_j}}{|\alpha_j|}\right]^{p_A - 1} \frac{\mathrm{sg}(x_j^* - a_j)}{\alpha_j} + \omega_H \left[\frac{|x_j^* - y_j^*|}{\|x^* - y^*\|_{p_H}}\right]^{p_H - 1} \frac{\mathrm{sg}(x_j^* - y_j^*)}{\alpha_j},$$

and

$$\omega_a \left[\frac{\sin \gamma_{b_i}}{|\alpha_i|}\right]^{p_B - 1} \frac{\mathrm{sg}(y_i^* - b_i)}{\alpha_i} - \omega_H \left[\frac{|x_i^* - y_i^*|}{\|x^* - y^*\|_{p_H}}\right]^{p_H - 1} \frac{\mathrm{sg}(x_i^* - y_i^*)}{\alpha_i} = \omega_a \left[\frac{\sin \gamma_{b_j}}{|\alpha_j|}\right]^{p_B - 1} \frac{\mathrm{sg}(y_j^* - b_j)}{\alpha_j} - \omega_H \left[\frac{|x_j^* - y_j^*|}{\|x^* - y^*\|_{p_H}}\right]^{p_H - 1} \frac{\mathrm{sg}(x_j^* - y_j^*)}{\alpha_j}.$$

Snell's like result

Corollary

If $d = 2$, $p_A = p_B = p_H = 2$ *and* $\mathcal{H} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$, *the points* x^*, y^* *satisfy one of the following conditions:*

1. $\omega_a \sin\theta_a = \omega_b \sin\theta_b = \omega_H \frac{|y_1^*|}{\|x^* - y^*\|_{p_H}}$ *and* $x^* \neq y^*$, *or*
2. $\omega_a \sin\theta_a = \omega_b \sin\theta_b$ *and* $x^* = y^*$,

where θ_a *is the angle between the vectors* $a - x^*$ *and* $(0, -1)$ *and* θ_b *the angle between* $b - y^*$ *and* $(0, 1)$.

Location if the hyperplane is endowed with third norm

$$\min_{x \in \mathbb{R}^d} \sum_{a \in A} \omega_a \, d_t(x, a) + \sum_{b \in B} \omega_b \, d_t(x, b). \qquad \text{(PT)}$$

Location if the hyperplane is endowed with third norm

Theorem

Assume that $\min\{|A|, |B|\} > 2$. *If the points in* A *or* B *are not collinear and* $p_B > 1$ *or* $p_A < +\infty$ *then Problem* (PT) *always has a unique optimal solution.*

Proposition

Let $A, B \subseteq \mathbb{R}^d$ *and* $\mathcal{H} = \{x \in \mathbb{R}^d : \alpha^t x = \beta\}$. *Then, if* $p_A \geq p_B \geq p_H$, *Problem* (PT) *reduces to Problem* (P).

Theorem

(PT) *can also be formulated as a SOCP Problem.*

Experiments: Comparisons

SOCP coded in Gurobi 5.6 (PC with an Intel Core i7 processor at 2x 2.40 GHz, and 4GB RAM). Barrier convergence tol. QCP: 10^{-10}.

$\|A \cup B\|$	$\mathcal{H}$	CPUTime (P)	f^* (P)	CPUTime†,‡	$f^{\dagger,\ddagger}$
4 †	$y = x$	0.037041	26.951942	49.62	26.951958
18 †	$y = 1.5x$	0.057064	112.350633	35.54	112.350702
30 ‡	$y = 0.5x$	0.056049	301.378686	8.25	301.491361
30 ‡	$y = x$	0.076050	265.971645	15.31	265.973315
30 ‡	$y = 1.5x$	0.074053	257.814199	16.94	257.814247
50 ‡	$y = 0.5x$	0.107079	1126.392248	35.00	1127.382313
50 ‡	$y = x$	0.116091	966.377027	30.61	966.377615
50 ‡	$y = 1.5x$	0.095062	939.487369	29.44	939.487629

†Parlar '94 ‡Zaferanieh, Taghizadeh, Brimberg & Wesolowsky, '08. $\ell_{p_H} = \frac{1}{4}\ell_\infty$

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$\lvert A \cup B \rvert$	$\mathcal{H}$	CPUTime(PT)	f^* (PT)	x^* (PT)
4 †	$y = x$	0.0000	20.5307	$(0.000000, 0.000001)$
18 †	$y = 1.5x$	0.0000	108.3362	$(8.811381, 7.119336)$
30 †	$y = 0.5x$	0.0156	254.7805	$(6.000000, 3.000000)$
30 ‡	$y = x$	0.0000	230.7513	$(5.234851, 5.234838)$
30 ‡	$y = 1.5x$	0.0156	244.4072	$(5.153294, 5.102873)$
50 ‡	$y = 0.5x$	0.0156	917.1736	$(11.923664, 5.961832)$
50 ‡	$y = x$	0.0156	808.2990	$(10.000020, 9.999995)$
50 ‡	$y = 1.5x$	0.0156	892.4482	$(10.521522, 9.571467)$

†Parlar '94 ‡Zaferanieh, Taghizadeh, Brimberg & Wesolowsky, '08.

Eilon, Watson-Gandy & Christofides data set

$\lvert A \cup B \rvert = 50$			$\mathcal{H} = \{y = 1.5x\}$ ($\lvert A \rvert = 15$)		$\mathcal{H} = \{y = x\}$ ($\lvert A \rvert = 18$)		$\mathcal{H} = \{y = 0.5x\}$ ($\lvert A \rvert = 39$)	
p_A	p_B	p_H	CPUTime	f^*	CPUTime	f^*	CPUTime	f^*
1.5	1		0.0000	230.8447	0.0313	212.9341	0.0156	200.6406
2	1		0.0158	227.9991	0.0156	202.6576	0.0000	185.9525
	1.5		0.0313	194.1881	0.0313	189.0401	0.0156	182.1283
3	1		0.0313	223.8203	0.0469	194.1612	0.0156	174.0444
	1.5		0.0156	192.0466	0.0469	180.9279	0.0313	170.3199
	2		0.0156	178.2223	0.0312	174.8964	0.0313	168.5066
∞	1		0.0000	219.8367	0.0000	182.1900	0.0000	161.2033
	1.5		0.0313	188.7783	0.0156	168.9589	0.0000	157.2146
	2		0.0156	175.4420	0.0156	163.6797	0.0000	155.6124
	3		0.0156	164.5924	0.0156	159.3740	0.0156	154.3965
1	1	1.5	0.0156	237.4732	0.0156	224.9178	0.0000	236.1300
		2	0.0000	237.3162	0.0156	218.9480	0.0000	235.4689
		3	0.0156	236.3904	0.0156	213.5591	0.0156	234.9807
		∞	0.0000	233.7967	0.0156	204.3500	0.0000	234.7300
1.5	1	2	0.0156	230.8165	0.0313	206.9512	0.0469	200.5514
		3	0.0625	228.5484	0.0938	201.5863	0.0156	200.3068
		∞	0.0313	225.9387	0.0156	192.4722	0.0156	200.1428
	1.5	2	0.0313	196.5559	0.0469	193.3584	0.0313	196.4864
		3	0.0469	196.5561	0.0469	188.3989	0.0313	196.3008
		∞	0.0156	196.5431	0.0469	179.3396	0.0313	196.1787
2	1	3	0.0156	225.7539	0.0313	197.2805	0.0156	185.9501
		∞	0.0156	223.1421	0.0156	188.1506	0.0156	185.9133
	1.5	3	0.0469	194.1881	0.0469	184.0770	0.0313	182.1271
		∞	0.0156	194.1881	0.0313	175.0117	0.0158	182.0955
	2	3	0.0156	180.1096	0.0156	178.0624	0.0156	180.1097
		∞	0.0156	180.1097	0.0156	169.7842	0.0156	180.0857
3	1	∞	0.0313	221.2011	0.0156	184.9957	0.0313	174.0442
	1.5		0.0313	192.0466	0.0313	171.8455	0.0313	170.3199
	2		0.0156	178.2223	0.0313	166.6027	0.0156	168.5066
	3		0.0312	166.8362	0.0469	162.3214	0.0313	166.8361

Experiments: Larger Instances

			$\lvert A \cup B \rvert = 5000$			$\lvert A \cup B \rvert = 10000$			$\lvert A \cup B \rvert = 50000$		
p_A	p_B	p_H	$d = 2$	$d = 3$	$d = 5$	$d = 2$	$d = 3$	$d = 5$	$d = 2$	$d = 3$	$d = 5$
1.5	1		3.2034	5.4599	10.1520	7.4852	9.2511	19.0804	40.9418	74.9246	115.2941
2	1		1.5939	2.2502	7.6415	5.1255	8.2040	14.0078	21.8708	25.9411	59.7786
	1.5		3.9692	6.0632	4.5474	8.1728	14.0797	23.8067	55.2635	83.8310	154.2883
3	1		3.9222	5.1412	6.9852	6.8132	9.4927	20.6114	42.9964	61.4724	116.4665
	1.5		5.4850	10.0950	13.4449	14.3149	21.0337	34.0574	91.9616	106.6900	206.6997
	2		7.9385	9.8603	10.1802	14.2672	17.7362	38.0629	95.3150	135.0647	180.6230
∞	1		0.3125	0.6940	9.4607	0.8750	1.6096	6.3288	6.0945	25.7856	89.7772
	1.5		1.2346	2.2502	8.6333	5.6724	4.9605	9.1259	18.8410	32.5503	54.0310
	2		0.8908	1.2188	15.9704	1.9534	2.7346	7.9853	18.8615	17.2053	40.5464
	3		3.4691	2.7346	12.0584	9.5637	6.7195	9.5323	71.7654	70.1868	49.5907
1	1	1.5	18.9396	28.7109	15.6735	37.5415	80.9833	401.8414	596.6057	878.6363	3171.6235
		2	13.7043	24.4318	13.2359	29.2056	68.3894	372.3283	354.3334	721.5562	3166.1511
		3	17.5702	25.1258	3.8570	39.3008	93.4990	415.0733	541.8219	1014.1090	3945.8234
		∞	4.9695	11.7517	3.1101	13.7673	26.7468	96.7260	133.7586	632.9736	2492.2830
1.5	1	2	5.2506	8.2509	4.6457	13.7986	16.0956	37.3793	105.4177	103.2694	273.0866
		3	6.2975	11.9545	4.0473	13.2135	24.9720	57.8267	96.9583	128.9880	326.7660
		∞	3.6722	5.5632	4.1409	7.0632	13.1580	31.0345	46.1239	81.3482	118.2435
	1.5	2	12.9546	15.8455	3.7347	23.3466	29.3155	46.6898	138.6629	200.2891	385.1307
		3	13.5232	14.9234	4.5473	22.2837	33.9099	53.9483	171.0538	175.6803	697.5071
		∞	12.0022	11.5482	3.9533	21.8464	22.1743	37.0102	111.1779	144.5975	241.2852
2	1	3	3.5316	7.6883	125.3288	9.8294	11.5794	41.0986	61.4067	62.9410	158.6635
		∞	1.7034	3.3288	145.9833	3.5629	7.7041	15.4610	22.8465	38.9976	98.4269
	1.5	3	5.6255	9.3605	105.3967	13.4234	19.0805	45.4697	71.1114	101.3439	269.3303
		∞	5.1256	5.4850	137.3159	7.6791	16.5075	24.8255	63.0027	85.4602	134.8291
	2	3	6.6725	9.4387	132.3028	12.1731	20.4003	39.2473	79.9453	121.0863	220.7875
		∞	4.6879	5.4607	153.6319	9.4696	14.5639	22.6620	68.1690	63.1358	118.4005
3	1	∞	3.7357	6.5511	17.7052	7.8602	10.1575	34.1457	37.1292	48.5630	140.3546
	1.5		7.7665	10.4455	17.7145	15.2061	26.2626	37.2546	84.7931	119.5438	235.1177
	2		7.6569	10.6885	17.4306	16.5483	23.6745	44.5896	99.2611	227.0411	219.4903
	3		9.8843	10.0948	19.1583	19.2838	21.8153	43.0209	129.5420	153.3979	243.4983

Extensions

- Norms for each demand points: Each point provided with two norms $||\cdot||_a^A$ and $||\cdot||_b^B$.

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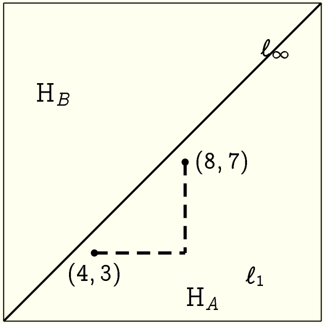

length $= 8$.

Extensions

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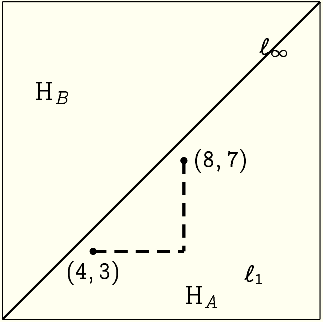
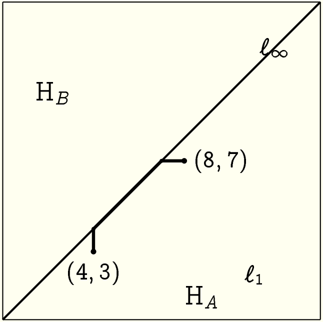


length = 8.



length = 6.

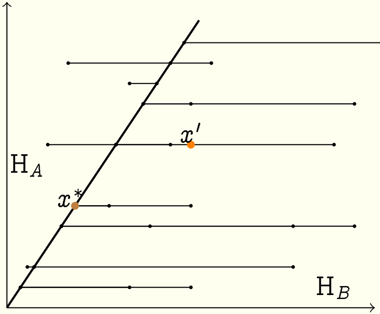
Extensions

$$
\begin{aligned}
\min \sum_{a \in A} \omega_a z_a &+ \sum_{b \in B} \omega_b(\theta_b + u_b) \\
\text{s.t. } z_a^1 &\geq \|x - a\|_{p_A}, \ \forall a \in A, \\
z_a^2 &\geq \|x - y_a^1\|_{p_A}, \ \forall a \in A, \\
z_a^3 &\geq \|y_a^1 - y_a^2\|_{p_H}, \ \forall a \in A, \\
z_a^4 &\geq \|y_a^2 - a\|_{p_A}, \ \forall a \in A, \\
\theta_b &\geq \|x - y_b\|_{p_A}, \ \forall b \in B, \\
u_b &\geq \|y_b - b\|_{p_B} \ \forall b \in B, \quad (\mathrm{P}_A^{\mathrm{EXT}}) \\
z_a &\geq z_a^1 + M_a(\delta_a - 1), \forall a \in A, \\
z_a &\geq z_a^2 + z_a^3 + z_4^a - M_a\delta_a, \forall a \in A, \\
\alpha^t x &\leq \beta, \\
\alpha^t y_a^j &= \beta, \ \forall j = 1, 2, \\
\alpha^t y_b &= \beta, \ \forall a \in A, \\
\delta_a &\in \{0, 1\}, \ \forall a \in A, \\
z_a^k &\geq 0, \ \forall a \in A, k = 1, 2, 3, 4 \\
\theta_b, u_b &\geq 0, \ \forall b \in B, \\
x, y_a^1, y_a^2, y_b &\in \mathbb{R}^d.
\end{aligned}
$$

$$
\begin{aligned}
\min \sum_{b \in B} \omega_b z_b &+ \sum_{a \in A} \omega_a(\theta_a + u_a) \\
\text{s.t. } z_b^1 &\geq \|x - b\|_{p_B}, \ \forall b \in B, \\
z_b^2 &\geq \|x - y_b^1\|_{p_B}, \ \forall b \in B, \\
z_b^3 &\geq \|y_b^1 - y_b^2\|_{p_H}, \ \forall b \in B, \\
z_b^4 &\geq \|y_b^2 - b\|_{p_B}, \ \forall b \in B, \\
\theta_a &\geq \|x - y_a\|_{p_A}, \ \forall a \in A, \\
u_a &\geq \|y_a - a\|_{p_B} \ \forall a \in A, \quad (\mathrm{P}_B^{\mathrm{EXT}}) \\
z_b &\geq z_b^1 + M_b(\delta_b - 1), \forall b \in B, \\
z_b &\geq z_b^2 + z_b^3 + z_4^a - M_b\delta_b, \forall b \in B, \\
\alpha^t x &\geq \beta, \\
\alpha^t y^j &= \beta, \ \forall j = 1, 2, \\
\alpha^t y_a &= \beta, \ \forall a \in A, \\
\delta_b &\in \{0, 1\}, \ \forall b \in B, \\
z_b^k &\geq 0, \ \forall a \in A, k = 1, 2, 3, 4 \\
\theta_b, u_b &\geq 0, \ \forall b \in B, \\
x, y_b^1, y_b^2, y_a &\in \mathbb{R}^d.
\end{aligned}
$$

Extensions

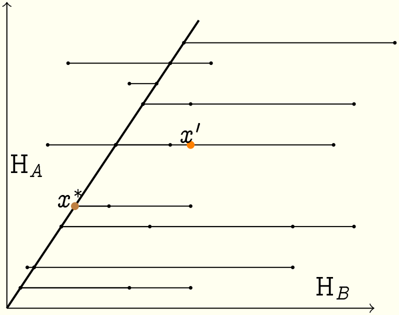
$$\ell_{p_H} = \frac{1}{4}\ell_\infty,\ \ell_{p_A} = \ell_{p_B} = \ell_1$$

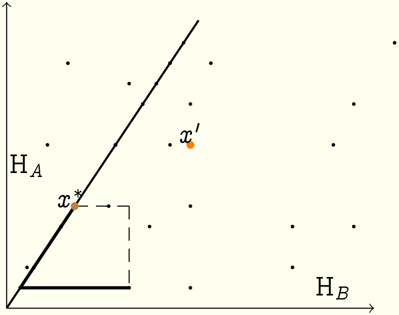


$f^* = 128.00$
$f' = 132.9166.$

Extensions

$$\ell_{p_H} = \frac{1}{4}\ell_\infty,\ \ell_{p_A} = \ell_{p_B} = \ell_1$$





$f^* = 128.00$
$f' = 132.9166.$

$d((6,1), x^*) = 6.3333$
$d_1((6,1), x^*) = 6.66666.$

Extensions

$$p_A = p_B = 1 \text{ and } \|\cdot\|_H = \frac{1}{4}\ell_\infty \text{ , } \mathcal{H} = \{(x,y) : y = \alpha_1 x\}$$

α_1	N	x'_T	f'_T	CPUTime $_T$	x^*_{Refl}	f^*_{Refl}	CPUTime $_{Refl}$	Improvement
0.5	4	(5, 2.5)	16.75	0.0000	(5, 2.5)	16.75	0.0156	0.00%
	18	(9, 4.5)	97.75	0.0000	(9, 4.5)	89.50	0.0313	9.22%
	30	(6, 3)	266.50	0.0000	(6, 3)	251.00	0.0313	6.18%
	50 †	(12, 6)	959.75	0.0000	(11, 5.5)	911.50 ⋆	>3600	5.29%
	50 ‡	(5.89, 2.945)	201.55	0.0000	(5.89, 2.945)	189.91 ⋆	>3600	6.13%
1	4	(5, 6)	24.17	0.0000	(4, 6)	23.67	0.0156	2.11%
	18	(9, 8)	132.92	0.0000	(3.3333, 5)	128.00	0.0156	3.84%
	30	(5, 5)	299.75	0.0000	(2.6667, 4)	269.75	0.0625	11.12%
	50 †	(11, 10)	1076.58	0.0156	(5.3333, 8)	1009.25 ⋆	>3600	6.67%
	50 ‡	(3.7133, 5.570)	206.37	0.0156	(3.5, 5.250)	195.52 ⋆	>3600	5.55%
1.5	4	(0, 0)	22.50	0.0000	(5, 5)	22.50	0.0156	0.00%
	18	(8, 8)	123.00	0.0000	(8, 8)	105.50	0.0781	16.59%
	30	(5, 5)	265.25	0.0000	(5, 5)	251.25	1.2971	5.57%
	50 †	(1, 10)	927.75	0.0000	(1, 10)	873.50 ⋆	>3600	6.21%
	50 ‡	(5, 5)	177.52	0.0000	(5.57, 5.57)	170.4 ⋆	>3600	4.18%

⋆ : Best Solution Found

†Zaferanieh, Taghizadeh Kakhki, Brimberg, J. & Wesolowsky, '08.

‡Eilon, Watson-Gandy & Christofides, '71.

Thank you!

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http://arxiv.org/abs/1404.3068