Exploiting symmetries in multifacility location

Seminario Nuevos Desafíos de la Matemática Combinatoria

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(joint work with J. Puerto and S. ElHaj-BenAli)

Desafíos de la Matemática Combinatoria
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Outline

Moments, SOS and SDP

Let's $\mathbf{K} = \{x \in \mathbb{R}^d : g_j(x) \geq 0,\ \forall j\}$; $\tilde{f}, g_j \in \mathbb{R}[X]$.

$$\left.\begin{array}{ll} \min & \tilde{f}(x) \\ & x \in \mathbf{K} \end{array}\right\}$$

Moments, SOS and SDP

Let's $\mathbf{K} = \{x \in \mathbb{R}^d : g_j(x) \geq 0,\ \forall j\}$; $\tilde{f}, g_j \in \mathbb{R}[X]$.

$$\left.\begin{array}{ll} \min & \tilde{f}(x) \\ & x \in \mathbf{K} \end{array}\right\} \overset{\text{(Lasserre, 2001)}}{\equiv} \left\{\begin{array}{ll} \displaystyle\min_{\mu \in \mathcal{M}(\mathbf{K})} & \displaystyle\int \tilde{f}\, d\mu \\ & \int d\mu = 1 \\ & \mu \geq 0 \end{array}\right. , \qquad (1)$$

$\mathcal{M}(\mathbf{K})$ is the vector space of finite, signed Borel measures supported on $\mathbf{K}$.

Moments, SOS and SDP

$$\mathbb{R}[X] = \mathbb{R}[1, X_1, ..., X_d].$$

$$\begin{aligned}\mathcal{B} &= [1, X_1, ..., X_d, X_1^2, X_1X_2, ..., X_1X_d, X_2^2, ...]\\ &= [X^\alpha]_{\alpha\in\mathbb{N}^d}\end{aligned}$$

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Then, for any sequence $\{y_\alpha\}_\alpha$ indexed in the same order that $\mathcal{B}$, and for any polynomial $\tilde{f} = \sum_{\alpha \in \mathbb{N}^d} \tilde{f}_\alpha X^\alpha$ We introduce the functional $L_y : \mathbb{R}[X] \longrightarrow \mathbb{R}$:

$$\tilde{f} \quad \longmapsto \quad \sum_{\alpha \in \mathbb{N}^d} \tilde{f}_\alpha y_\alpha$$

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Example: For $\mathbb{R}[x_1, x_2]$ and $r = 2$, $\mathcal{B} = \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, \ldots\}$, so for $f = x_1^2 - x_2^3 + 2x_1x_2 \in \mathbb{R}[x_1, x_2]$: $L_y(f) = y_{2,0} - y_{0,3} + 2y_{1,1}$

Moment and Locaizing Matrices

The moment matrix of order r, $\mathrm{M}_r(y)$, is defined as $\mathrm{M}_r(y)_{ij} = L_y(b_i \cdot b_j)$, where b_i are the elements in $\mathcal{B}$ such that $deg(b_i), deg(b_j) \leq r$.

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$$M_2(y) = \begin{array}{c} \\ 1 \\ X_1 \\ X_2 \\ X_1^2 \\ X_1X_2 \\ X_2^2 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & X_1 & X_2 & X_1^2 & X_1X_2 & X_2^2 \end{array} \\ \begin{pmatrix} y_{0,0} & y_{1,0} & y_{0,1} & y_{2,0} & y_{1,1} & y_{0,2} \\ y_{1,0} & y_{2,0} & y_{1,1} & y_{3,0} & y_{2,1} & y_{1,2} \\ y_{0,1} & y_{1,1} & y_{0,2} & y_{2,1} & y_{1,2} & y_{0,3} \\ y_{2,0} & y_{3,0} & y_{2,1} & y_{4,0} & y_{3,1} & y_{2,2} \\ y_{1,1} & y_{2,1} & y_{1,2} & y_{3,1} & y_{2,2} & y_{1,3} \\ y_{0,2} & y_{1,2} & y_{0,3} & y_{2,2} & y_{1,3} & y_{0,4} \end{pmatrix} \end{array}$$

Moment and Localizing Matrices

For any $g \in \mathbb{R}[X]$, the localizing matrix of g of order r, $\mathrm{M}_r(g\,y)$, is defined as $\mathrm{M}_r(g\,y)_{ij} = L_y(g \cdot b_i \cdot b_j)$, where b_i are the elements in $\mathcal{B}$ such that $deg(g) + deg(b_i) + deg(b_j) \leq 2r$.

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For $g(x_1, x_2) = 1 - x_1 x_2$ and $r = 1$:

$$M_1(gy) = \begin{array}{c} \\ 1 \\ X_1 \\ X_2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad 1 \quad & \quad X_1 \quad & \quad X_2 \quad \end{array} \\ \begin{pmatrix} y_{0,0} - y_{1,1} & y_{1,0} - y_{2,1} & y_{0,1} - y_{1,2} \\ y_{1,0} - y_{2,1} & y_{2,0} - y_{3,1} & y_{1,1} - y_{2,2} \\ y_{0,1} - y_{1,2} & y_{1,1} - y_{2,2} & y_{0,2} - y_{1,3} \end{pmatrix} \end{array}$$

Moments, SOS and SDP

If **K** satisfies Arquimedean Property (also called Putinar's Property):

$$\exists u \in \mathbb{R}[x]: \ \{x : u(x) \geq 0\} \text{ is compact and}$$

$$u \ = \ \sigma_0 + \sum_{j=1}^{\ell} \sigma_j \, g_j, \ \text{being } \sigma_j \text{ s.o.s. polynomials} \tag{2}$$

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Then,

$$\left.\begin{array}{ll} \inf & \langle \tilde{f}, \mu \rangle \\ s.t. & \langle 1, \mu \rangle = 1, \\ & \mu \geq 0, \\ & \mu \in \mathcal{M}(\mathbf{K}) \end{array}\right\} \equiv \left.\begin{array}{ll} \inf & L_y(\tilde{f}) \\ s.t. & \{y_\alpha\} \text{ has} \\ & \text{a representing} \\ & \text{measure} \end{array}\right\}$$

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$$\nu = \max\{\left\lceil \frac{deg \tilde{f}}{2} \right\rceil, \left\lceil \frac{deg g_j}{2} \right\rceil\}$$

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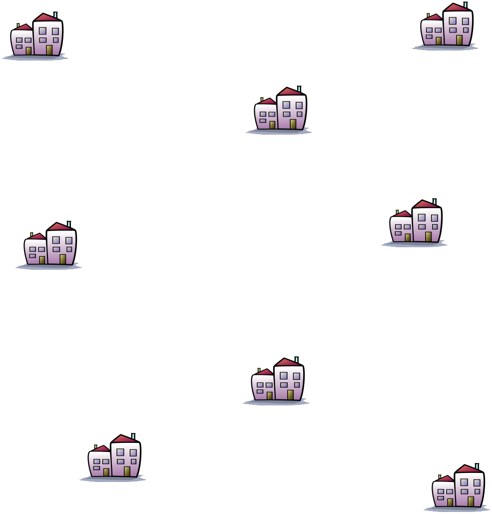
Solving for a fixed r_0 it gives a relaxation of the original polynomial optimization problem by a "SDP"

Continuos 1-F OM Location



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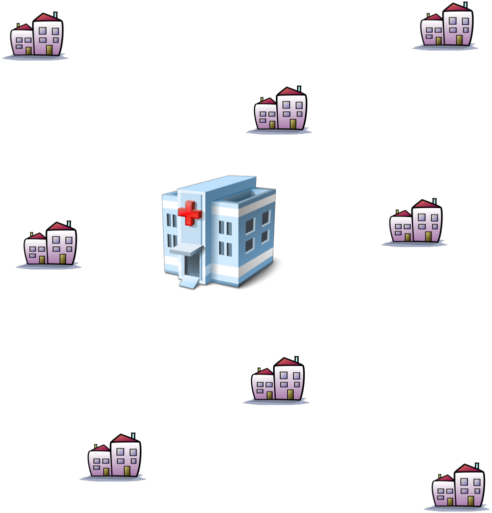

Generalized location problems with rational objective

We are given n point $\{a_1, \ldots, a_n\} \subset \mathbb{R}^d$ endowed with an ℓ_α-norm, $\alpha = r/s \in \mathbb{Q}$, m.c.d.$(r, s)$=1.

- $f_j := \frac{p_j}{q_j} : \mathbb{R}^d \to \mathbb{R}$ are polynomials or piecewise polynomials for $j = 1, \ldots, m$.

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- $f_j := \frac{p_j}{q_j} : \mathbb{R}^d \to \mathbb{R}$ are polynomials or piecewise polynomials for $j = 1, \ldots, m$.
- $K = \{(x, u) \in \mathbb{R}^d \times \mathbb{R}^n_+ : g_j(x, u) \geq 0,\ j = 1, \ldots, m, q_j > 0$
  $u_i = \begin{cases} \|x - a_i\|_\alpha,\ or \\ \min\limits_{x \in X,\ |X|=p} \|x - a_i\|_\alpha \end{cases},\ i = 1, ..., n\}$

  In most cases, we add the redundant constraint $\|x\|_2^2 \leq M$.

  K is a closed, compact semi-algebraic set.

We shall define the dependence of f_j to the decision variable $x \in \mathbb{R}^d$ via $u = (u_1, \ldots, u_n)$, where $u_i : \mathbb{R}^d \mapsto \mathbb{R}$, $u_i(x) := \|x - a_i\|_\alpha$, $i = 1, \ldots, n$. Therefore, the j-th component of the ordered median objective function of our problems reads as:

$$\begin{array}{rccl} \tilde{f}_j(x): & \mathbb{R}^d & \mapsto & \mathbb{R} \\ & x & \mapsto & \tilde{f}_j(x) := f_j(\|x - a_1\|_\alpha, \ldots, \|x - a_n\|_\alpha) \end{array}$$

Consider the following problem:

$$(\textbf{LOCOMRF}) \quad \rho_\lambda := \min_x \{ \sum_{j=1}^m \lambda_j(x) \tilde{f}_{(j)}(x) : x \in \mathbf{K} \}, \tag{3}$$

- $\mathbf{K} \subseteq \mathbb{R}^d$ satisfies Putinar's property.

- $f(u_1, \ldots, u_n) = \sum_{i=1}^n u_i$ if
$$u_i = \begin{cases} \|x - a_i\|_\alpha, & \text{Weber problem} \\ \min\limits_{x \in X,\ |X| = p} \|x - a_i\|_\alpha, & \text{Multifacility Weber problem} \end{cases},$$
- $f(u_1, \ldots, u_n) = \sum_{i=1}^n u_i^q$ the centroid problem,
- $f(u_1, \ldots, u_n) = \max\limits_{1 \leq i \leq n} u_i \equiv \min\limits_{u_i \leq z} z$, center problem,
- $f(u_1, \ldots, u_n) = \sum\limits_{i=1}^n \lambda_i u_{(i)} \equiv \min\limits_{\sum_{i=1}^n \lambda_1 u_{\sigma(i)} \leq z, \forall \sigma} z$, ordered median problem,

- $f(u_1, \ldots, u_n) = \sum_{i<j}^{n} |u_i - u_j|$, absolute deviation or envy problem
- $f(u_1, \ldots, u_n) = \sum_{i=1}^{n} (u_i - \bar{u})^2$, variance problem ...
- $f(u_1, \ldots, u_n) = \sum_{j=1}^{n} \frac{w_j}{u_j^2}$, obnoxious facility location
- $f(u_1, \ldots, u_n) = \sum_{j=1}^{n} \frac{b_j}{1 + h_j |u_j|^{\lambda}}$, Huff competitive location
- Gradual covering, acceleration-deceleration distance, inventory gradual covering ...

Equivalent problem

$$\overline{\rho}_\lambda = \min_{x,w,u,v} \quad \sum_{j=1}^{m} \lambda_j(x) \sum_{i=1}^{m} f_i(u) w_{ij} \tag{4}$$

$$\text{s.t.} \quad \sum_{j=1}^{m} w_{ij} = 1, \ \text{for } i = 1, \ldots, m$$

$$\sum_{i=1}^{m} w_{ij} = 1, \ \text{for } j = 1, \ldots, m$$

$$\sum_{i=1}^{m} w_{ij} f_i(u) \geq \sum_{i=1}^{m} w_{ij+1} f_i(u), j = 1, \ldots, m \tag{5}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{m} w_{ij}^2 - w_{ij} = 0,$$

$$v_{ij}^s \geq x_j - a_{ij}, \ i = 1, \ldots, n, \ j = 1, \ldots, d \tag{6}$$

$$v_{ij}^s \geq a_{ij} - x_j, \ i = 1, \ldots, n, \ j = 1, \ldots, d \tag{7}$$

$$z_i^r = \sum_{j=1}^{d} v_{ij}^r, \ i = 1, \ldots, n, \tag{8}$$

$$u_i = z_i^s, \ i = 1, \ldots, n,$$

$$\sum_{i,j=1}^{m} w_{ij}^2 \leq m, \ w_{ij} \in \mathbb{R}, \ x \in \mathbf{K}.$$

SDP Relaxation

For $r \geq \max\{r_0, \nu_0\}$ where $r_0 := \max_{k=1,\ldots,\ell} \xi_k$ and $\nu_0 := \max\{\max_{j=0,\ldots,m} \nu_j, \overbrace{\max_{j=1,\ldots,m} \nu_j'}^{=1}\} = \max_{j=0,\ldots,m} \nu_j$, we introduce the following hierarchy of semidefinite programs:

$$
\begin{array}{lll}
\min_{\mathbf{y}} & \mathrm{L}_{\mathbf{y}}(p_\lambda) & \\
\text{s.t.} & \mathrm{M}_r(\mathbf{y}; I(0)) \succeq 0, & \\
& \mathrm{M}_{r-\xi_k}(g_k\mathbf{y}; I(0)) \succeq 0, & k = 1,\ldots,\ell, \\
& \mathrm{M}_r(\mathbf{y}; I(0) \cup I(j) \cup I(j+1)) \succeq 0, & j = 1,\ldots,m, \\
& \mathrm{M}_{r-\nu_j}(h_j\mathbf{y}; I(0) \cup I(j) \cup I(j+1)) \succeq 0, & j = 1,\ldots,m-1, \\
& \mathrm{M}_{r-1}(h_j'\mathbf{y}; I(0) \cup I(j) \cup I(j+1)) \succeq 0, & j = 1,\ldots,m, \\
& \mathrm{L}_y(\sum_{i=1}^m w_{ij} - 1) = 0, & j = 1,\ldots,m, \\
& \mathrm{L}_y(\sum_{j=1}^m w_{ij} - 1) = 0, & i = 1,\ldots,m, \\
& \mathrm{L}_y(w_{ij}^2 - w_{ij}) = 0, & i,j = 1,\ldots,m, \\
& \mathrm{L}_y(q_\lambda) = 1, &
\end{array}
\qquad (\mathbf{Q}_r)
$$

with optimal value denoted $\min \mathbf{Q}_r$.

Some Results

Theorem (B., Puerto, ElHaj-BenAli, 2012)

1. *Let (x) be a feasible solution of (**LOCOMF**) then there exists a solution (x, u, v, w) for ($\mathrm{MFOMP1}_\lambda$) such that their objective values are equal. Conversely, if (x, u, v, w) is a feasible solution for ($\mathrm{MFOMP1}_\lambda$) then there exists a solution (x) for (**LOCOMF**) having the same objective value. In particular $\varrho_\lambda = \overline{\varrho}_\lambda$. Moreover, if $K \subset \mathbb{R}^d$ satisfies Putinar's property then $\overline{\mathbf{K}} \subset \mathbb{R}^{d+m^2+n(d+2)}$ also satisfies Putinar's property.*

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2. *Let $\overline{\mathbf{K}} \subset \mathbb{R}^{d+m^2+n(d+2)}$ (compact) be the feasible domain of Problem $(\mathrm{MFOMP1}_\lambda)$. Let $\mathbf{Q}_r$ be the semidefinite program $(\mathbf{Q1}_r)$ with (g_k), $(h_j) \subset \mathbb{R}[x, u, v, w]$ the polynomial functions defining the constraints of $\overline{\mathbf{K}}$. Then:*

   (a) $\inf \mathbf{Q}_r \uparrow \rho$ *as* $r \to \infty$.

   (b) *Let $\mathbf{y}^r$ be an optimal solution of the SDP relaxation $\mathbf{Q}_r$ in $(\mathbf{Q1}_r)$. If*

$$\operatorname{rank} M_r(\mathbf{y}^r) = \operatorname{rank} M_{r-r_0}(\mathbf{y}^r) = t \qquad (9)$$

*then $\min \mathbb{Q}_r = \rho$ and one may extract t points $(x^*(k), u^*(k), v^*(k), w^*(k))_{k=1}^t \subset \overline{\mathbf{K}}$, all global minimizers of the **MOMRF** problem.*

B. & ElHaj-BenAli & Puerto, CORS 2013

Convex OMP

$$\min_{x \in \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \|x - a_{\sigma(i)}\|_\tau. \tag{10}$$

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Theorem (B., Puerto, ElHaj-BenAli, 2013)

For any set of lambda weights satisfying $\lambda_1 \geq \ldots \geq \lambda_n$ and $\tau = \frac{r}{s}$ such that $r, s \in \mathbb{N} \setminus \{0\}$, $r > s$ and $\gcd(r, s) = 1$, Problem (10) can be represented as a semidefinite programming problem with $n^2 + n(2d + 1)$ linear constraints and at most $4nd \log r$ positive semidefinite constraints.

B. & Puerto & ElHaj-BenAli, Preprint 2013

Constrained Case

Theorem

Consider the restricted problem:

$$\min_{x \in \mathbf{K} \subset \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \|x - a_{\sigma(i)}\|_\tau . \tag{11}$$

Assume that the hypothesis of Theorem 2 holds. In addition, any of the following conditions holds:

1. $g_i(x)$ *are concave for* $i = 1, \ldots, \ell$ *and* $-\sum_{i=1}^{\ell} \mu_i \nabla^2 g_i(x) \succ 0$ *for each dual pair* (x, μ) *of the problem of minimizing any linear functional* $c^t x$ *on* **K** *(Positive Definite Lagrange Hessian (PDLH)).*
2. $g_i(x)$ *are sos-concave on* **K** *for* $i = 1, \ldots, \ell$ *or* $g_i(x)$ *are concave on* **K** *and strictly concave on the boundary of* **K** *where they vanish, i.e.* $\partial \mathbf{K} \cap \partial \{x \in \mathbb{R}^d : g_i(x) = 0\}$, *for all* $i = 1, \ldots, \ell$.
3. $g_i(x)$ *are strictly quasi-concave on* **K** *for* $i = 1, \ldots, \ell$.

Then, there exists a constructive finite dimension embedding, which only depends on τ *and* g_i, $i = 1, \ldots, \ell$, *such that (18) is a semidefinite problem.*

Constrained Case

$$
\begin{aligned}
(\mathbf{Q}_N): \min \quad & \sum_{k=1}^{n} v_k + \sum_{i=1}^{n} w_i && (12)\\
s.t. \quad & v_i + w_k \geq \lambda_k z_i, \qquad \forall i, k = 1, \ldots, n, && (13)\\
& y_{ij} - x_j + a_{ij} \geq 0, \qquad \forall i = 1, \ldots, n,\ j = 1, \ldots, d. && (14)\\
& y_{ij} + x_j - a_{ij} \geq 0, \qquad \forall i = 1, \ldots, n,\ j = 1, \ldots, d. &&\\
& y_{ij}^r \leq u_{ij}^s z_i^{r-s}, \qquad \forall i = 1, \ldots, n,\ j = 1, \ldots, d, && (15)\\
& \omega_i^{\frac{r}{s}} \sum_{j=1}^{d} u_{ij} \leq z_i, \qquad \forall i = 1, \ldots, n, && (16)\\
& \mathrm{M}_N(\kappa) \succeq 0, && (17)\\
& \mathrm{M}_{N-\xi_k}(g_k, \kappa) \succeq 0, \qquad k = 1, \ldots, \ell, && (18)\\
& \\
& L_\kappa(x_j) = x_j, \qquad j = 1, \ldots, d, &&\\
& L_\kappa(z_i) = z_i, \qquad i = 1, \ldots, n, &&\\
& L_\kappa(v_i) = v_i, \qquad i = 1, \ldots, n, &&\\
& L_\kappa(w_i) = w_i, \qquad i = 1, \ldots, n, &&\\
& L_\kappa(u_{ij}) = u_{ij}, \qquad i = 1, \ldots, n,\ j = 1, \ldots, d, &&\\
& L_\kappa(y_{ij}) = y_{ij}, \qquad i = 1, \ldots, n,\ j = 1, \ldots, d, &&\\
& \kappa_0 = 1 &&\\
& u_{ij} \geq 0, \qquad \forall\ i = 1, \ldots, n,\ j = 1, \ldots, d. && (19)
\end{aligned}
$$

with optimal value denoted $\min \mathbf{Q}_N$.

Constrained Case

Theorem

Consider ρ_λ defined as the optimal value of the problem:

$$\rho_\lambda = \min_{x \in \mathbf{K} \subset \mathbb{R}^d} \sum_{i=1}^{n} \lambda_i \omega_{\sigma(i)} \|x - a_{\sigma(i)}\|_\tau . \qquad (20)$$

Then, with the notation above:
(a) $\min \mathbf{Q}_N \uparrow \rho_\lambda$ *as* $N \to \infty$*.*
(b) Let κ^N *be an optimal solution of Problem* $(\mathbf{Q}_N)$*. If*

$$\operatorname{rank} \mathrm{M}_N(\kappa^N) = \operatorname{rank} \mathrm{M}_{N-N_0}(\kappa^r) = \vartheta$$

then $\min \mathbf{Q}_N = \rho_\lambda$ *and one may extract* ϑ *points*

$$(x^*(i), z^*(i), v^*(i), w^*(i), u^*(i), y^*(i))_{i=1}^{\vartheta} \subset \mathbf{K},$$

all global minimizers of Problem (27).

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$$\operatorname{rank} \mathrm{M}_N(\kappa^N) = \operatorname{rank} \mathrm{M}_{N-N_0}(\kappa^r) = \vartheta$$

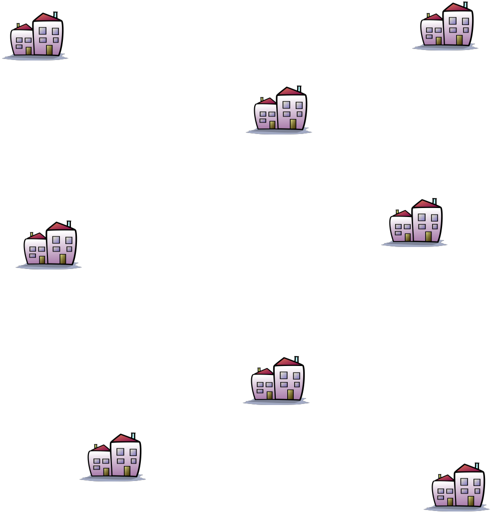
then $\min \mathbf{Q}_N = \rho_\lambda$ *and one may extract* ϑ *points*

$$(x^*(i), z^*(i), v^*(i), w^*(i), u^*(i), y^*(i))_{i=1}^{\vartheta} \subset \mathbf{K},$$

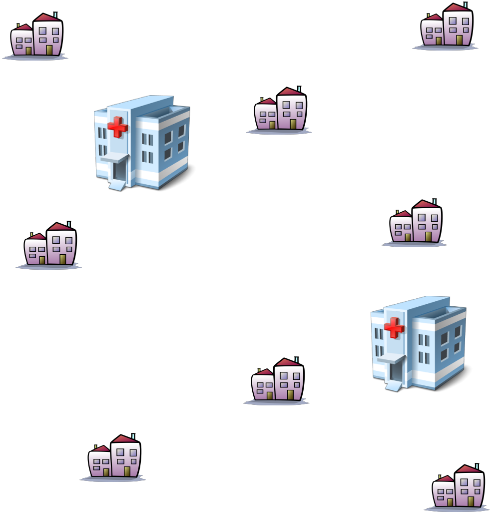
all global minimizers of Problem (27).

B. & Puerto & ElHaj-BenAli, Preprint 2013

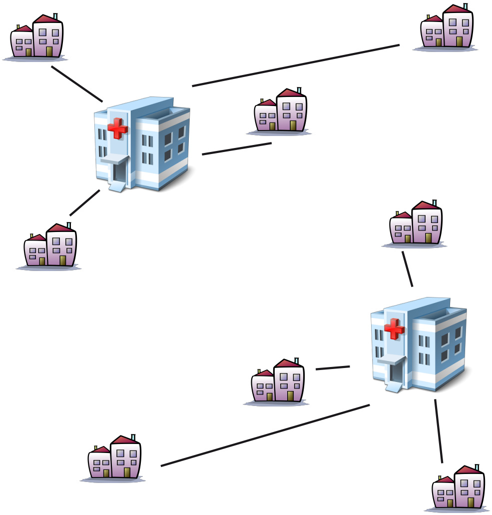
Continuos MF OM Location



Continuos MF OM Location



Continuos MF OM Location



Continuous MF OMP

We are given a set of demand points $S = \{a_1, ..., a_n\}$ and two sets of scalars $\Omega := \{\omega_1, ..., \omega_n\}$, $\omega_i \geq 0$, $\forall\ i \in \{1, \ldots, n\}$ and $\Lambda := \{\lambda_1, ..., \lambda_n\}$ where $\lambda_1 \geq ... \geq \lambda_n \geq 0$. The elements ω_i are weights corresponding to the importance given to the existing facilities $a_i, i \in \{1, ..., n\}$.

Continuous MF OMP

$$\rho_\lambda := \min_x \{\sum_{i=1}^{n} \lambda_i \tilde{f}_{(i)}(x) : x = (x_1, \dots, x_p),\ x_j \in \mathbf{K},\ \forall j = 1, \dots, p\}, \qquad \textbf{(LOCOMF)}$$

where:

• $\mathbf{K} \subseteq \mathbb{R}^d$ satisfies the Archimedian property. Without loss of generality we shall assume that we know $M > 0$ such that $\sum_{j=1}^{p} \|x_j\|_2^2 \leq M$.

• $\tau := \frac{r}{s} \geq 1$, $r, s \in \mathbb{N}$, $r \geq s$ and $gcd(r,s) = 1$.

• $\lambda_\ell \geq 0$ for all $\ell = 1, \dots, n$.

Poly Opt Formulation

$$\overline{\rho}_\lambda = \min_{x,y,w,u,v} p_\lambda(x, w, t) := \sum_{\ell=1}^{n} \lambda_\ell \sum_{i=1}^{n} t_i w_{i\ell} \qquad (\mathrm{MFOMP1}_\lambda)$$

$$\text{s.t. } h_i^1 := \sum_{\ell=1}^{n} w_{i\ell} - 1 = 0, \text{ for } i = 1, \ldots, n, \qquad (21)$$

$$h_\ell^2 := \sum_{i=1}^{n} w_{i\ell} - 1 = 0, \text{ for } \ell = 1, \ldots, n, \qquad (22)$$

$$h_\ell^3 := \sum_{i=1}^{n} w_{i\ell} t_i - \sum_{i=1}^{n} w_{i\ell+1} t_i \geq 0, \ell = 1, \ldots, n-1, \qquad (23)$$

$$h_{ij}^4 := w_{i\ell}^2 - w_{i\ell} = 0, \text{ for } i, \ell = 1, \ldots, n, \qquad (24)$$

$$h_\ell^5 := 1 - \sum_{i=1}^{n} w_{i\ell}^2 \geq 0, \ell = 1, \ldots, n \qquad (25)$$

$$h_{ijk}^6 := v_{ijk}^s - (x_{jk} - a_{ik})^r \geq 0, \; i = 1, \ldots, n, \; j = 1, \ldots, p, \; k = 1, \ldots, d \qquad (26)$$

$$h_{ijk}^7 := v_{ijk}^s - (a_{ik} - x_{jk})^r \geq 0, \; i = 1, \ldots, n, \; j = 1, \ldots, p, \; k = 1, \ldots, d \qquad (27)$$

$$h_{ij}^8 := \left(\sum_{k=1}^{d} v_{ijk}\right)^s - u_{ij}^r \geq 0, \; i = 1, \ldots, n, \; j = 1, \ldots, p \qquad (28)$$

$$h_{ij}^9 := u_{ij} - t_i \geq 0, \; i = 1, \ldots, n, \; j = 1, \ldots, p \qquad (29)$$

$$h_{ij}^{10} := t_i - z_{ij} u_{ij} \geq 0, \; i = 1, \ldots, n, \; j = 1, \ldots, p \qquad (30)$$

$$h_i^{11} := \sum_{j=1}^{p} z_{ij} - 1 = 0, \; i = 1, \ldots, n, \qquad (31)$$

$$h_{ij}^{12} := z_{ij}^2 - z_{ij} \geq 0, \text{ for } i = 1, \ldots, n, j = 1, \ldots, p, \qquad (32)$$

The p-median Euclidean case

$$\min_{x=(x_1,\ldots,x_p)\in\mathbb{R}^{pd}} \sum_{i=1}^{m} w_i \min_{j=1,\ldots,p} \|x_j - a_i\|_2 \quad \equiv \quad \left\{ \begin{array}{ll} \min & \displaystyle\sum_{i=1}^{n} w_i t_i \\ \text{s.t.:} & t_i^2 \geq \displaystyle\sum_{j=1}^{p}\sum_{k=1}^{d} z_{ij}(a_{ik} - x_{jk})^2, \quad i=1,\ldots,n \\ & \displaystyle\sum_{j=1}^{p} z_{ij} = 1, \quad i=1,\ldots,n \\ & \displaystyle\sum_{j=1}^{p}(z_{ij} - z_{ij}^2) \leq 0, \quad i=1,\ldots,n \\ & \displaystyle\sum_{i=1}^{n} t_i^2 + \sum_{i=1}^{p}\sum_{j=1}^{p} z_{ij}^2 + \sum_{j=1}^{p}\sum_{k=1}^{d} x_{jk}^2 \leq M \\ & t_i \geq 0,\ z_{ij} \in [0,1],\ x_j \in K,\ \forall i=1,\ldots,n,\ j=1,\ldots \end{array} \right.$$

Theorem

Let x be a feasible solution of **LOCOMF** *then there exists a solution (x, z, u, v, w, t) for* $\mathrm{MFOMP1}_{\lambda}$ *such that their objective values are equal. Conversely, if (x, z, u, v, w, t) is a feasible solution for* $\mathrm{MFOMP1}_{\lambda}$ *then there exists a solution (x) for* **LOCOMF** *having the same objective value. In particular $\rho_{\lambda} = \overline{\rho}_{\lambda}$. Moreover, if $\mathbf{K} \subset \mathbb{R}^{d}$ satisfies Archimidean's property then $\overline{\mathbf{K}} \subset \mathbb{R}^{pd+np+np+npd+n^2+n}$ also satisfies Archimidean's property.*

The Moment approach

Let $h_0(x, z, u, v, w, t) := p_\lambda(x, w, t)$, and denote $\xi_j := \lceil(\deg g_j)/2\rceil$ and $\nu_j := \lceil(\deg h_j)/2\rceil$, where $\{g_1, \ldots, g_{n_{\mathbf{K}}}\}$, and $\{h_0, h_1, \ldots, h_{nc1}\}$ are, respectively, the polynomial constraints that define $\mathbf{K}$ and $\overline{\mathbf{K}} \setminus \mathbf{K}$ in $\mathrm{MFOMP1}_\lambda$. For $r \geq r_0 := \max\{\max_{k=1,\ldots,nK} \xi_k, \max_{j=0,\ldots,nc} \nu_j\}$, introduce the hierarchy of semidefinite programs:

$$\begin{array}{lll} \min_{\mathbf{y}} & \mathrm{L}_{\mathbf{y}}(p_\lambda) & \\ \text{s.t.} & \mathrm{M}_r(\mathbf{y}) \succeq 0, & \\ & \mathrm{M}_{r-\xi_k}(g_k, \mathbf{y}) \succeq 0, & k = 1, \ldots, n_{\mathbf{K}}, \\ & \mathrm{M}_{r-\nu_j}(h_j, \mathbf{y}) \succeq 0, & j = 1, \ldots, nc1, \\ & y_0 = 1, & \end{array} \qquad (\mathbf{Q1}_r)$$

with optimal value denoted $\inf \mathbf{Q1}_r$ (and $\min \mathbf{Q1}_r$ if the infimum is attained).

The Moment approach

Theorem

Let $\overline{\mathbf{K}} \subset \mathbb{R}^{pd+np+np+npd+n^2+n}$ *(compact) be the feasible domain of Problem* $\mathrm{MFOMP1}_\lambda$*. Let* $\inf \mathbf{Q1}_r$ *be the optimal value of the semidefinite program* $\mathbf{Q1}_r$*. Then, with the notation above:*

(a) $\inf \mathbf{Q1}_r \uparrow \rho_\lambda$ *as* $r \to \infty$*.*

(b) Let $\mathbf{y}^r$ *be an optimal solution of the SDP relaxation* $\mathbf{Q1}_r$*. If*

$$\operatorname{rank} \mathrm{M}_r(\mathbf{y}^r) = \operatorname{rank} \mathrm{M}_{r-r_0}(\mathbf{y}^r) = \varphi$$

then $\min \mathbb{Q}1_r = \rho_\lambda$ *and one may extract* φ *points* $(x_1^*(k), \ldots, x_p^*(k), z^*(k), u^*(k), v^*(k), w^*(k), t^*(k))_{k=1}^{\varphi} \subset \overline{\mathbf{K}}$*, all global minimizers of the* $\mathrm{MFOMP1}_\lambda$ *problem.*

SOC Programming Formulation

$$\hat{\rho}_\lambda = \min \sum_{\ell=1}^{n} \lambda_\ell \theta_\ell \qquad (\mathrm{MFOMP2}_\lambda)$$

$$\begin{aligned}
\text{s.t. } & (28), (29), (38), && && (36)\\
& t_i \leq \theta_\ell + UB_i(1 - w_{i\ell}), && i = 1, \ldots, n,\ \ell = 1, \ldots, n, && (37)\\
& \theta_\ell \geq \theta_{\ell+1}, && \ell = 1, \ldots, n-1, && (38)\\
& u_{ij} \leq t_i + M_i(1 - z_{ij}), && \forall\ i = 1, \ldots, n,\ j = 1, \ldots, p, && (39)\\
& v_{ijk} - x_{jk} + a_{ik} \geq 0, && i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, ..., d, && (40)\\
& v_{ijk} + x_{jk} - a_{ik} \geq 0, && i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, .., d, && (41)\\
& v_{ijk}^r \leq d_{ijk}^s u_{ij}^{r-s}, && i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, \ldots, d, && (42)\\
& \sum_{k=1}^{d} d_{ijk} \leq u_{ij}, && i = 1, \ldots, n,\ j = 1, \ldots, p, && (43)\\
& w_{i\ell} \in \{0, 1\}, \theta_\ell \in \mathbb{R}^+ && \forall\ i, \ell = 1, \ldots, n, && (44)\\
& z_{ij} \in \{0, 1\}, && \forall\ i = 1, \ldots, n, j = 1, \ldots, p, && (45)\\
& t_i \in \mathbb{R}^+,\ v_{ijk}, d_{ijk} \in \mathbb{R}^+, u_{ij} \in \mathbb{R}^+, && i = 1, \ldots, n, j = 1, \ldots, p, k = 1, \ldots, d, && (46)\\
& x_j \in \mathbf{K}, && j = 1, \ldots, p. && (47)
\end{aligned}$$

Theorem

Let x be a feasible solution of **LOCOMF** *then there exists a solution $(x, z, u, v, w, t, \theta, \varsigma, d)$ for $\mathrm{MFOMP2}_\lambda$ such that their objective values are equal. Conversely, if $(x, z, u, v, w, t, \theta, \varsigma, d)$ is a feasible solution for $\mathrm{MFOMP2}_\lambda$ then there exists a solution (x) for* **LOCOMF** *having the same objective value. $\rho_\lambda = \hat{\rho}_\lambda$.*

The p-median Euclidean case

$$\min_{x \in \mathbb{R}^{pd}} \sum_{i=1}^{m} w_i \min_{j=1,\ldots,p} \|x_j - a_i\|_2 \quad \equiv \quad \left\{ \begin{array}{lll} \min & \displaystyle\sum_{i=1}^{n} w_i t_i & \\ \text{s.t.:} & u_{ij}^2 \geq \displaystyle\sum_{k=1}^{d} (a_{ik} - x_{jk})^2, & i = 1, \ldots, n;\ j = 1, \ldots, p, \\ & u_{ij} \geq t_i + M(1 - z_{ij}), & i = 1, \ldots, n;\ j = 1, \ldots, p, \\ & \displaystyle\sum_{j=1}^{p} z_{ij} \geq 1, & i = 1, \ldots, n \\ & z_{ij} \in \{0, 1\}, & i = 1, \ldots, n;\ j = 1, \ldots, p, \\ & t_i \geq 0,\ u_{ij} \geq 0, & i = 1, \ldots, n;\ j = 1, \ldots, p, \\ & x = (x_1, \ldots, x_p) \in \mathbb{R}^{pd}. & \end{array} \right.$$

The SDP-relaxation Approach

Let $\mathbf{y} = (y_\alpha)$ be a real sequence indexed in the monomial basis $(x^\beta z^\eta u^\gamma v^\delta w^\zeta t^\alpha \theta^\varsigma d^\psi)$ of $\mathbb{R}[x, z, u, v, w, t, \theta, d]$ (with $\alpha = (\beta, \eta, \gamma, \delta, \zeta, \alpha, \varsigma, \psi) \in \mathbb{N}^{pd} \times \mathbb{N}^{np} \times \mathbb{N}^{np} \times \mathbb{N}^{npd} \times \mathbb{N}^{n^2} \times \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^{npd}$).
Let $h_0(\theta) := \sum_{\ell=1}^m \lambda_\ell \theta_\ell$, and denote $\xi_j := \lceil (\deg g_j)/2 \rceil$ and $\nu_j := \lceil (\deg h_j)/2 \rceil$, where $\{g_1, \ldots, g_{n_\mathbf{K}}\}$, and $\{h_1, \ldots, h_{nc2}\}$ are, respectively, the polynomial constraints that define $\mathbf{K}$ and $\hat{\mathbf{K}} \setminus \mathbf{K}$ in $\mathrm{MFOMP2}_\lambda$. For $r \geq r_0 := \max\{\max_{k=1,\ldots,n_\mathbf{K}} \xi_k, \max_{j=0,\ldots,nc2} \nu_j\}$, introduce the hierarchy of semidefinite programs:

$$\begin{array}{lll} \min\limits_{\mathbf{y}} & \mathrm{L}_\mathbf{y}(p_\lambda) & \\ \mathrm{s.t.} & \mathrm{M}_r(\mathbf{y}) \succeq 0, & \\ & \mathrm{M}_{r-\xi_k}(g_k, \mathbf{y}) \succeq 0, & k = 1, \ldots, n_\mathbf{K}, \\ & \mathrm{M}_{r-\nu_j}(h_j, \mathbf{y}) \succeq 0, & j = 1, \ldots, nc2, \\ & y_0 = 1, & \end{array} \qquad (\mathbf{Q2}_r)$$

with optimal value denoted $\inf \mathbf{Q2}_r$ (and $\min \mathbf{Q2}_r$ if the infimum is attained).

The SDP-relaxation Approach

Theorem

Let $\hat{\mathbf{K}} \subset \mathbb{R}^{d+np(d+2)+n^2+2n+npd}$ *(compact) be the feasible domain of Problem* $\mathrm{MFOMP2}_\lambda$*. Let* $\inf \mathbf{Q2}_r$ *be the optimal value of the semidefinite program* $\mathbf{Q2}_r$*. Then, with the notation above:*

(a) $\inf \mathbf{Q2}_r \uparrow \rho_\lambda$ *as* $r \to \infty$*.*

(b) Let $\mathbf{y}^r$ *be an optimal solution of the SDP relaxation* $\mathbf{Q2}_r$*. If*

$$\operatorname{rank} \mathrm{M}_r(\mathbf{y}^r) = \operatorname{rank} \mathrm{M}_{r-r_0}(\mathbf{y}^r) = \varphi$$

then $\min \mathbf{Q2}_r = \rho_\lambda$ *and one may extract* φ *points* $(x_1^*(k), \ldots, x_p^*(k), z^*(k), u^*(k), v^*(k), w^*(k), t^*(k), \theta^*(k), d^*(k))_{k=1}^{\varphi} \subset \hat{\mathbf{K}}$*, all global minimizers of the* $\mathrm{MFOMP2}_\lambda$ *problem.*

The SDP-relaxation Approach

Theorem

Let $\hat{\mathbf{K}} \subset \mathbb{R}^{d+np(d+2)+n^2+2n+npd}$ (compact) be the feasible domain of Problem $\mathrm{MFOMP2}_\lambda$. Let $\inf \mathbf{Q2}_r$ be the optimal value of the semidefinite program $\mathbf{Q2}_r$. Then, with the notation above:

(a) $\inf \mathbf{Q2}_r \uparrow \rho_\lambda$ *as* $r \to \infty$.

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$$\operatorname{rank} \mathrm{M}_r(\mathbf{y}^r) = \operatorname{rank} \mathrm{M}_{r-r_0}(\mathbf{y}^r) = \varphi$$

then $\min \mathbf{Q2}_r = \rho_\lambda$ *and one may extract* φ *points* $(x_1^*(k), \ldots, x_p^*(k), z^*(k), u^*(k), v^*(k), w^*(k), t^*(k), \theta^*(k), d^*(k))_{k=1}^{\varphi} \subset \hat{\mathbf{K}}$*, all global minimizers of the* $\mathrm{MFOMP2}_\lambda$ *problem.*

Bottleneck: $N = n^2 + 2np + pd + n + npd$ variables $\Rightarrow$ SDP Matrix Size:

$$\binom{N+r}{r}$$

Reduction (I): Sparsity

Assuming that:

1. There is $M > 0$ such that $\|x\|_2^2 < M$ for all $x \in \mathbf{K}$.
2. The index sets $I = \{1, \ldots, d\}$ and $J = \{1, \ldots, m\}$ are partitioned into sets $\{I_k\}_{k=1}^{\pi}$ and $\{J_k\}_{k=1}^{\pi}$ respectively, satisfying:
   1. $\{J_\ell\}$ are disjoint sets.
   2. For every $j \in J_k$, the constraint $g_j(x) \geq 0$ is only concerned with the variables $X(I_k) = \{x_i : i \in I_k\}$.
   3. The objective function f can be written as $f = \sum_{k=1}^{\pi} f_k$ where $f_k \in \mathbb{R}[X(I_k)]$ for $k = 1, \ldots, \pi$.
   4. For every $k = 1, \ldots, \pi - 1$ $I_{k+1} \cap \bigcup_{j=1}^{k} I_j \subseteq I_s$ for some $s \leq k$.

Reduction (I): Sparsity

Then, we consider the following semidefinite program:

$$
Q_r^{\mathrm{sp}}: \quad \begin{array}{rcl} \multicolumn{3}{l}{\inf_y \mathrm{L}_{\mathbf{y}}(f)} \\ \mathrm{M}_r(\mathbf{y}; I_k) & \succeq & 0, \; k = 1, \ldots, \pi, \\ \mathrm{M}_{r - \lceil \deg g_j/2 \rceil}(g_j \, \mathbf{y}; I_k) & \succeq & 0, \; j \in J_k, \; k = 1, \ldots, \pi, \quad 1 \leq j \leq m \\ y_0 = 1 & & \end{array} \tag{48}
$$

Theorem (Lasserre, 2006)

With the notation above, $\lim_{r \to \infty} \inf Q_r^{\mathrm{sp}} = \min\{f(x) : x \in \mathbf{K}\}$*. Furthermore, if* y^r *is a feasible solution of* Q_r^{sp} *with* $L_{y^r}(f) \leq \inf Q_r^{\mathrm{sp}} + \frac{1}{r}$ *and* $\hat{y}^r = \{y_\alpha^r : \sum_{i=1}^r \alpha_i = 1\}$*, then* $\lim_{r \to \infty} \hat{y}^r = x^*$*, if* $x^* \in \mathbf{K}$ *is the unique global minimizer of the polynomial optimization problem.*

Reduction (I): Sparsity

Let $\tilde{I}(0,0) = I^x \cup I^w \cup I^t$ and $\tilde{I}(j,\ell) = I^x(j) \cup I^z(j) \cup I^u(j) \cup I^v(j) \cup I^w(\ell) \cup I^t$ for all $\ell = 1, \ldots, n-1$, $j = 1, \ldots, p$.

Reduction (I): Sparsity

Let $\tilde{I}(0,0) = I^x \cup I^w \cup I^t$ and $\tilde{I}(j,\ell) = I^x(j) \cup I^z(j) \cup I^u(j) \cup I^v(j) \cup I^w(\ell) \cup I^t$ for all $\ell = 1, \ldots, n-1$, $j = 1, \ldots, p$.
Observe that

$$\tilde{I}(j+1, \ell+1) \cap \bigcup_{j' \leq j, \ell' \leq \ell} \tilde{I}(j,\ell) \subseteq \tilde{I}(0,0), \quad \forall j \geq 0,\ \ell \geq 0. \tag{49}$$

Reduction (I): Sparsity

For $r \geq \max\{r_0, \nu_0\}$ where $r_0 := \max_{k=1,\ldots,\ell} \xi_k$ and $\nu_0 := \max_{j=0,\ldots,12} \nu_\ell^j$,:

$$\begin{array}{lll}
\inf_{\mathbf{y}} & \mathrm{L}_{\mathbf{y}}(\sum_{\ell=1}^n \sum_{i=1}^n \lambda_i(x) t_i w_{i\ell}) & \\
\text{s.t.} & \mathrm{M}_r(\mathbf{y}; \tilde{I}(0,0)) & \succeq 0, \\
& \mathrm{M}_{r-\xi_k}(g_k \mathbf{y}; \tilde{I}(0,0)) & \succeq 0,\ k = 1, \ldots, n_{\mathbf{K}}, \\
& \mathrm{M}_r(\mathbf{y}; \tilde{I}(j,\ell)) & \succeq 0,\ j = 1, \ldots, p,\ \ell = 1, \ldots, n-1, \\
& \mathrm{M}_{r-\nu_\ell 3}(h_\ell^3 \mathbf{y}; \tilde{I}(j,\ell)) & \succeq 0,\ j = 1, \ldots, p,\ \ell = 1, \ldots, n-1, \\
& \mathrm{M}_{r-\nu_\ell^5}(h_\ell^5 \mathbf{y}; \tilde{I}(j,\ell)) & \succeq 0,\ j = 1, \ldots, p,\ \ell = 1, \ldots, n-1, \\
& \mathrm{M}_{r-\nu_{ijk}^6}(h_{ijk}^6 \mathbf{y}; \tilde{I}(j,\ell)) & \succeq 0,\ i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, \ldots, d,\ \ell = 1, \ldots, n-1, \\
& \mathrm{M}_{r-\nu_{ijk}^7}(h_{ijk}^7 \mathbf{y}; \tilde{I}(j,\ell)) & \succeq 0,\ i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, \ldots, d,\ \ell = 1, \ldots, n-1, \\
& \mathrm{M}_{r-\nu_{ij}^8}(h_{ij}^8 \mathbf{y}; \tilde{I}(j,\ell)) & \succeq 0,\ i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, \ldots, d,\ \ell = 1, \ldots, n-1, \\
& \mathrm{M}_{r-\nu_{ij}^9}(h_{ij}^9 \mathbf{y}; \tilde{I}(j,\ell)) & \succeq 0,\ i = 1, \ldots, n,\ j = 1, \ldots, p,\ \ell = 1, \ldots, n-1, \\
& \mathrm{M}_{r-\nu_{ij}^1 0}(h_{ij}^{10} \mathbf{y}; \tilde{I}(j,\ell)) & \succeq 0,\ i = 1, \ldots, n,\ j = 1, \ldots, p, \\
& \mathrm{L}_y(\sum_{i=1}^n w_{i\ell} - 1) & = 0,\ \ell = 1, \ldots, n, \\
& \mathrm{L}_y(\sum_{\ell=1}^n w_{i\ell} - 1) & = 0,\ i = 1, \ldots, n, \\
& \mathrm{L}_y(w_{i\ell}^2 - w_{i\ell}) & = 0,\ i, \ell = 1, \ldots, n, \\
& \mathrm{L}_y(\sum_{j=1}^p z_{ij} - 1) & = 0,\ i, \ell = 1, \ldots, n, \\
& \mathrm{L}_y(z_{ij}^2 - z_{ij}) & = 0,\ i = 1, \ldots, n,\ j = 1, \ldots, p,
\end{array} \qquad (\mathbf{Q1}_r^{\mathrm{sp}})$$

with optimal value denoted inf $\mathbf{Q1}_r^{\mathrm{sp}}$.

Reduction (I): Sparsity

Theorem

Let $\overline{\mathbf{K}} \subset \mathbb{R}^{pd+n^2+np+npd+n^2+n}$ *be the feasible domain of* $\mathrm{MFOMP1}_\lambda$*. Then, with the notation above:*

(a) $\inf \mathbf{Q1}_r^{\mathrm{sp}} \uparrow \rho_\lambda$ *as* $r \to \infty$.

(b) *Let* $\mathbf{y}^r$, *be an optimal solution of the SDP relaxation* $\mathbf{Q1}_r^{\mathrm{sp}}$*. If*

$$\begin{aligned} \operatorname{rank} \mathrm{M}_r(\mathbf{y}^r; I^x) &= \operatorname{rank} \mathrm{M}_{r-r_0}(\mathbf{y}^r; I^x) \\ \operatorname{rank} \mathrm{M}_r(\mathbf{y}^r; \tilde{I}(j,\ell)) &= \operatorname{rank} \mathrm{M}_{r-\nu_0}(\mathbf{y}^r; \tilde{I}(j,\ell)) \; \ell = 1, \ldots, n, j = 1, \ldots, p \end{aligned} \tag{50}$$

and if

$$\operatorname{rank}(\mathrm{M}_r(\mathbf{y}^r; I^x \cup (I^z(j) \cup I^u(j) \cup I^v(j) \cup I^w(\ell) \cup I^t) \cap (I^z(j') \cup I^u(j') \cup I^v(j') \cup I^w(\ell') \cup I^t))) = 1$$

for all $(j,\ell) \neq (j',k')$ *then* $\inf \mathbf{Q1}_r^{\mathrm{sp}} = \rho_\lambda$*.*

Moreover, let $\Delta_{j,\ell} := \{(x^*(j,\ell), z^*(j,\ell), u^*(j,\ell), v^*(j,\ell), w^*(j,\ell)), t^*(j,\ell)\}$ *be the set of solutions obtained by the application of the condition (57). Then, every* $(x^*, z^*, u^*, v^*, w^*, t^*)$ *such that* $(x^*_{jk}, z^*_{ij}, u^*_{ij}, v^*_{ijk}, w^*_{ij}, t^*_i)_{((j,k),(i,j),(i,j),(i,j,k),(i,j),i) \in \tilde{I}(j',k')} = (x^*(j',k'), z^*(j',k'), u^*(j',k'), v^*(j',k'), w^*(j',k'), t^*(j',k'))$ *for some* $\Delta_{j',k'}$ *is an optimal solution of Problem* $\mathbf{MOMRF}_\lambda$*.*

Reduction (II): Symmetry

We will apply the symmetry results when permuting the j-indices in the set of variables $\Upsilon = \{x, z, u, v\}$.

Reduction (II): Symmetry

We will apply the symmetry results when permuting the *j*-indices in the set of variables $\Upsilon = \{x, z, u, v\}$.
We consider the following action φ over $\mathbb{R}^p$:

$$\varphi : \mathcal{S}_p \times \mathbb{R}^p \to \mathbb{R}^p$$

defined as $\varphi(\sigma, (y_1, \ldots, y_p)) = (y_{\sigma(1)}, \ldots, y_{\sigma(p)})$ for any $\sigma \in \mathcal{S}_p$ and $y \in \mathbb{R}^p$.

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$$\varphi : \mathcal{S}_p \times \mathbb{R}^p \to \mathbb{R}^p$$

defined as $\varphi(\sigma, (y_1, \ldots, y_p)) = (y_{\sigma(1)}, \ldots, y_{\sigma(p)})$ for any $\sigma \in \mathcal{S}_p$ and $y \in \mathbb{R}^p$.

$$\varphi_\Upsilon : \mathcal{S}_p \times \mathbb{R}^{Np+M} \to \mathbb{R}^{Np+M}$$

defined such that φ_Υ maps $(\sigma, (x, z, u, v, w, t)$ into $(\varphi(\sigma, x(I^x(1))), \ldots, \varphi(\sigma, v(I^v(i,k))), w(I^w), t(I^t))$, i.e., permuting the indices associated with facilities in the decision variables (the j-index).

Reduction (II): Symmetry

We will apply the symmetry results when permuting the j-indices in the set of variables $\Upsilon = \{x, z, u, v\}$.
We consider the following action φ over $\mathbb{R}^p$:

$$\varphi : \mathcal{S}_p \times \mathbb{R}^p \to \mathbb{R}^p$$

defined as $\varphi(\sigma, (y_1, \ldots, y_p)) = (y_{\sigma(1)}, \ldots, y_{\sigma(p)})$ for any $\sigma \in \mathcal{S}_p$ and $y \in \mathbb{R}^p$.

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$\varphi_\Upsilon = \varphi \oplus \overset{N}{\cdots} \oplus \varphi \oplus 1_M.$

Reduction (II): Symmetry

By Maschke's Theorem (see [**?**, Thm 1.5.3]), every G-module V is a direct sum of irreducible G-submodules of V, i.e.,

$$V \;\cong\; \bigoplus_{i=1}^{s} V_i \;\text{ with irreducible } G\text{-submodules } V_i \,. \tag{51}$$

Each irreducible G-submodule might occur several times in the direct sum.

$$Q_r^{\mathrm{sym}}: \quad \begin{array}{rcll} \inf_y \mathrm{L}^{\mathrm{sym}}(p,\mathbf{y}) & & & \\ \mathrm{M}_r^{\mathrm{sym}}(\mathbf{y}) & \succeq & 0\,, & \\ \mathrm{M}_{r-\lceil \deg g_j/2\rceil}^{\mathrm{sym}}(g_j\,\mathbf{y}) & \succeq & 0\,, & 1 \le j \le m \end{array} \tag{52}$$

with optimal value denoted by $\inf Q_r^{\mathrm{sym}}$.

Theorem ([?])

Assume that the Archimedean Property holds and let $(Q_r^{\mathrm{sym}})_{r\geq r_0}$ be the hierarchy of SDP-relaxations defined in (59). Then $(\inf Q_r^{\mathrm{sym}})_{r\geq r_0}$ is a monotone non-decreasing sequence that converges to ρ^*.

Reduction (II): Symmetry

Lemma

Let $\mathcal{B}_k(Y)$ be a symmetry-adapted basis of $\mathbb{R}[Y_1, \ldots, Y_p]$ of degree at most k and $\mathcal{B}^{st}(X)$ the standard monomial basis of $\mathbb{R}[w(I^w), t(I^t)]$ with degree at most k. Then, the elements of a symmetry-adapted basis of $\mathbb{R}[x, z, u, v, w, t]$ are of the form:

$$b = b^{x_1} \cdots b^{v_{n,d}} \cdot b'$$

where $b^{x_k} \in \mathcal{B}_k(x(I^x(k)))$, $b^{z_i} \in \mathcal{B}_i(z(I^z(i)))$, $b^{u_i} \in \mathcal{B}_k(u(I^u(i)))$, $b^{v_{i,k}} \in \mathcal{B}_k(v(I^v(i,k)))$, for $i = 1, \ldots, n$, $k = 1, \ldots, d$, and $b' \in \mathcal{B}^{st}(X)$ and such that $deg(b) \leq k$.

Reduction (II): Symmetry

Lemma

Let T be a generalized Young tableau with shape $\lambda \vdash p$ and content μ^β. The generalized Specht polynomials $S_{(t_\lambda, T)}$ generate an $\mathcal{S}_p$-submodule of $\mathbb{R}\{Y^\beta\}$ which is isomorphic to the Specth module S^λ.

With the above results, we get the following result which is proven in [**?**].

Theorem

Let $\beta \in \mathbb{N}_0^p$ with $\displaystyle\sum_{i=1}^{p} \beta_i = r$ and shape μ^β. Then:

$$\mathbb{R}\{Y^\beta\} = \bigoplus_{\lambda \trianglerighteq \mu^\beta} \bigoplus_{T \in \mathcal{T}_{\lambda,\mu}} \mathbb{R}\{S_{t_\lambda, T}\}$$

where t_λ denotes the unique λ-tableau with increasing rows and columns and $\mathcal{T}_{\lambda,\mu}$ the set of semistandard generarized Young tableaux of shape λ and content μ.

Example

For $n = 3$ demand points, in the plane ($d = 2$), $p = 2$ facilities to be located and relaxation order $k = 2$:

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First, the components in the symmetry-adapted basis are indexed by the partitions $\lambda \vdash (2)$, thus $\lambda \in \{(2), (1,1)\}$. The β to take into account are $\beta \in \{(0,0), (1,0), (2,0), (1,1)\}$ with shapes μ equal to (2), (1, 1), (1, 1) and (2), respectively. Thus, the semistandard generalized Young tableaux for each of these shapes and contents are:

- $\mu = (2)$: $\boxed{1 \mid 1}$ and $\boxed{1 \mid 2}$
- $\mu = (1,1)$: No semistandard generalized Young tableaux exists in this case.

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- $\mu = (2)$: $\boxed{1 \mid 1}$ and $\boxed{1 \mid 2}$
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Hence, there is only one irreducible component in this case (for $\mu = (2)$), so the symmetry-adapted basis in $\mathbb{R}[Y_1, Y_2]$ is

$$\{1, Y_1 + Y_2, Y_1^2 + Y_2^2, Y_1 Y_2\}$$

We observe that the standard monomial basis for this set of two variables has 6 monomials while this basis has only four elements.