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Banzhaf index for multiple voting systems. An application to the European Union

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Abstract Multi-criteria simple games constitute an extension of the basic framework of voting systems and collective decision-making. The study of power index plays an important role in the theory of multi-criteria simple games. Thus, in this paper, we propose the extended Banzhaf index for these games, as the natural generalization of this index in conventional simple games. This approach allows us to compare various criteria simultaneously. An axiomatic characterization of this power index is established. The Banzhaf index is computed by taking into account the minimal winning coalitions of each class. Since this index depends on the number of ways in which each player can effect a swing, one of the main difficulties for finding this index is that it involves a large number of computations. We propose a combinatorial procedure, based on generating functions, to obtain the Banzhaf index more efficiently for weighted multi-criteria simple games. As an application, the distribution of voting power in the European Union is calculated.

Keywords Multi-criteria simple games · Voting systems · Power indices · Generating functions

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1 Introduction

Voting systems and collective decision-making in which a number of voters are required to collectively accept or reject a given single proposal have been analyzed within the framework of simple games. The two most commonly used power indices in scalar simple games are the Shapley–Shubik power index (Shapley and Shubik 1954) and the Banzhaf power index (Banzhaf 1965) which have frequently been applied to evaluate numerous political situations (Felsenthal and Machover 1998; Grilli di Cortona et al. 1999). While the Shapley–Shubik index considers the order in which a winning coalition can be obtained, the Banzhaf index is the number of swings normalized by the total number of swings. The model of coalition formation assumed has no regard for the order of the players, and therefore each swing receives equal importance.

Frequently, however, it is unreasonable to consider isolated issues. For instance, in many political processes the problem is not to pick one from among a set of alternatives, but to decide how many of a set of motions will be passed. In the literature, we find several attempts to incorporate many possible actions for the players into theoretical game models and voting situations, in order to more accurately model the behavior of players in a real situation. Thus, in Felsenthal and Machover (1997, 1998), Fishburn (1973), Rubinstein (1980) the failure of classic simple games to admit abstention as a distinguished alternative is indicated, and voting systems with abstention are studied in these papers. Other authors consider more than three alternatives in their analysis, and their primary concern is to introduce power indices (Shapley–Shubik index and/or Banzhaf index) for the games they define (Amer et al. 1998; Bolger 1993; Freixas 2005; Freixas and Zwicker 2003; Hsiao and Raghavan 1993). Related work can be found in Pongou et al. (2011) where the notion of influence relation is extended to institutions which allow more than two levels of participation.

Although in these models the players face multiple alternatives, they must cast a single vote in favor of one particular option. Therefore, these settings do not allow voters to simultaneously choose more than one of these alternatives.

The extension of simple games to multi-criteria simple games (Monroy and Fernández 2009) provides a very natural way of modeling decision problems when the decision-makers consider multiple qualitative criteria simultaneously. These games constitute a formal framework to deal with a wide range of qualitative group-decision problems, as well as a generalization of the above-cited models. When modeling voting systems as multi-criteria simple games, a more satisfactory evaluation of the influence on the outcome by the strategic position of each player, is obtained. This analysis provides the importance of each player with respect to the aggregated criterion, and with respect to each criterion. Thus, the decisiveness of players and/or criteria is revealed. Example 4.1 illustrates this fact.

Monotonic multi-criteria simple games are completely defined by the sets of minimal winning coalitions of the positive classes. When there are many of these coalitions and/or a great number of players, it is worth knowing the power of each player in the game. Therefore, one major goal of this paper is to show the importance of the natural generalization of the classic Banzhaf power index to the multi-criteria simple game framework, as well as the good behavior of this extension, in order to provide a more satisfactory evaluation of the real power of each player. Thus, we define the Banzhaf index for multi-criteria simple games by following an axiomatic procedure similar to Dubey and Shapley's axiomatization for simple games (Dubey and Shapley 1979). In addition, we propose procedures based on generating functions to obtain this index more efficiently.

The paper is organized as follows. In Sect. 2 the model together with basic concepts, the canonical representation and the unanimity multi-criteria simple games are introduced.

In Sect. 3 the classic Banzhaf power index is extended to multi-criteria simple games by using an axiomatic approach. When the multi-criteria simple game is given in a weighted representation, the generating function of the extended Banzhaf index is provided in Sect. 4. Section 5 is devoted to conclusions.

2 Multi-criteria simple games

In this section we summarize the model together with definitions and results that can be consulted in Monroy and Fernández (2009, 2011).

Let $N = \{1, 2, \dots, n\}$ be the set of players, where every subset S of N is a coalition, $\mathcal{P}(N)$ is the set of all coalitions and we denote the k qualitative criteria by $C_1, C_2, \dots, C_k$. These criteria are evaluated by the players in each coalition S, thereby yielding *the valuation criteria space* denoted by $\mathcal{C}$.

Definition 2.1 A multi-criteria qualitative game is a weighted hypergraph $(N, \mathcal{E}, \phi)$ where vertices, N, are the players, hyperedges, $\mathcal{E} \subseteq \mathcal{P}(N)$, represent the coalitions and the weight function, $\phi : \mathcal{E} \to \mathcal{C}$, represents the relative significance of coalitions.

Agents in S can value any number of criteria. The mapping function ϕ summarizes the values of the criteria for the agents in S with a value. Coalitions are classified in accordance with different established conditions on the values $\phi(S)$. Thus, a classification of the $2^n - 1$ coalitions in $\mathcal{P}(N)$ is $U = \{U_1, U_2, \dots, U_h\}$ where $\bigcup_{i=1}^{h} U_i = \mathcal{P}(N)$, and each class $U_j \in U$ is the set of the coalitions whose aggregated values verify those established conditions.

Definition 2.2 A multi-criteria simple game is defined by the tuple $(N, \mathcal{E}, \mathcal{C}, \phi, U, r)$ where $(N, \mathcal{E}, \phi)$ is a multi-criteria qualitative game, $U = \{U_1, U_2, \dots, U_h\}$ is a classification on $\mathcal{P}(N)$, and $r : \mathcal{C} \to \mathcal{P}(U)$ is a rule which maps each coalition aggregated value, $\phi(S)$, to a set of classes.

Henceforth, the multi-criteria simple game will be denoted by (N, v) where $v : \mathcal{P}(N) \to \mathcal{P}(U)$ is defined by $v(S) = r \circ \phi(S) = r(\phi(S)) = \{U_l,\ S \in U_l\}$.

Example 2.1 (Simultaneous multiple voting) Consider a multiple vote of k candidates and n voters, where each voter has $p \le k$ votes. All voters can divide their own votes as they please, perhaps giving all votes to one candidate or distributing them among the candidates as they see fit. A candidate $j \in \{1, 2, \dots, k\}$ is chosen if he obtains, at least, q_j votes.

In order to describe this voting process as a multi-criteria simple game, k criteria are considered: C_j, $j \in \{1, 2, \dots, k\}$, where criterion C_j represents the candidate j. The classification of the different coalitions is given by $U = \{U_1, U_2, \dots, U_k, U_{k+1}\}$, where U_j, $j \in \{1, 2, \dots, k\}$, is the set of coalitions for which candidate j is chosen and U_{k+1} is the set of the remaining coalitions.

The votes obtained by the candidate j from the voters in coalition S are denoted by $n_j(S)$, $j = 1, 2, \dots, k$. The function $\phi : \mathcal{P}(N) \to \mathbb{R}^k$ assigns the k-tuple $\phi(S) = (n_1(S), \dots, n_k(S))$ to each coalition S. The function $r : \mathbb{R}^k \to \mathcal{P}(U)$ establishes which of the values $\phi(S)$ verify the different conditions of the classes, that is,

$$r\big(n_1(S),\dots,n_k(S)\big) \ni \begin{cases} U_1 & \text{if } n_1(S) \geq q_1 \\ U_2 & \text{if } n_2(S) \geq q_2 \\ \vdots & \\ U_k & \text{if } n_k(S) \geq q_k \\ U_{k+1} & \text{otherwise.} \end{cases}$$

Therefore, we have defined a multi-criteria simple game, (N, v), where the value function, $v = r \circ \phi$, assigns to each coalition the set of classes to which it belongs.

For instance, consider a particular simultaneous multiple voting game of three candidates ($k = 3$) and three voters ($n = 3$), where each voter has two votes which can be given to one candidate or to two of the candidates. A candidate $j \in \{1, 2, 3\}$ is chosen if he obtains, at least, two votes ($q_j = 2$). Suppose that player 1 gives two votes to candidate 1, player 2 gives one vote each to candidate 1 and candidate 2, and player 3 gives one vote each to candidate 1 and candidate 2. In this situation, $U_1 = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, $U_2 = \{\{2, 3\}, \{1, 2, 3\}\}$, $U_3 = \emptyset$, $U_4 = \{\{2\}, \{3\}\}$. Thus, for coalition $S = \{2, 3\}$, $v(S) = \{U_1, U_2\}$ since the number of votes obtained by candidate 1 and candidate 2 form voters in S is 2.

The special class of monotonic multi-criteria simple games constitute an important tool for the modeling of voting systems.

Definition 2.3 Let (N, v) be a multi-criteria simple game with classification $U = \{U_1, U_2, \dots, U_h\}$.

1. A class $U_i \in U$ is a positive class if $S_1 \in U_i$, and $S_1 \subset S_2 \subset N$, imply $S_2 \in U_i$.
2. A class $U_i \in U$ is a negative class if $S_2 \in U_i$, and $S_1 \subset S_2 \subset N$, imply $S_1 \in U_i$.

Example 2.2 (2.1 continued) We now show that class U_2 in the particular simultaneous multiple voting game is a positive class. Consider $S_1 \in U_2$ and $S_1 \subset S_2$. Since $S_1 \in U_2$, then at least candidate 2 has obtained 2 or more votes from the voters in S_1. Thus, any super-coalition of S_1, S_2, will collect at least the same number of votes for candidate 2 as in S_1, and therefore $S_2 \in U_2$. Class U_4 is a negative class. Consider the coalition $S_2 \in U_4$, then no candidate has obtained 2 votes from voters in S_2. Obviously, any sub-coalition of S_2, S_1, will collect fewer votes than in S_2, and therefore no candidate will obtain more votes from S_1 than from S_2. Hence $S_1 \in U_4$.

From Definition 2.3, the concepts of winning coalitions and losing coalitions in scalar simple games can be extended to the multi-criteria case as follows:

Let (N, v) be a multi-criteria simple game with classification $U = \{U_1, U_2, \dots, U_h\}$.

Definition 2.4 A coalition in a positive class is called a winning coalition for the class. A coalition is an absolute winning coalition when it belongs to all positive classes.

Definition 2.5 A coalition S is a losing coalition for a positive class U_j if $S \notin U_j$. A coalition is a losing coalition for the game if is a losing coalition for all positive classes.

Example 2.3 (2.1 continued) In the particular simultaneous multiple voting game, coalition $S = \{1, 3\}$ is a winning coalition for class U_1, and coalition $S = \{3\}$ is a losing coalition for the game.

In this paper we focus in decisive monotonic multi-criteria simple games since they represent efficient group decision rules without inconsistencies.

Definition 2.6 A multi-criteria simple game (N, v) with classification $U = \{U_1, U_2, \dots, U_h\}$ is monotonic if

$$v(S) \subseteq v(T), \quad \text{if } S \subset T,\ S \in U_i, \text{ and } U_i \text{ is a positive class.}$$
$$v(T) \supseteq v(S), \quad \text{if } T \subset S,\ S \in U_i, \text{ and } U_i \text{ is a negative class.}$$

The following result characterizes monotonic multi-criteria simple games.

Theorem 2.1 (Monroy and Fernández 2009) *A multi-criteria simple game* (N, v) *with classification* $U = \{U_1, U_2, \dots, U_h\}$ *is monotonic if and only if the value of each coalition,* $v(S)$, *is given by either only positive classes or by only negative classes.*

Example 2.4 (2.1 continued) Consider the particular simultaneous multiple voting game whose classification has two positive classes U_1, U_2, and one negative class, U_4. (Since there are no coalitions in U_3, then it makes no sense to analyze U_3.) From the conditions on the criteria which define each class, it follows that $U_i \cap U_4 = \emptyset$, $i = 1, 2$. Therefore, for each coalition $S \in P(N)$, $v(S)$ is given by positive classes or by the negative class. Thus, simultaneous multiple voting is a monotonic multi-criteria simple game.

The result in Theorem 2.1, together with the types of problems that will be analyzed in this framework, suggest the possibility of considering monotonic multi-criteria simple games where all the classes of the classification have the same nature except one class, which has the opposite nature and contains all the coalitions which do not belong to any of the former classes. This class is called the residual class, denoted by $\mathcal{R}$, and will be omitted when no needed. Consider a multi-criteria simple game (N, v) with classification $U = \{U_1, \dots, U_h, \mathcal{R}\}$.

Definition 2.7 A coalition $S \in U_i$ such that $N \backslash S \in \mathcal{R}$, is a decisive coalition for the class U_i.

Definition 2.8 A coalition $S \in \mathcal{R}$ such that $N \backslash S \in U_i$, is a strictly losing coalition.

Definition 2.9 A multi-criteria simple game is a decisive multi-criteria simple game if all its winning coalitions are decisive coalitions, and all its losing coalitions are strictly losing coalitions.

In a monotonic multi-criteria simple game, the notion of minimal winning coalition is established as follows:

Definition 2.10 A coalition S is a minimal winning coalition for the positive class U_j if $S \in U_j$ and for each sub-coalition S' of S, $S' \notin U_j$. A coalition S is an absolute minimal winning coalition if S is a minimal winning coalition for U_i, $\forall i$.

Definition 2.11 A coalition S is a maximal losing coalition for the positive class U_i if $S \notin U_j$ and for each super-coalition S' of S, $S' \in U_j$.

Example 2.5 (2.1 continued) In the particular simultaneous multiple voting game, coalition $S = \{2, 3\}$ is a minimal winning coalition for class U_1 since $S \in U_1$, but no sub-coalition, $S' = \{2\}$, $S' = \{3\}$, belongs to U_1. Note that $S = \{2, 3\}$ is also a minimal winning coalition for class U_2, and therefore $S = \{2, 3\}$ is an absolute minimal winning coalition.

Coalition $S = \{3\}$ is a maximal losing coalition for class U_1, since $S = \{3\} \notin U_1$, and $\{1, 3\}, \{2, 3\}, \{1, 2, 3\} \in U_1$.

2.1 Canonical representation of a multi-criteria simple game

Given a multi-criteria simple game (N, v) with classification $U = \{U_1, U_2, \dots, U_h\}$, a canonical representation of this game is given by the hypergraph (N, ε, ν), where $\nu : \mathcal{P}(N) \to \{0, 1\}^h$ with

$$\nu(S) = \big(\nu_i(S)\big)_{i=1,\dots,h} \quad \text{and} \quad \begin{cases} \nu_i(S) = 1 & \text{if } S \in U_i \\ \nu_i(S) = 0 & \text{if } S \notin U_i. \end{cases}$$

Henceforth, the canonical representation of the multi-criteria simple game (N, v) will be denoted by (N, ν).

Example 2.6 (2.1, continued) The canonical representation for the particular simultaneous multiple voting game is (N, ν) where $\nu : \mathcal{P}(N) \to \{0, 1\}^4$, and therefore $\nu(S) \in \{0, 1\}^4$. Thus, for coalition $S = \{2, 3\}$, since $v(S) = \{U_1, U_2\}$, it follows that $\nu(S) = (1, 1, 0, 0)$.

The canonical representation (N, ν) of a multi-criteria simple game, (N, v) with classification $U = \{U_1, U_2, \dots, U_h\}$, induces h component scalar simple games (N, ν_i), $i = 1, \dots, h$, defined by $\nu_i : \mathcal{P}(N) \to \{0, 1\}$ such that $\nu_i(S) = 1$ or 0. If $\nu_i(S) = 1$ then $S \in U_i$ and S is a winning coalition in the scalar game (N, ν_i). If $\nu_i(S) = 0$ then $S \notin U_i$ and S is a losing coalition in the scalar game (N, ν_i).

Definition 2.12 The canonical representation (N, ν) of a multi-criteria simple game, (N, v), with classification $U = \{U_1, U_2, \dots, U_h\}$, is monotonic if[1]

$$\nu(S) \le \nu(T), \quad \text{if } S \subset T,\ S \in U_i, \text{ and } U_i \text{ is a positive class.}$$
$$\nu(T) \ge \nu(S), \quad \text{if } T \subset S,\ S \in U_i, \text{ and } U_i \text{ is a negative class.}$$

Proposition 2.1 *The canonical representation (N, ν) of a multi-criteria simple game (N, v) is monotonic if and only if the multi-criteria simple game is monotonic.*

When the canonical representation of a monotonic multi-criteria simple game is considered, the notion of minimal winning coalition is established as follows:

Definition 2.13 A coalition S is a minimal winning coalition for the positive class U_i if $\nu_i(S) = 1$ and $\nu_i(S') = 0$ for each sub-coalition S' of S.

$\mathcal{SGC}_N^h$ denotes the set of multi-criteria monotonic simple games in canonical representation with the set of players N and whose classifications have the same number of positive classes, h, and only one residual class, $\mathcal{R}$. In this case, if $\nu(S) = (0, \dots, 0)$ then $S \in \mathcal{R}$. The set $\mathcal{SGC}_N^h$ is a distributive lattice.

[1] For $x, y \in R^k$ we denote $x \ge y \Leftrightarrow x_i \ge y_i, x \ne y$.

2.2 Unanimity multi-criteria simple games

Let $N=\{1,2,\dots,n\}$ be the set of players, and consider k qualitative criteria $C_1, C_2, \dots, C_k$. Let $U=\{U_1, U_2, \dots, U_h, \mathcal{R}\}$ be a classification in $\mathcal{P}(N)$ given by conditions established on the criteria.

For any class $U_q \in U$ and for any coalition $S \subseteq N$, the unanimity multi-criteria simple game $(N, u_{(S,q)})$, where $u_{(S,q)} : \mathcal{P}(N) \to \mathcal{P}(U)$, is defined by

$$u_{(S,q)}(T) = \begin{cases} U_q & \text{if } S \subseteq T \\ \mathcal{R} & \text{otherwise.} \end{cases}$$

There are $(2^n - 1) \times h$ unanimity multi-criteria simple games.

The canonical representation of the unanimity multi-criteria simple game $(N, u_{(S,q)})$, denoted by $(N, \nu_{(S,q)})$, is given by

$$\nu_{(S,q)}(T) = \begin{cases} (0,\dots,1,\dots,0) & \text{if } S \subseteq T \\ (0,\dots,0,\dots,0) & \text{otherwise} \end{cases}$$

where number 1 is in position q.

Note that the canonical representation of a unanimity multi-criteria simple game, whose classification has h classes, is equivalent to a unanimity multi-criteria game with h criteria.

Let $(N,\nu) \in \mathcal{SGC}_N^h$ be the canonical representation of a monotonic multi-criteria simple game with classification $U=\{U_1, U_2, \dots, U_h\}$. Denote by $W_m^q = \{S_1^q, \dots, S_{k_q}^q\}$ the set of minimal winning coalitions for the class U_q, $q \in \{1,\dots,h\}$.

Proposition 2.2 *Each component ν_q, $q \in \{1,\dots,h\}$, of the canonical representation ν can be expressed as $\nu_q = \nu^q_{(S_1,q)} \vee \nu^q_{(S_2,q)} \vee \cdots \vee \nu^q_{(S_{k_q},q)}$.*

3 The Banzhaf index for multi-criteria simple games

In this section we propose a natural extension of the Banzhaf power index of scalar simple games (Banzhaf 1965), when the canonical representation of a monotonic multi-criteria simple game is considered. We introduce definitions and results, which are needed to provide an axiomatic characterization of the index, and we state the theorem which proves its existence and uniqueness.

Definition 3.1 A power index in $\mathcal{SGC}_N^h$ is a map $\phi : \mathcal{SGC}_N^h \to \mathbb{R}^{h\times n}$ such that

$$\Phi(\nu) = \begin{pmatrix} \Phi_1^1(\nu) & \cdots & \Phi_n^1(\nu) \\ \Phi_1^2(\nu) & \cdots & \Phi_n^2(\nu) \\ \vdots & \vdots & \vdots \\ \Phi_1^h(\nu) & \cdots & \Phi_n^h(\nu) \end{pmatrix} = \big(\Phi_1(\nu), \dots, \Phi_n(\nu)\big).$$

The power (or the value) of player i in the class U_j is $\Phi_i^j(\nu)$. Thus, the column $\Phi_i(\nu) \in \mathbb{R}^h$ is the power of player i in each class, and the row $\Phi^j(\nu) \in \mathbb{R}^n$ represents the power of each player in the class U_j.

Consider a monotonic multi-criteria simple game, (N, v) with classification $U = \{U_1, U_2, \dots, U_h\}$ and canonical representation (N,ν).

Definition 3.2 Let π be any permutation of the set N. The canonical representation $(N, \pi\nu)$ is given by $\pi\nu(\pi S) = \nu(S)$, where $\pi S = \{\pi(i_1), \ldots, \pi(i_S)\}$, for all $S = \{i_1, \ldots, i_S\} \subseteq N$.

Definition 3.3 A player $i \in N$ is a *dummy for the class* $U_j \in U$, if for all coalition S in U_j, $S \cup \{i\}$ is in the class U_j. A dummy player for all the classes is called a *dummy*.

The key element in the construction of the Banzhaf index, sometimes known as the Banzhaf value, is a swing.

Definition 3.4 A swing for player $i \in N$ in the class $U_j \in U$ is a pair of sets $(S, S \setminus \{i\})^j$, such that S is a winning coalition for the class U_j and $S \setminus \{i\}$ is not.

For each $i \in N$, we denote $\eta_{ij}(\nu)$ as the number of swings for $\{i\}$ in the class U_j and $\bar{\eta}_j(\nu)$ as the total number of swings of the class U_j, $\bar{\eta}_j(\nu) = \sum_{i \in N} \eta_{ij}(\nu)$.

We denote the swing-vector of player $\{i\}$ by $\eta_i(\nu)$, the swing-matrix of the players by $\eta(\nu)$, and the total swing-vector by $\bar{\eta}(\nu)$:

$$\eta_i(\nu) = \begin{pmatrix} \eta_{i1}(\nu) \\ \eta_{i2}(\nu) \\ \cdots \\ \eta_{ih}(\nu) \end{pmatrix}, \qquad \eta(\nu) = \begin{pmatrix} \eta_{11}(\nu) & \eta_{21}(\nu) & \cdots & \eta_{n1}(\nu) \\ \eta_{12}(\nu) & \eta_{22}(\nu) & \cdots & \eta_{n2}(\nu) \\ \cdots & & & \\ \eta_{1h}(\nu) & \eta_{2h}(\nu) & \cdots & \eta_{nh}(\nu) \end{pmatrix}$$

$$\bar{\eta}(\nu) = \begin{pmatrix} \sum_{i \in N} \eta_{i1}(\nu) \\ \sum_{i \in N} \eta_{i2}(\nu) \\ \cdots \\ \sum_{i \in N} \eta_{ih}(\nu) \end{pmatrix} = \sum_{i \in N} \begin{pmatrix} \eta_{i1}(\nu) \\ \eta_{i2}(\nu) \\ \cdots \\ \eta_{ih}(\nu) \end{pmatrix} = \sum_{i \in N} \eta_i(\nu).$$

When the number of swings of a player $\{i\}$ in a class U_j is zero, $\eta_{ij}(\nu) = 0$, then the player $\{i\}$ is a *dummy for the class*, since this player is never needed to help a coalition win.

However, if the number of swings of a player $\{i\}$ in a class U_j coincides with the total number of swings of the class U_j, $\eta_{ij}(\nu) = \bar{\eta}_j(\nu)$, then player $\{i\}$ is a dictator for the class U_j. A player can be a dictator for several classes. If a player is a dictator for all the classes then the player is an absolute dictator.

The swing vectors, $\eta_i(\nu)$ $\forall i \in N$, are the vector indices of Banzhaf and they assign the sum of each player's marginal contributions towards the coalitions to which the player belongs. Therefore:

$$\eta_i(\nu) = \begin{pmatrix} \sum_{S: i \in S \subset N} [\nu_1(S) - \nu_1(S \backslash \{i\})] \\ \sum_{S: i \in S \subset N} [\nu_2(S) - \nu_2(S \backslash \{i\})] \\ \cdots \\ \sum_{S: i \in S \subset N} [\nu_h(S) - \nu_h(S \backslash \{i\})] \end{pmatrix}.$$

Since the interpretation of the ratios of these vectors is of more interest than their magnitudes, it is common practice to normalize these vectors:

$$\beta_i(\nu) = \left(\frac{\eta_{i1}(\nu)}{\bar{\eta}_1(\nu)}, \frac{\eta_{i2}(\nu)}{\bar{\eta}_2(\nu)}, \ldots, \frac{\eta_{ih}(\nu)}{\bar{\eta}_h(\nu)} \right)^t \quad i = 1, \ldots, n$$

which provide a matrix with h rows:

$$\beta(\nu) = \big(\beta_1(\nu), \beta_2(\nu), \ldots, \beta_n(\nu)\big) = \left(\frac{\eta_{1j}(\nu)}{\bar{\eta}_j(\nu)}, \frac{\eta_{2j}(\nu)}{\bar{\eta}_j(\nu)}, \ldots, \frac{\eta_{nj}(\nu)}{\bar{\eta}_j(\nu)} \right) \quad j = 1, \ldots, h.$$

Definition 3.5 The normalized extended index of Banzhaf of the game (N, v), with canonical representation (N, ν), is

$$\beta(\nu) = \big(\beta_1(\nu), \beta_2(\nu), \ldots, \beta_n(\nu)\big) = \begin{pmatrix} \frac{\eta_{11}(\nu)}{\bar{\eta}_1(\nu)} & \frac{\eta_{21}(\nu)}{\bar{\eta}_1(\nu)} & \cdots & \frac{\eta_{n1}(\nu)}{\bar{\eta}_1(\nu)} \\ \frac{\eta_{12}(\nu)}{\bar{\eta}_2(\nu)} & \frac{\eta_{22}(\nu)}{\bar{\eta}_2(\nu)} & \cdots & \frac{\eta_{n2}(\nu)}{\bar{\eta}_2(\nu)} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\eta_{1h}(\nu)}{\bar{\eta}_h(\nu)} & \frac{\eta_{2h}(\nu)}{\bar{\eta}_h(\nu)} & \cdots & \frac{\eta_{nh}(\nu)}{\bar{\eta}_h(\nu)} \end{pmatrix}.$$

Another, more natural, normalization is obtained by taking into account the probability that a player is a swinger:

$$\big(\beta'_1(\nu), \beta'_2(\nu), \ldots, \beta'_n(\nu)\big) = \frac{1}{2^{n-1}} \begin{pmatrix} \eta_{11}(\nu) & \eta_{21}(\nu) & \cdots & \eta_{n1}(\nu) \\ \eta_{12}(\nu) & \eta_{22}(\nu) & \cdots & \eta_{n2}(\nu) \\ \cdots & \cdots & \cdots & \cdots \\ \eta_{1h}(\nu) & \eta_{2h}(\nu) & \cdots & \eta_{nh}(\nu) \end{pmatrix}$$

which results from the following probabilistic model. Suppose that each player tosses a coin to decide whether to vote yes or no about r issues. The set of "yes" votes, S, is then an r-dimensional random variable which gives a probability of $\frac{1}{2^{n-1}}$ for each subset of N in each of the r classes. If S is a winning coalition then the r issues are passed. For each player $\{i\}$ and for each class U_j, the winning coalitions correspond with the number of swings $(S \cup \{i\}, S \setminus \{i\})^j$. That is, the probability that a player is a swinger is $(\beta'_1(\nu), \beta'_2(\nu), \ldots, \beta'_n(\nu))$.

The following theorem characterizes the extended Banzhaf index for multi-criteria simple games.

Theorem 3.1 *There exists a unique map* $\Phi : \mathcal{SGC}_N^h \to \mathbb{R}^{h\times n}$ *satisfying the following four axioms*:

A1: *If* i *is a dummy then* $\Phi_i(\nu) = (0, \ldots, 0)^t$.
A2: $\sum_{i\in N} \Phi_i(\nu) = \bar{\eta}(\nu)$.
A3: *For any permutation* π *of* N, $\Phi_{\pi(i)}(\pi\nu) = \Phi_i(\nu)$.
A4: *For any* (N, ν_1) *and* (N, ν_2) *in* $\mathcal{SGC}_N^h$, $\Phi(\nu_1 \vee \nu_2) + \Phi(\nu_1 \wedge \nu_2) = \Phi(\nu_1) + \Phi(\nu_2)$.

Moreover, $\Phi(\nu) = \eta(\nu)$, *for all* (N, ν) *in* $\mathcal{SGC}_N^h$.

Proof For any class $U_q \in U$ and for any coalition $S \subseteq N$, we have defined $(2^n - 1) \times h$ unanimity multi-criteria simple games, $(N, u_{(S,q)})$, whose canonical representation, $(N, \nu_{(S,q)})$, is given by

$$\nu_{(S,q)}(T) = \begin{cases} (0, \ldots, 1, \ldots, 0) & \text{if } S \subseteq T \\ (0, \ldots, 0, \ldots, 0) & \text{otherwise} \end{cases}$$

where number 1 is in position q.

Each player i in $N \setminus S$ is a dummy in $\nu_{(S,q)}$ for all classes U_q, $q \in \{1, \ldots, h\}$, then, from A1, $\Phi_i(\nu_{(S,q)}) = (0, \ldots, 0)^t$. In addition, if π is the permutation that interchanges i and k (for any $i \in S$ and $k \in S$) and leaves the other players fixed, then $\pi(\nu_{(S,q)}) = \nu_{(S,q)}$ and thus, from A3, $\Phi_i(\nu_{(S,q)}) = \Phi_k(\nu_{(S,q)})$. Therefore, from A2,

$$\sum_{i\in N} \Phi_i(\nu_{(S,q)}) = |S| \cdot \Phi_i(\nu_{(S,q)}) = \big(0, \ldots, \bar{\eta}(\nu_{(S,q)}), \ldots, 0\big)$$

and $\Phi_i(\nu_{(S,q)})$ is uniquely determined, if Φ exists, and is given by

$$\Phi_i(\nu_{(S,q)}) = \begin{cases} (0,\dots,\frac{\bar{\eta}(\nu_{(S,q)})}{|S|},\dots,0) & \text{if } i \in S \\ (0,\dots,0,\dots,0) & \text{if } i \notin S \end{cases}$$

where

$$\frac{\bar{\eta}(\nu_{(S,q)})}{|S|} = 2^{|N-S|}.$$

In order to prove the uniqueness of the map Φ, an induction on the number of minimal winning coalitions and on the cardinal of those coalitions is performed.

Consider $(N,\nu) \in \mathcal{SGC}_N^h$ with classification $U = \{U_1, U_2, \dots, U_h\}$. Denote by $W_m^q = \{S_1^q, \dots, S_{k_q}^q\}$ the set of minimal winning coalitions for the class U_q, $q \in \{1,\dots,h\}$. From Proposition 2.2, each component ν_q, $q \in \{1,\dots,h\}$, of ν can be expressed as $\nu_q = \nu^q_{(S_1,q)} \vee \nu^q_{(S_2,q)} \vee \cdots \vee \nu^q_{(S_{k_q},q)}$.

Suppose, by induction hypothesis, that $\Phi(\nu_{(S,q)})$ is uniquely determined if the number of minimal winning coalitions is lower than k_q. It remains to be proved that $\Phi(\nu_{(S,q)})$ is also uniquely determined if the number of minimal winning coalitions is k_q.

If ν is not a unanimity game, then for each component ν_q, $q \in \{1,\dots,h\}$, $k_q > 1$, and ν_q can be expressed as $\nu_q = \nu'_q \vee \nu''_q$, where ν'_q and ν''_q have fewer minimal winning coalitions for the class U_q than ν_q. For instance, consider $\nu'_q = \nu^q_{(S_1,q)}$ and $\nu''_q = \nu^q_{(S_2,q)} \vee \cdots \vee \nu^q_{(S_{k_q},q)}$. Analogously, $\nu'_q \wedge \nu''_q$ has even fewer minimal winning coalitions for the class U_q. Thus, applying the induction hypothesis yields that $\Phi(\nu'_q \wedge \nu''_q)$, $\Phi(\nu'_q)$, and $\Phi(\nu''_q)$ are uniquely determined, and by using A4:

$$\Phi(\nu_q) = \Phi(\nu'_q \vee \nu''_q) = \Phi(\nu'_q) + \Phi(\nu''_q) - \Phi(\nu'_q \wedge \nu''_q).$$

Hence, $\Phi(\nu_q)$ is uniquely determined and therefore, $\Phi(\nu)$ is also uniquely determined.

In order to prove the existence of Φ, note that the proof of uniqueness implicitly contains a recursive construction of Φ thereby establishing existence. However, it is simpler to prove directly that the function η satisfies the axioms A1–A4. In fact, A1–A3 are obvious.

The expression of $\eta_i(\nu)$ shows that $\eta(\nu)$ can be extended to a linear map in $\mathcal{SGC}_N^h$, and since $\nu + \nu' = (\nu \vee \nu') + (\nu \wedge \nu')$, for $(N,\nu), (N,\nu') \in \mathcal{SGC}_N^h$, then A4 is satisfied. □

Example 3.1 (2.1 continued) Consider the particular simultaneous multiple voting in Example 2.1. In this game, due to the symmetry of the players, if the Banzhaf index is computed *ex-ante*, then the three players have the same power in each class, which is 1/3. However, the real power of the players is obtained *ex-post*, since it depends on the votes given to the candidates. Thus, for classes $U_1 = \{\{1\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$, $U_2 = \{\{2,3\},\{1,2,3\}\}$, $U_3 = \emptyset$, $U_4 = \{\{2\},\{3\}\}$, the extended Banzhaf index for this game becomes:

$$\beta(\nu) = \big(\beta_1(\nu), \beta_2(\nu), \beta_3(\nu)\big) = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Since there are no coalitions in U_3, then it makes no sense to consider the Banzhaf index of the players in this class.

Nevertheless, a player can analyze his power *ex-ante* taking into account his choices and the other players' choices, where each of these latter choices has the same probability.

The Banzhaf index has been computed by taking into account the minimal winning coalitions of each class. If the multi-criteria simple game is given in a weighted representation, then it is possible to apply this approach, since the minimal winning coalitions can be obtained from this representation. However, in this situation, generating functions, by using techniques of combinatorial analysis, become efficient tools which facilitate the computation of this index.

4 Weighted multi-criteria simple games and generating functions

Weighted systems constitute an alternative way of representing multi-criteria simple games. The notion of weighted representation for this type of game was introduced in Monroy and Fernández (2007). In that paper, it is shown that any multi-criteria simple game admits two different types of weighted representations. In the first one, each class of the multi-criteria game is defined by a family of quotas, and in the second one each class of the classification is given by only one quota. Since the generating function method was used by Brams and Affuso (1976) for computing the Banzhaf index in weighted voting games, in the literature can be found several procedures, based on generating functions, which allow the computation of different power indices (Alonso-Mejide and Bowles 2005; Fernández et al. 2002). In this section we provide the generating function of the Banzhaf index in weighted multi-criteria simple games, which eases the computation of this index. We first introduce some definitions and results.

Consider a multi-criteria simple game (N, v) with k criteria and classification $U = \{U_1, U_2, \ldots, U_h\}$.

Definition 4.1 A weighted representation of (N, v) is $[\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_h | \overrightarrow{w}_1, \overrightarrow{w}_2, \ldots, \overrightarrow{w}_n]$ where $\mathcal{Q}_j$ is the family of quotas which defines the class U_j, $j \in \{1, 2, \ldots, h\}$ and $\overrightarrow{w}_i \in R^k$, $i \in \{1, 2, \ldots, n\}$, is a set of weight vectors.

Definition 4.2 A weighted representation of a multi-criteria simple game is a canonical weighted representation of the game if each class of the classification is defined by only one quota vector.

Theorem 4.1 (Monroy and Fernández 2007) *Any multi-criteria simple game admits a canonical weighted representation and its dimension is upper-bounded by the cardinal number of the family of the maximal losing coalitions of the game.*

The proof of this theorem provides a procedure to obtain this representation by constructing a weighted majority game for each maximal losing coalition L_k^j of the class U_j, such that L_k^j is a losing coalition and all the other coalitions are winning coalitions. If p_j is the number of maximal losing coalitions of the class U_j, $\forall j = 1, \ldots, h$, then the canonical weighted representation of order p, $p = \sum_{j=1}^{h} p_j$, is

$$\left[\begin{array}{c|c|c|c||cccc}
q_1^1 & 0 & \cdots & 0 & w_{11}^1 & w_{12}^1 & \cdots & w_{1n}^1 \\
q_2^1 & 0 & \cdots & 0 & w_{21} & w_{22} & \cdots & w_{2n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
q_{p_1}^1 & 0 & \cdots & 0 & w_{p_1 1}^1 & w_{p_1 2}^1 & \cdots & w_{p_1 n}^1 \\
0 & q_1^2 & \cdots & 0 & w_{11}^2 & w_{12}^2 & \cdots & w_{1n}^2 \\
0 & q_2^2 & \cdots & 0 & w_{21}^2 & w_{22}^2 & \cdots & w_{2n}^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & q_{p_2}^2 & \cdots & 0 & w_{p_2 1}^2 & w_{p_2 2}^2 & \cdots & w_{p_2 n}^2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & q_1^h & w_{11}^h & w_{12}^h & \cdots & w_{1n}^h \\
0 & 0 & \cdots & q_2^h & w_{21}^h & w_{22}^h & \cdots & w_{2n}^h \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & q_{p_h}^h & w_{p_h 1}^h & w_{p_h 2}^h & \cdots & w_{p_h n}^h
\end{array}\right].$$

In this weighted representation, each class U_j is given by a vector-weighted system, which can be considered as the intersection of p_j weighted majority games:

$$v_1^j = [q_1^j; w_{11}^j, w_{12}^j, \dots, w_{1n}^j], \quad v_2^j = [q_2^j; w_{21}^j, w_{22}^j, \dots, w_{2n}^j], \quad \dots,$$
$$v_{p_j}^j = [q_{p_j}^j; w_{p_j 1}^j, w_{p_j 2}^j, \dots, w_{p_j n}^j].$$

In the literature (see Algaba et al. 2003), this intersection game is considered as a weighted multiple-majority game, denoted by $v_1^j \wedge \cdots \wedge v_{p_j}^j$, where

$$\left(v_1^j \wedge \cdots \wedge v_{p_j}^j\right)(S) = \begin{cases} 1 & \text{if } w_t^j(S) \geq q_t^j, \ 1 \leq t \leq p_j \\ 0 & \text{otherwise} \end{cases}$$

with $w_t^j(S) = \sum_{i \in S} w_{ti}^j$.

Next, we consider generating functions which are particularly useful for solving counting problems.

Definition 4.3 A generating function of the sequence $\{a_0, a_1, a_2, \dots,\}$ is a formal power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$.

Since the Banzhaf index of a player depends on the number of swings for the player, in Brams and Affuso (1976) generating functions method is applied to obtain the normalized Banzhaf index. Thus, in Brams and Affuso (1976) is established that, for a weighted voting game $v = [q; w_1, \dots, w_n]$, the number of swings for player $i \in N$ satisfies $\eta_i(v) = \sum_{k=q-w_i}^{q-1} b_k^i$, where b_k^i is the number of coalitions $S \subseteq N$ such that $i \notin S$ and weight $w(S) = k$.

Proposition 4.1 (Brams and Affuso 1976) *Let (N, v) be a weighted voting game given by $v = [q; w_1, \dots, w_n]$. For a player $i \in N$, the generating functions of numbers $\{b_k^i\}$ are given by*:

$$B_i(x) = \prod_{j=1, j \neq i}^{n} \left(1 + x^{w_j}\right).$$

We now present generating functions for the computation of the Banzhaf power index in weighted multi-criteria simple games.

Let (N, v) be the canonical representation of the multi-criteria simple game, (N, v), with classification $U = \{U_1, \dots, U_h, \mathcal{R}\}$ and consider a canonical weighted representation, where all the weights and the quotas are positive integers. In Algaba et al. (2003) generating functions for the computation of the Banzhaf index for weighted multiple-majority games are provided. We apply these results to each class in the classification U in order to obtain the Banzhaf index for a multi-criteria simple game.

Proposition 4.2 *Let $v_j = v_1^j \wedge \cdots \wedge v_{p_j}^j$ be a weighted p_j-majority game for the class U_j. For each $i \in N$,*

1. *The number of swings of player i in the class U_j is given by*:
$$\eta_{ij}(v) = \sum_{\substack{k_t^j = q_t^j - w_{ti}^j \\ 1 \le t \le p_j}}^{w_t^j(N \setminus i)} b^i_{k_1^j \dots k_{p_j}^j} - \sum_{\substack{k_t^j = q_t^j \\ 1 \le t \le p_j}}^{w_t^j(N \setminus i)} b^i_{k_1^j \dots k_{p_j}^j}$$
where $b_{k_1^j \dots k_{p_j}^j}$ is the number of coalitions in U_j such that $i \notin S$ and $w_t^j(S) = k_t^j$ for all $1 \le t \le p_j$.
2. *The generating function for numbers $\{b^i_{k_1^j \dots k_{p_j}^j}\}_{k_1^j \dots k_{p_j}^j \ge 0}$, are given by*
$$B_i^j(x_1, \dots, x_{p_j}) = \prod_{r=1, r \ne i}^{n} \left(1 + x_1^{w_{1r}^j} \dots x_{p_j}^{w_{p_j r}^j}\right).$$

Note that for $p_j = 1$ this result coincides with that of scalar weighted voting games.

Example 4.1 (Algaba et al. 2003) A classic application in the literature of power indices is to the European Union. The Council of Ministers of the EU represents the national governments of the member states. The council uses a voting system of a qualified majority to pass new legislation. The Nice European Council in December 2000 established the decision rule for the EU, whose members have been augmented to 27 countries.

The voting rule prescribed by the Treaty of Nice indicates that the result will be favorable if it counts on the support of 2/3 of the countries, that is 18 countries, with at least 255 votes, and with at least 62 % of the population. In order to analyze this problem as a multi-criteria simple game, we consider the set of players N, given by the 27 countries, and three criteria. Criterion C_1 represents votes of countries, criterion C_2 represents countries, and criterion C_3 represents the population of each country as a percentage of the total EU population, multiplied by 1000. The classification of the different coalitions is given by $U = \{U_1, U_2, U_3, \mathcal{R}\}$, where

1. U_1 is the set of coalitions S such that the total number of votes given by players in S is at least 255.
2. U_2 is the set of coalitions S such that the number of players in S is at least 18.
3. U_3 is the set of coalitions S such that the sum of the percentages of the population of each country in S is at least 62 %.
4. R is the set of the remaining coalitions.

For each coalition $S \in \mathcal{P}(N)$ we denote $\phi_1(S)$ as the total votes of players in S, $\phi_2(S)$ as the cardinal of S, and $\phi_3(S)$ as the sum of the percentages of the population of each country in S. Thus,

$$v(S)=r\big(\{\phi_1(S),\phi_2(S),\phi_3(S)\}\big)\ni\begin{cases}U_1 & \text{if } \phi_1(S)\geq 255\\ U_2 & \text{if } \phi_2(S)\geq 18\\ U_3 & \text{if } \phi_3(S)\geq 620\\ \mathcal{R} & \text{otherwise.}\end{cases}$$

Taking into account the weighted system of each class, the canonical weighted representation for this multi-criteria simple game is:

$$\left[\begin{array}{c|c|c||cccccccccccccccccccc} 255 & 0 & 0 & 29 & 29 & 29 & 29 & 27 & 27 & 14 & 13 & 12 & \overset{4}{\overbrace{12}} & 10 & \overset{2}{\overbrace{10}} & \overset{3}{\overbrace{7}} & \overset{2}{\overbrace{7}} & 4 & 4 & 4 & 4 & 4 & 3\\ 0 & 18 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 620 & 170 & 123 & 122 & 120 & 82 & 80 & 47 & 33 & 22 & 21 & 18 & 17 & 11 & 8 & 5 & 4 & 3 & 2 & 1 & 1\end{array}\right]$$

where $\overset{a}{\overbrace{\quad}}$ indicates that this column is repeated "a" times.

In order to calculate the normalized extended Banzhaf index for this multi-criteria simple game, we apply the procedure based on generating functions to each class of the game. Thus, for class U_1 and player 1 we obtain:

$$\begin{aligned}B_1^1(x)&=\big(1+x^{29}\big)^3\big(1+x^{27}\big)^2\big(1+x^{14}\big)\big(1+x^{13}\big)\big(1+x^{12}\big)^5\big(1+x^{10}\big)^3\\ &\quad\times\big(1+x^7\big)^5\big(1+x^4\big)^5\big(1+x^3\big)\\ &=x^{316}+x^{313}+5x^{312}+10x^{309}+10x^{308}+\cdots+25\,985x^{254}+28\,467x^{253}\\ &\quad+30\,561x^{252}+32\,264x^{251}+35\,905x^{250}+38\,053x^{249}+40\,403x^{248}+43\,816x^{247}\\ &\quad+47\,157x^{246}+5023x^{245}+52\,758x^{244}+58\,005x^{243}+61\,053x^{242}+64\,214x^{241}\\ &\quad+69\,571x^{240}+73\,476x^{239}+78\,077x^{238}+82\,208x^{237}+88\,087x^{236}\\ &\quad+92\,948x^{235}+97\,365x^{234}+104\,534x^{233}+108\,758x^{232}\\ &\quad+115\,626x^{231}+121\,728x^{230}+127\,494x^{229}+134\,839x^{228}\\ &\quad+141\,015x^{227}+149\,067x^{226}+\cdots+10x^7+5x^4+x^3+1,\end{aligned}$$

which is a polynomial of degree 316, with 310 terms. The number of swings for player 1 is given by the sum of the coefficients from the 226th term to the 254th term, that is, $\eta_{11}=\sum_{k=226}^{254}=2\,193\,664$. Players 2, 3 and 4 have the same number of swings as player 1 since they have the same weights in class U_1. Analogously, the number of swings of the other players in this class as well as the number of swings of the players in the other two classes are obtained, yielding the total number of swings for each class: $\eta_1=28\,186\,428$, $\eta_2=84\,362\,850$, $\eta_3=132\,043\,208$.

Note that the result given by a coalition S will be favorable if $S\in U_i$, $i=1,2,3$, that is, if the coalition belongs to the intersection of the three classes. If we aggregate the classes by intersection, then a new multi-criteria simple game is obtained, whose classification is $U'=\{U_1,U_2,U_3,\bigcap_{i=1}^3 U_i,\mathcal{R}\}$. The weighted representations for U_1, U_2 and U_3 are the same as in the original game, and the weighted representation for the new class, $\bigcap_{i=1}^3 U_i$, is a vector-weighted system of order 3:

$$\left[\begin{array}{c|cccccccccccccccccccc} 255 & 29 & 29 & 29 & 29 & 27 & 27 & 14 & 13 & 12 & \overset{4}{\overbrace{12}} & 10 & \overset{2}{\overbrace{10}} & \overset{3}{\overbrace{7}} & \overset{2}{\overbrace{7}} & 4 & 4 & 4 & 4 & 4 & 3\\ 18 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 620 & 170 & 123 & 122 & 120 & 82 & 80 & 47 & 33 & 22 & 21 & 18 & 17 & 11 & 8 & 5 & 4 & 3 & 2 & 1 & 1\end{array}\right].$$

The following table contains the Banzhaf power indices of the countries for each class and for the intersection class:

Countries	U_1	U_2	U_3	$\bigcap_{i=1}^{3} U_i$
Germany	0.0778	0.0370	0.1750	0.0665
United Kingdom	0.0778	0.0370	0.1239	0.0665
France	0.0778	0.0370	0.1229	0.0665
Italy	0.0778	0.0370	0.1207	0.0665
Spain	0.0742	0.0370	0.0802	0.0631
Poland	0.0742	0.0370	0.0780	0.0631
Romania	0.0426	0.0370	0.0467	0.0407
The Netherlands	0.0397	0.0370	0.0326	0.0386
Greece	0.0368	0.0370	0.0217	0.0366
Czech Republic	0.0368	0.0370	0.0207	0.0366
Belgium	0.0368	0.0370	0.0207	0.0366
Hungary	0.0368	0.0370	0.0207	0.0366
Portugal	0.0368	0.0370	0.0207	0.0366
Sweden	0.0309	0.0370	0.0177	0.0325
Bulgaria	0.0309	0.0370	0.0169	0.0325
Austria	0.0309	0.0370	0.0169	0.0325
Slovak Republic	0.0218	0.0370	0.0108	0.0263
Denmark	0.0218	0.0370	0.0108	0.0263
Finland	0.0218	0.0370	0.0108	0.0263
Ireland	0.0218	0.0370	0.0079	0.0263
Lithuania	0.0218	0.0370	0.0079	0.0263
Latvia	0.0126	0.0370	0.0049	0.0198
Slovenia	0.0126	0.0370	0.0039	0.0198
Estonia	0.0126	0.0370	0.0030	0.0198
Cyprus	0.0126	0.0370	0.0020	0.0198
Luxembourg	0.0126	0.0370	0.0010	0.0198
Malta	0.0094	0.0370	0.0010	0.0177

One of the several advantages of studying the European Union system as a multi-criteria simple game is shown in this example. When the component games are considered, the importance of the selected criteria is revealed. For instance, the power of each country with respect to the second criterion, which is given by the Banzhaf index of class U_2, is the same for all countries, and therefore it holds no relevance in the determination of the real power of the countries. However, this criterion is being considered in European Union system. On the other hand, countries with different powers in class U_3 but with the same power in class U_1, end up with the same power in the intersection class. Thus, the most decisive criterion is the first criterion. We conclude that we present a valuable analysis, since it provides the power of each country not only with respect to the aggregated criterion, but also with respect to each one of the criteria considered.

5 Conclusions

The extended Banzhaf index has been introduced, which provides a notion of solution for group-decision problems where multiple players simultaneously deal with multiple qualitative criteria. An axiomatic characterization, similar to that of the classic index, has been established. It yields a quantitative measure of the power of the players in each class of the multi-criteria simple game. The extended Banzhaf index permits us to maintain the multi-dimensional nature of each player's decision, since in order to obtain this index, no weights have to be given to players nor to criteria. Thus, it is not sensitive to changes in their possible numeric evaluations, and therefore, it better reflects the real power of the players.

The combinatorial analysis is a useful tool which facilitates the computation of the Banzhaf index when a weighted representation of the multi-criteria simple game is considered. Therefore, a combinatorial method based on generating functions for computing the Banzhaf index efficiently has been introduced.

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