A moment method to solve multiobjective linear programs

Blanco, P., *Some algebraic methods for solving multiobjective polynomial integer programs*, J Symb. Comput. (2011).
Blanco, P., Ben Ali. *A semidefinite programming approach for solving Multiobjective Linear Programming*. J. Glob. Optim. (2013).

Desafíos de la Matemática Combinatoria
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FQM-5849

Outline

From single to multiobjective optimization

Let $f : \mathbb{R}^n \to \mathbb{R}^k$ and $S \subset \mathbb{R}^n$ compact.

$$\begin{aligned} v - \min\ & f(x) := (f^1(x), \ldots, k^k(x)) \\ & s.t.\ x \in S \end{aligned} \tag{VOP}$$

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$$v - \min f(x) := (f^1(x), \dots, k^k(x)) \quad \text{(VOP)}$$
$$s.t. \; x \in S$$

Definition

A decision vector $x^* \in S$ is a Pareto-optimal solution for VOP if there does not exist another decision vector $x \in S$ such that $f^i(x) \leq f^i(x^*)$ for all $i = 1, \dots, k$ and $f^j(x) < f^j(x^*)$ for at least one index j.
If x^* is a Pareto-optimal solution $f(x^*)$ is said to be an efficient point of VOP.

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Find the entire set of PO solutions.

Many applications in different fields: Economics, Game Theory, Spacial Analysis ...

MOLP: Very modest aspiration: The linear case

$$\begin{aligned} v-\min\ Cx &:= (c^1x,\ldots,c^kx) \\ s.t.\ Ax &\geq b \\ x &\geq 0 \end{aligned} \tag{MOLP}$$

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Definition

A decision vector $x^* \geq 0$ such that $Ax \geq b$ is a Pareto-optimal solution for MOLP if there does not exist another decision vector $x \geq 0$ with $Ax \geq b$ such that $c^i x \leq c^i x^*$ for all $i = 1, \ldots, k$ and $c^j x < c^j x^*$ for at least one index j.
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If x^* is a Pareto-optimal solution $f(x^*)$ is said to be an efficient point of MOLP.

Find the entire set of PO solutions.

Facial Structure of the solution set

The solution set is a connected union of faces (of any dimension)
Computing PO-set is $\#P$-hard

Motivation I: LP and MOLP

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Active Set (A.S.) Primal	A.S. Dual
A.S. Primal-Dual	Non-Active Set

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Our goals:

- New parallelism between LP and MOLP.
- Adapt the techniques of polynomial optimization (Lasserre 2009) to MOLP.

An illustrative example

Example

Consider problem MOLP with the data:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix},\ b = \begin{pmatrix} 4 \\ 3 \\ 4 \\ -5 \\ -5 \end{pmatrix}.$$

The problem is:

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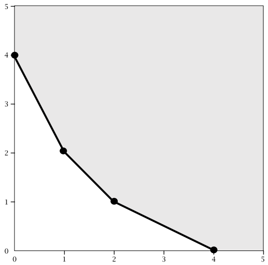

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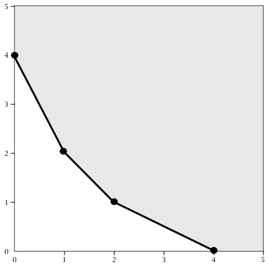

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Observe that the last two constraints refer to the upper bound constraints $x_1 \leq 5$ and $x_2 \leq 5$, so they are not considered as rows of the matrix A but as the sets of upper bounds in the polynomial constraints $p_j(x)$ in Theorem 3. Moreover, by the form of DLP_λ we can use $ub_i^D = 1$ for $i = 1, 2, 3$ as valid upper bounds for the variables in the dual problems.

From MOLP to Polynomial inequalities

$$v - \min\ Cx := (c^1 x, \ldots, c^k x) \qquad \text{(MOLP)}$$
$$s.t.\ Ax \geq b$$
$$x \geq 0$$

Lemma (Gass & Saaty 1955; Zadeh 1963; Geoffrion 1968)

x^* is a Pareto-optimal solution of MOLP *if and only if there exists a weighting vector $\lambda \in \Lambda = \{\omega \in \mathbb{R}^k_+, \sum_{i=1}^k \omega_i = 1\}$ such that x^* is a solution of the following scalar problem:*

$$\min \sum_{i=1}^{k} \lambda_i c^i(x) \qquad \text{(SP)}$$
$$s.t.\ Ax \geq b$$
$$x \geq 0$$

From MOLP to Polynomial inequalities

For a fixed $\lambda \in \Lambda$ we need to solve:

$$\min \sum_{\ell=1}^{k} \lambda_\ell c^\ell x \qquad (\mathrm{LP}_\lambda)$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0.$$

whose dual problem is:

$$\max \sum_{j=1}^{m} u_j b_j \qquad (\mathrm{DLP}_\lambda)$$
$$\text{s.t. } u^t A \leq \sum_{i=1}^{k} \lambda_i c^i$$
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Lemma (Strong Duality Theorem/Complementary Slackness Property)

Let x^* be a feasible solution of LP_λ and let u^* be a feasible solution of DLP_λ. Then, the following statements are equivalent:

1. *x^* is an optimal solution of LP_λ and u^* is an optimal solution of DLP_λ.*
2. *$c^t x^* = b^t u^*$.*
3. *x^* and u^* satisfy $u^{*t}(b - Ax^*) = 0$ and $(u^{*t}A - c^t)x^* = 0$.*

From MOLP to Polynomial inequalities

Hence, a solution of MOLP must be a solution of:

$$
\begin{aligned}
& u^t(b - Ax) = 0 \\
& (\sum_{i=1}^{k} \lambda_i c^i - u^t A)x = 0 \\
& Ax \geq b \\
& u^t A \leq \sum_{i=1}^{k} \lambda_i c^i \\
& \sum_{i=1}^{k} \lambda_i = 1 \\
& \lambda, u, x \geq 0.
\end{aligned}
\qquad (\mathrm{Sys}_1)
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Exploit facial structure of PO-set

System with continuum of solutions: Resort to extreme point PO-solutions

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Exploit facial structure of PO-set

System with continuum of solutions: Resort to extreme point PO-solutions

Extensions to other problems: integer case, convex continuous... (??)

Other MO problems...

- MO linear problems in integer variables: Short rational generating functions of lattice points in polyhedra (De Loera, Hemmecke and Köppe 2009, Blanco and P., 2012)

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- MO convex problems, Non-convex problems ...

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B: basis of A and N: columns of A not in B.
c_B^ℓ: ℓ-th objective function that corresponds to variables in the basis B.

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$$Bx = b, \tag{Sys-B}$$

$$\sum_{\ell=1}^{k} \lambda_\ell c^\ell - u^t A \geq 0, \tag{1}$$

$$\sum_{\ell=1}^{k} \lambda_\ell c_B^\ell B^{-1} A_{.j} - \sum_{\ell=1}^{k} \lambda_\ell c_j^\ell \leq 0, \quad \forall j \in N,$$

$$\sum_{\ell=1}^{k} \lambda_\ell = 1,$$

where $A_{i.}$ is the i-th row, $A_{.j}$ is the j-th column and A_{ij} is the (i,j) element of A, respectively.
K: least common multiple of all the determinants of full rank submatrices of (A, I).
K': least common multiple of all the determinants of full rank submatrices of 1, for all B.

From MOLP to Polynomial inequalities

Theorem

If x is a Pareto-optimal solution and extreme point of the feasible region of MOLP *then $K\,x$ is the projection onto the first n-components of a solution of the system* (Sys_2)*:*

$$h_0^0(\lambda) := \sum_{\ell=1}^{k} \lambda_\ell - K' = 0,$$

$$h_1^0(x,u) := u^t(K\,b - Ax) = 0,$$

$$h_2(x,u) := (\sum_{\ell=1}^{k} \lambda_\ell c^\ell - u^t A)x = 0,$$

$$g_s^0(x) := A_{s\cdot}x - Kb_s \geq 0,$$

$$g_j(u,\lambda) := \sum_{\ell=1}^{k} \lambda_\ell c_j^\ell - u^t A_{\cdot j} \geq 0,$$

$$p_j(x) := \prod_{\ell=0}^{ub_j^P K} (x_j - \ell) = 0,$$

$$q_s(u) := \prod_{\ell=0}^{ub_s^D K'} (u_s - \ell) = 0,$$

$$t_r(\lambda) := \prod_{\ell=0}^{K'} (\lambda_r - \ell) = 0.$$

Conversely, any of the finitely many solutions of System (Sys_2) *induces a Pareto-optimal solution of* MOLP *and all the Pareto-optimal extreme points are included among them.*

Example: Continuation

The System applied to the example is:

$$
(\mathrm{Sys}_2)\left\{
\begin{aligned}
h_0^0 &= \lambda_1 + \lambda_2 - 6 && = 0 \\
h_1^0 &= u_1(24 - 2x_1 - x_2) + u_2(18 - x_1 - x_2) + u_3(24 - x_1 - 2x_2) && = 0 \\
h_2^1 &= (-2u_1 - u_2 - u_3 + \lambda_1)x_1 + (-u_1 - u_2 - 2u_3 + \lambda_2)x_2 && = 0 \\
g_1^0 &= 2x_1 + x_2 - 24 && \geq 0 \\
g_2^0 &= x_1 + x_2 - 18 && \geq 0 \\
g_3^0 &= x_1 + 2x_2 - 24 && \geq 0 \\
g_1^1 &= -2u_1 - u_2 - u_3 + \lambda_1 && \geq 0 \\
g_2^1 &= -u_1 - u_2 - 2u_3 + \lambda_2 && \geq 0 \\
p_1 &= \prod_{\ell=1}^{30}(x_1 - \ell) && = 0 \\
p_2 &= \prod_{\ell=1}^{30}(x_2 - \ell) && = 0 \\
q_1 &= \prod_{\ell=1}^{6}(u_1 - \ell) && = 0 \\
q_2 &= \prod_{\ell=1}^{6}(u_2 - \ell) && = 0 \\
q_3 &= \prod_{\ell=1}^{6}(u_3 - \ell) && = 0 \\
t_1 &= \prod_{\ell=1}^{6}(\lambda_1 - \ell) && = 0 \\
t_2 &= \prod_{\ell=1}^{6}(\lambda_2 - \ell) && = 0
\end{aligned}
\right.
$$

MOLP and SDP

Theorem

The entire set of Pareto-optimal extreme point solutions of MOLP *is encoded in the optimal solutions,* $\mathbf{y} = (y_{\alpha\beta\gamma}) \subset \mathbb{R}$*, of the semidefinite program* $SDP - N^*$*, for some* $N^* \in \mathbb{N}$*.*

$$
\begin{aligned}
\min \ & y_0 := 1 && (SDP - N^*) \\
s.t. \ & \mathrm{M}_{N^*}(\mathbf{y}) \succeq 0, \\
& \mathrm{M}_{N^*-1}(h_0^0 \mathbf{y}) = 0, \\
& \mathrm{M}_{N^*-1}(h_1^0 \mathbf{y}) = 0, \\
& \mathrm{M}_{N^*-1}(h_2 \mathbf{y}) = 0, \\
& \mathrm{M}_{N^*-1}(g_s^0 \mathbf{y}) \succeq 0, \ s = 1, \ldots, m, \\
& \mathrm{M}_{N^*-1}(g_j \mathbf{y}) \succeq 0, \ j = 1, \ldots, n, \\
& \mathrm{M}_{N^*-\zeta_j}(p_j \mathbf{y}) = 0, \ \ j = 1, \ldots, n, && (2) \\
& \mathrm{M}_{N^*-\eta_s}(q_s \mathbf{y}) = 0, \ \ s = 1, \ldots, n, && (3) \\
& \mathrm{M}_{N^*-\nu_r}(t_r \mathbf{y}) = 0, \ \ r = 1, \ldots, k. && (4)
\end{aligned}
$$

Any generic solution of the above problem (for instance obtained using interior point methods) shall give full rank to the moment matrix.

Getting the PO-set

We can transform (Sys_2) into an algebraic set adding slack variables.
Let

$$\hat{J} = \langle h_0^0, h_1^0, h_2, g_1^0, \ldots, g_m^0, g_1, \ldots, g_n, p_1, \ldots, p_n, q_1, \ldots, q_m, t_1, \ldots, t_k \rangle$$

the zero-dimensional ideal in $\mathbb{R}[x, u, \lambda]$, generated by all the polynomial equations defining the System. The variety $V_{\mathbb{R}}(\hat{J})$ is finite. (Apply Lasserre, Laurent, Rostalski (2008): $\sqrt[\mathbb{R}]{\hat{J}} = \langle \text{Ker}\mathrm{M}_s(y) \rangle$ for some s.)

Since $\hat{J}$ is zero-dimensional $\mathbb{R}[x, u, \lambda]/\hat{J}$ is a finite dimensional $\mathbb{R}$-vector space with the usual addition and scalar product. Let $\mathcal{B}_{\hat{J}} = \{b_1, \ldots, b_N\}$ be a basis.

Furthermore, $\mathbb{R}[x, u, \lambda]/\hat{J}$ is an algebra with multiplication $[f][g] = [fg]$.
For any $h \in \mathbb{R}[x, u, \lambda]$:

$$\begin{array}{rccc} m_h : & \mathbb{R}[x, u, \lambda]/\hat{J} & \rightarrow & \mathbb{R}[x, u, \lambda]/\hat{J} \\ & f & \rightarrow & m_h([f]) := [fh] \end{array}$$

Let $\hat{\mathrm{M}}_h$ be the *multiplication matrix* associated with the linear operator m_h expressed in the basis $\mathcal{B}_{\hat{J}}$.

MOLP and SDP

For any $v \in V_{\mathbb{R}}(\hat{J})$, let $r_v := (b_\ell(v))_{1 \leq \ell \leq N} \in \mathbb{R}^N$ be the evaluation of the point v by the polynomials that define the basis $\mathcal{B}_{\hat{J}}$.

Matrices $\hat{\mathrm{M}}_h$ satisfy:

$$\hat{\mathrm{M}}_h r_v = h(v) r_v \text{ for all } v \in V_{\mathbb{R}}(\hat{J})) \quad \text{(Stickelberger Theorem)}$$

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Theorem

For t large enough, there exists $d \leq s \leq t$ such that:

$$\operatorname{rank} \mathrm{M}_s(\mathbf{y}) = \operatorname{rank} \mathrm{M}_{s-d}(\mathbf{y}) = |V_{\mathbb{R}}(\hat{J})|, \quad \mathbf{y} = (y_{\alpha\beta\gamma})_{|\alpha\beta\gamma| \leq 2t} \in R(SDP\text{-}t).$$

Moreover, one can obtain the coordinates of all $(x, u, \lambda) \in V_{\mathbb{R}}(\hat{J})$, as the eigenvalues of multiplication matrices $\hat{\mathrm{M}}_{x_\ell}$, $\hat{\mathrm{M}}_{u_j}$, $\hat{\mathrm{M}}_{\lambda_s}$ for all $\ell = 1, \dots, n$, $j = 1, \dots, m$, $s = 1 \dots, k$.

Example: Continuation

We use `Gloptipoly 3` and $N^* = 4$, the rank condition of Theorem 5 is satisfied, i.e. $\mathrm{rank}\,\mathrm{M}_4(x, u, \lambda) = \mathrm{rank}\,\mathrm{M}_1(x, \mu, \lambda) = 6$. Thus, we extract the following solutions of *SDP − N**:

	Solutions		
	x	u	λ
Sol. #1	$(6, 12)$	$(2, 0, 0)$	$(4, 2)$
Sol. #2	$(0, 24)$	$(2, 0, 0)$	$(4, 2)$
Sol. # 3	$(6, 12)$	$(0, 3, 0)$	$(3, 3)$
Sol. # 4	$(12, 6)$	$(0, 3, 0)$	$(3, 3)$
Sol. # 5	$(12, 6)$	$(0, 0, 3)$	$(3, 3)$
Sol. # 6	$(24, 0)$	$(0, 0, 2)$	$(2, 4)$

Note that the number of moments involved in the SDP problem that had to be solved was 6435. In this problem, the moment matrix $\mathrm{M}_{N^*}(x, u, \lambda)$ has size 330×330.

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Thus, projecting the set of extracted solutions onto the x-coordinates and dividing by K, we get the set of extreme Pareto-optimal solutions of the problem, $X_E = \{(4,0),(1,2),(2,1),(0,4)\}$.

The PO-set

These Pareto-optimal solutions and the complete Pareto-optimal set are shown in Fig. ?? (black dots and black segments, respectively).

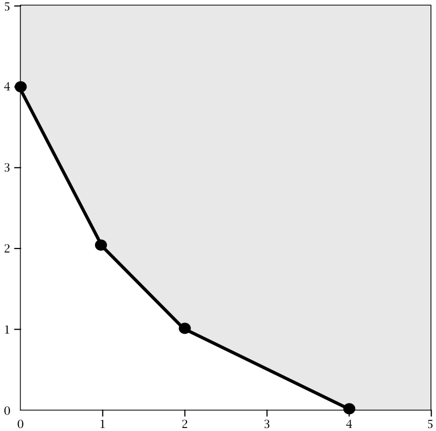

Figure: Pareto-optimal set of Example.

Conclusions

- We present moment approach to find the set of extreme PO extreme points of a MOLP.
- We explicitly give an SDP problem the solutions of which encode all the PO extreme points of MOLP.
- We show how all these points can be obtained by applying the so called moment matrix algorithm.
- The main drawback is the size of the SDP problem which is not polynomial in the input size of MOLP.
- Our results also show the power of some techniques developed in the field of polynomial optimization to be applied in apparently different areas such as Multiobjective Optimization.

Thanks for your attention!!!

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- V. Blanco, J. Puerto, *A new complexity result on multiobjective linear integer programming using short rational generating functions*, Optimization Letters, (2012).
- V. Blanco, J. Puerto, S. El Haj-Ben Ali. *A semidefinite programming approach for solving Multiobjective Linear Programming*. J. Glob. Optim. (2013).