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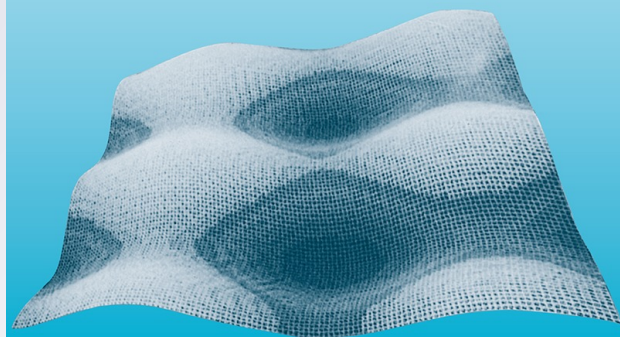
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Abstract Several algorithms are available in the literature for finding the entire set of Pareto-optimal solutions of Multiobjective Linear Programmes (MOLPs). However, all of them are based on active-set methods (simplex-like approaches). We present a different method, based on a transformation of any MOLP into a unique lifted Semidefinite Program (SDP), the solutions of which encode the entire set of Pareto-optimal extreme point solutions of any MOLP. This SDP problem can be solved, among other algorithms, by interior point methods; thus unlike an active set-method, our method provides a new approach to find the set of Pareto-optimal solutions of MOLP.

Keywords Multiobjective Linear Programming · Semidefinite Programming · Polynomial optimization · Moment problem

1 Introduction

Although already more than 60 years old, Linear Programming is still nowadays one of the most important areas of research and application in Mathematical Programming/Operations Research. There are many different algorithms to obtain its solutions but roughly speaking all of them can be classified into active-set methods such as primal simplex, dual simplex and

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V. Blanco
Department of Quantitative Methods for Economics and Business,
Universidad de Granada, Granada, Spain
e-mail: vblanco@ugr.es

J. Puerto (✉) · S. El Haj Ben Ali
IMUS, Universidad de Sevilla, Seville, Spain
e-mail: puerto@us.es

primal-dual simplex; and interior point methods including affine-scaling, path-following, or potential-reduction methods, among others.

Although not at the same level as standard linear programming, it is also commonly accepted that Multiobjective Linear Programming is another very important area of activity within the optimization field. Multiobjective Optimization is motivated by the need to consider multiple, conflicting, objectives in real world decision making problems. Multiple objective linear programming has been a subject of research since the 1960s for its relevance in practice and as a mathematical topic in its own right. Its development has come in parallel to the scalar counterpart and its theory and algorithms have been developed continuously over the years with important contributions also in the last years. Although there are several notions of efficiency, in this paper we assume that solving a MOLP amounts to obtain the entire set of Pareto-optimal solutions. By analogy with the scalar case, one can find in the specialized literature several algorithms to find these solutions. In the multiobjective case there are primal simplex-like algorithms (Steuer [27], Yu and Zeleny [30] and the references therein), primal-dual simplex-like algorithms (Ehrgott et al. [7]) and dual simplex-like algorithms (Benson [2], Ehrgott et al. [6]). Moreover, there are some partial approaches that use interior point methods to approximate or to find some Pareto-optimal points (see e.g. Fliege [10,11]), but no interior point-based algorithm that finds all Pareto-optimal solutions has been proposed so far. Quoting [6]: “No interior point algorithm that finds all efficient solutions has been proposed”.

Due to the natural parallelism between these two areas, namely scalar and Multiobjective Linear Programming, different authors, among them we cite Ehrgott et al. [6], wondered whether unlike the simplex-like approaches (primal, dual or primal-dual) there would exist an alternative approach to generate the complete Pareto-optimal set of a MOLP. This paper gives an affirmative answer to this question.

The goal of this paper is to develop a Semidefinite Programming method to generate the entire Pareto-optimal set of a MOLP. The interest of this research is twofold: (1) It strengthens the parallelism between these two areas of Mathematical Programming, namely Linear Programming and Multiobjective Linear Programming; (2) It is theoretically appealing because it shows how to adapt some tools available nowadays in the field of polynomial optimization [15,16] to be applied in a completely different field as is Multiobjective Linear Programming.

From the results in this paper, we show that it is possible to find the entire set of Pareto-optimal solutions using a SDP based approach. Indeed, we will explicitly describe an algorithm to perform that task. Our approach goes beyond a trivial application of Interior Point Methods (IPM) to solve scalarized single objective linear problems since by this methodology one would need to either solve an infinite number of linear problems or to identify ‘a priori’ the partition of the parametric space of scalarized problems that correspond to extreme Pareto-optimal solutions of the MOLP. None of the above mentioned two cases are, in general, possible. Our method consists of constructing one lifted semidefinite program based on the original MOLP so that their solutions encode all the extreme Pareto-optimal solutions of the MOLP. The reader may find some parallelism between this approach and the semidefinite relaxations of combinatorial optimization problems. In both cases, the goal is to embed the original problem in a sufficiently high dimension space so that the non-connected solutions of the original problem can be recovered from some sort of projection of the optimal solutions of the lifted SDP relaxation. (The reader is referred to [24] for further details on the relationship between SDP and Integer Programming.) This recovering is not always possible unless the semialgebraic set encoding the solutions satisfies

some algebraic properties. In the case covered in this paper, such a construction is possible and it is based on the moment matrix algorithm by Lasserre [15]. Nevertheless, the size of the necessary SDP is not polynomial in the input size of our original MOLP. Finding the entire Pareto-optimal set of a MOLP is, in general, NP-hard since it may be equivalent to enumerate the vertices of the feasible region of the problem (see Khachiyan et al [14]). Thus if our construction were polynomial in the input size, interior point algorithms would prove polynomiality of the problem. Needless to say that even though our construction is explicit, we do not claim that this approach is computationally competitive with other approaches available nowadays, as for instance the variant of the outer algorithm of Benson developed recently in [6]. Nevertheless, it is of theoretical interest because it proposes the first method which is not based on an active-set approach for obtaining the entire Pareto-optimal set, it provides a completely different scheme to address this problem and develops another pseudo-polynomial algorithm to deal with that computation. It is worth mentioning a different framework where a similar approach is applicable: Multiobjective Polynomial Integer Programming. We point out that the use of algebraic tools for solving Multiobjective Optimization problems is not new. The interested reader is referred to [3,4] for further details.

The rest of the paper is organized as follows. Section 2 describes the general Multiobjective Optimization problem, together with the solution concept considered in this paper, namely the Pareto-optimal set of solutions. In addition, it recalls the main results that are needed for the rest of the paper. Section 3 presents the theoretical results that imply that the entire set of Pareto-optimal solutions of a MOLP can be recovered from the optimal solutions of a SDP in a lifted space. Here, we explicitly describe a system of polynomial equations that encodes the entire set of Pareto-optimal extreme points which is the basis of our next results. Section 4 reduces the problem of finding the entire set of Pareto-optimal extreme points to project the solutions of one explicit SDP problem. It is a feasibility problem since its objective function is constant. In addition, we also show how to extract all the, finitely many, projections of its solutions by applying the moment matrix algorithm [15]. This construction is illustrated with an example taken from the literature [6]. In the final section of the paper we draw some conclusions.

2 Multiobjective Linear Programming

In this section, for the sake of completeness, we recall the main theoretical results for the development in this paper. We begin by describing the general framework to cast the problem to be handled. A *Multiobjective Optimization Problem (MOP)* consists of:

$$\begin{aligned} v - \min(f_1(x), \dots, f_k(x)) \\ \text{s.t. } x \in S \end{aligned} \tag{MOP}$$

where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, \dots, k$ are the *objective functions* and $S \subseteq \mathbb{R}^n$ is the *feasible region*. The symbol $v - \min$ means that we want to minimize all the objective functions simultaneously. If there is no conflict between the objective functions, then, a solution can be found where every objective function attains its optimum. In such a case, no special methods are needed. Otherwise, first we need to state what we understand by a solution of the above problem. It is commonly agreed that the solution concept is related with the notion of Pareto-optimal points. The definition of Pareto-optimal solution is due to

Pareto [22,23]. However, that name was first used by Little in [19], as recently reported by Ehrgott [9].

Definition 1 A decision vector $x^* \in S$ is a *Pareto-optimal solution* for MOP if there does not exist another decision vector $x \in S$ such that $f_i(x) \leq f_i(x^*)$ for all $i = 1, \dots, k$ and $f_j(x) < f_j(x^*)$ for at least one index j .

The set of Pareto-optimal solutions is called the *Pareto-optimal set*. (PO set, for short.)
 If x^* is a Pareto-optimal solution $f(x^*)$ is said to be an *efficient point* of MOP.

In this paper, by solving MOP we understand finding the entire set of Pareto-optimal solutions. There are some other definitions of vector optimality for Multiobjective Optimization Problems, such as local, weak, proper or strong Pareto optimality (see [8,20]). As mentioned in the Introduction, the motivation of this paper comes from developing a new semidefinite problem whose solutions encode the entire set of Pareto-optimal solutions of any Multiobjective Linear Program. Therefore, we will restrict ourselves to consider as solution concept for the MOLP this set, namely the set of *standard* Pareto-optimal solutions, although similar approaches, to the one adopted in this paper, would be also valid for the rest of the vector-optimality definitions in Multiobjective Optimization.

One of the main methods for describing the Pareto-optimal set of a Multiobjective Optimization problem is by *scalarization*, that is, transforming the multiobjective problem into a single or a family of single-objective problems with a real-valued objective function, depending on some parameters. This enables the use of the theory and methods of scalar optimization to be applicable to get the solutions of MOP. The importance of these methods rests on the fact that Pareto-optimal solutions of MOP can be characterized, in most cases, as solutions of certain single objective optimization problems. There are several of these scalarization methods for solving MOP (see [20]). Among them, we consider the weighting method. The idea of this method is to associate each objective function with a weighting coefficient so as to minimize the weighted sum of the objective functions. The weighting method can be used so that the decision maker specifies a weighting vector representing his preference information. However, this method can also be used to generate iteratively several solutions of MOP by modifying adequately the weighting coefficients. The success of this approach is based on the following result by Gass and Saaty [12] or Zadeh [31].

Lemma 2 *Let f_i be convex for all $i = 1, \dots, k$ and S be a convex set. Then, if $x^* \in S$ is a Pareto-optimal solution of MOP, there exists a weighting vector $\lambda \in \mathbb{R}_+^k \setminus \{0\}$, $\sum_{i=1}^k \lambda_i = 1$ such that x^* is a solution of the following scalar problem:*

$$\begin{aligned} \min \sum_{i=1}^k \lambda_i f_i(x) \\ \text{s.t. } x \in S. \end{aligned} \tag{SP}$$

Note that from the above result, in particular if f_i is linear for all $i = 1, \dots, k$ and S is a convex polyhedron, all the Pareto optimal solutions of MOP can be found by the weighting method. Scalarization results are rather appealing since they reduce Multiobjective Linear Programming to single objective counterparts. The main drawback is that one would need to solve infinitely many single objective problems. Clearly, this is not an ‘efficient’ approach.

In this paper we are interested in solving a special class of Multiobjective Optimization problems where all the objective functions are linear and the feasible region is described by a set of linear inequalities, that is:

$$\begin{aligned}
 v - \min Cx &:= (c^1x, \dots, c^kx) \\
 \text{s.t. } Ax &\geq b \\
 x &\geq 0
 \end{aligned}
 \tag{MOLP}$$

with c^i the i th row of $C \in \mathbb{Q}^{k \times n}$, $i = 1, \dots, k$, $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. We assume w.l.o.g. that A , b and C have integer coefficients (the above problem is equivalent when multiplying by the least common multiple of the corresponding denominators), that the system $Ax \geq b$, $x \geq 0$, has non-redundant inequalities and that $0 \notin \text{conv}(c^1, \dots, c^k)$ since otherwise all the feasible region is Pareto-optimal and the problem is trivial. Under our assumption, the Pareto-optimal set is included in the boundary of the feasible region and it is well-known that it is edge connected but not necessarily convex, see e.g. [27]. With these settings we say that a Pareto-optimal solution for MOLP is an *extreme point solution* if it is a vertex of the polyhedron defining the feasible region of MOLP.

In addition and in order to simplify our presentation, we will assume w.l.o.g. that the feasible region is a polytope and that we are given redundant upper bounds on the values of the x variables, namely we are given ub_j^P such that $x_j \leq ub_j^P$, $j = 1, \dots, n$. Note that by the fact that the feasible region is a polytope these bounds can always be obtained for sufficiently large ub^P values and they are redundant.

Lemma 2 ensures that to solve MOLP it suffices to apply the above weighting method. In doing that, this problem is transformed to a family of parametric linear programming problems. For this reason, we recall here some results about linear programming that will be useful in the next sections.

Consider the following pair of dual linear programming problems:

$$\begin{array}{ll}
 \min c^t x & \max b^t u \\
 \text{s.t. } Ax \geq b & \text{s.t. } A^t u \leq c \\
 x \geq 0 & u \geq 0
 \end{array}
 \tag{LP} \qquad \tag{DLP}$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$.

By our assumption on the feasible region of the primal problem LP, we observe that the dual problem can be also assumed to be bounded and that we can assume that we are also given redundant upper bounds on the feasible values of DLP. Namely, we know ub_i^D such that $u_i \leq ub_i^D$ for all $i = 1, \dots, m$.

The following classical result, the proof of which can be found in [25,29], gives the relationship between the optimal solutions of the above problems:

Lemma 3 (Strong Duality Theorem/Complementary Slackness Property) *Let x^* be a feasible solution of LP and let u^* be a feasible solution of DLP. Then, the following statements are equivalent:*

1. x^* is an optimal solution of LP and u^* is an optimal solution of DLP.
2. $c^t x^* = b^t u^*$.
3. x^* and u^* satisfy $u^{*t}(b - Ax^*) = 0$ and $(u^{*t}A - c^t)x^* = 0$.

In the next section we will show how the entire set of Pareto-optimal solutions of a Multiobjective Linear Problem MOLP can be extracted from the solutions of a SDP problem via an appropriate projection of its variables. Since IPM's are nowadays among the most popular algorithms to solve SDP, thus we can conclude that interior point-based methods can be effectively used to determine the Pareto-optimal set of MOLP.

3 A polynomial system of inequalities encoding the Pareto-optimal set of MOLP

Note that by Lemma 2, solving MOLP is equivalent to solve, for all the infinitely many $\lambda \in \Lambda := \{\lambda \in \mathbb{R}_+^k : \sum_{\ell}^k \lambda_{\ell} = 1\}$, the following parametric family of single objective problems:

$$\begin{aligned} \min & \sum_{\ell=1}^k \lambda_{\ell} c^{\ell} x \\ \text{s.t.} & Ax \geq b \\ & x \geq 0. \end{aligned} \tag{LP}_{\lambda}$$

Next, for each $\lambda \in \Lambda$, the dual of DLP_{λ} is:

$$\begin{aligned} \max & \sum_{j=1}^m u_j b_j \\ \text{s.t.} & u^t A \leq \sum_{i=1}^k \lambda_i c^i \\ & u \geq 0. \end{aligned} \tag{DLP}_{\lambda}$$

Hence, by Lemma 3, a solution of MOLP must be a solution of the following system of polynomial equations/inequalities:

$$\begin{aligned} & u^t (b - Ax) = 0 \\ & \left(\sum_{i=1}^k \lambda_i c^i - u^t A \right) x = 0 \\ & Ax \geq b \\ & u^t A \leq \sum_{i=1}^k \lambda_i c^i \\ & \sum_{i=1}^k \lambda_i = 1 \\ & \lambda, u, x \geq 0. \end{aligned} \tag{Sys}_1$$

Solutions of this system are triplets (x, u, λ) such that x is a Pareto-optimal solution of MOLP, and optimal solution of DLP_{λ} ; and u is an optimal solution of DLP_{λ} . We will call such a triplet a ‘valid triplet’. However, we observe that the above system may have an infinite number of solutions because there may be a continuum of solutions in x and u since the Pareto-optimal set is a connected union of faces of the polyhedron $Ax \geq b, x \geq 0$. Nevertheless by the edge-connectedness of the Pareto-optimal set, it suffices to know the Pareto-optimal points that are extreme points in the feasible region

to reconstruct the entire set (see for instance [1,5]). Therefore, our goal is to reduce the solutions of the above system to a finite number of the original ones (extreme points) that are enough to reconstruct the entire set of Pareto-optimal solutions. Hence, we shall transform System Sys₁ so as to characterize only extreme points of the original feasible region.

It is worth mentioning that a similar scheme could be applied to general multiobjective convex optimization although some problems would arise. Indeed, in that case we have to replace the use of System Sys₁ by using Kuhn-Tucker optimality conditions on the parametric family of weighted problems. The only inconvenience is that in that case the system may have a continuum of solutions and in general, one cannot build the entire Pareto-optimal set with a finite set of representatives, as we do in the linear case with the extreme solutions. This implies that finiteness results cannot be obtained. Another different framework where a similar approach is applicable is in Multiobjective Polynomial Combinatorial Optimization. The interested reader is referred to [3,4] for further details.

As it is usual, let B denote a basis of A , i.e. a full rank submatrix of A . In addition and when there is no possible confusion, we will also use B as the set of the indices of its columns. Analogously, N denotes the set of columns of A not in B and c_B^ℓ are the coefficients of the ℓ th objective function that corresponds to variables in the basis B . Finally, with x_B and x_N , we refer to the variables corresponding to the columns in B and N , respectively.

Let B be an arbitrary full rank submatrix of (A, I) (here I stands for the identity matrix of size $m \times m$). Let us denote by Sys-B the system:

$$\begin{aligned} & \sum_{\ell=1}^k \lambda_\ell c^\ell - u^t A \geq 0 \\ & \sum_{\ell=1}^k \lambda_\ell c_B^\ell B^{-1} A_{.j} - \sum_{\ell=1}^k \lambda_\ell c_j^\ell \leq 0, \quad \forall j \in N, \\ & \sum_{\ell=1}^k \lambda_\ell = 1, \end{aligned} \tag{Sys-B}$$

where $A_i.$ is the i th row, $A_{.j}$ is the j th column and A_{ij} is the (i, j) element of A , respectively.

Finally, let K be the least common multiple of all the determinants of full rank submatrices of (A, I) and K' be the least common multiple of all the determinants of full rank submatrices of Sys-B, for all B .

The following result shows a one-to-one correspondence between the extreme Pareto-optimal solutions of MOLP and an explicit compact semialgebraic set (described by a set of polynomial equations/inequalities and with a finite number of solutions). This identification will be crucial for the SDP approach presented in the next section of the paper.

Theorem 4 *If x is a Pareto-optimal solution and extreme point of the feasible region of MOLP then Kx is the projection onto the first n -components of a solution of the system Sys₂:*

$$h_0^0(\lambda) := \sum_{\ell=1}^k \lambda_\ell - K' = 0, \tag{1}$$

$$h_1^0(x, u) := u^t(Kb - Ax) = 0, \tag{2}$$

$$h_2(x, u) := \left(\sum_{\ell=1}^k \lambda_\ell c^\ell - u^t A \right) x = 0, \tag{3}$$

$$g_s^0(x) := A_s \cdot x - Kb_s \geq 0, \quad s = 1, \dots, m, \tag{4}$$

$$g_j(u, \lambda) := \sum_{\ell=1}^k \lambda_\ell c_j^\ell - u^t A_{\cdot j} \geq 0, \quad j = 1, \dots, n, \tag{5}$$

$$p_j(x) := \prod_{\ell=0}^{ub_j^P K} (x_j - \ell) = 0, \quad j = 1, \dots, n, \tag{6}$$

$$q_s(u) := \prod_{\ell=0}^{ub_s^D K'} (u_s - \ell) = 0, \quad s = 1, \dots, m, \tag{7}$$

$$t_r(\lambda) := \prod_{\ell=0}^{K'} (\lambda_r - \ell) = 0, \quad r = 1, \dots, k. \tag{8}$$

(Sys₂)

Conversely, any of the finitely many solutions of System Sys₂ induces a Pareto-optimal solution of MOLP and all the Pareto-optimal extreme points are included among them.

Proof Let x be a Pareto-optimal extreme point solution of MOLP. Being an extreme point of the feasible region $Ax \geq b, x \geq 0$ means that there exists a basis B which defines x such that $x = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$ and $x_B = B^{-1}b \geq 0$. Next, observe that if K is the least common multiple of all the determinants of full rank submatrices of (A, I) then, by Cramer's rule, $Kx = \begin{pmatrix} Kx_B \\ 0 \end{pmatrix}$, has integer coordinates in the range $[0, ub^P K]$.

Besides, by Lemma 2, x must be optimal for the problem $LP_{\lambda(x)}$ for some $\lambda(x)$ (Note that this weighting coefficient may depend on x). The optimality condition of x (nonnegative reduced cost for the non basic variables) translates into the following necessary and sufficient condition for any valid $\lambda(x)$:

$$\sum_{\ell=1}^k \lambda_\ell(x) c^\ell B^{-1} A_{\cdot j} - \sum_{\ell=1}^k \lambda_\ell(x) c_j^\ell \leq 0, \quad \forall j \in N, \tag{9}$$

$$\sum_{\ell=1}^k \lambda_\ell(x) = 1. \tag{10}$$

Moreover, by Lemma 3, each pair $(x, \lambda(x))$, where $\lambda(x)$ is defined by (9)–(10), has associated an optimal extreme solution of $DLP_{\lambda(x)}$ such that $(x, u(x), \lambda(x))$ is a valid triplet, i.e. it satisfies System Sys_1 . By the fact that $u(x)$ is an extreme solution of $DLP_{\lambda(x)}$ it must satisfy:

$$u^t A - \sum_{\ell=1}^k \lambda_{\ell} c^{\ell} \leq 0. \tag{11}$$

Hence, there is always one of these valid triplets for x so that $(u(x), \lambda(x))$ is an extreme solution of the system of linear inequalities defined by (9), (10) and (11). This system written in matrix form is:

$$L \begin{bmatrix} u \\ \lambda \end{bmatrix} := \begin{bmatrix} \mathbf{0} & C_B B^{-1} N - C_N \\ A^t & C \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} \begin{matrix} \leq \\ \leq \\ = \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Next, if $(u(x), \lambda(x))$ is an extreme solution of this system there must exist B_L , a full rank submatrix of (L, I) , such that $(u(x), \lambda(x))^t = B_L^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then, if K' is the least common multiple of all determinants of full rank submatrices of (L, I) , by Cramer's rule, the vector $(K'u(x), K'\lambda(x))$ has integer coordinates. Moreover, since the original variables u_j in $(DLP_{\lambda(x)})$ are bounded above by ub_j^D for all $j = 1, \dots, m$ and $\lambda_s \leq 1$ for all $s = 1, \dots, k$ we obtain that

$$K'u(x)_j \in [0, K'ub_j^D] \cap \mathbb{Z}, \forall j = 1, \dots, m, \quad K'\lambda_s \in [0, K'] \cap \mathbb{Z}, \forall s = 1, \dots, k. \tag{12}$$

Now, let us consider (x, u, λ) to be one of such valid triplets characterized above, namely (u, λ) is an extreme solution of (9), (10), (11). Recall that we have proven that this triplet must also satisfy (12). Next, let $(\hat{x}, \hat{u}, \hat{\lambda})$ be the vector such that $\hat{x} = (Kx_1, \dots, Kx_n)$, $\hat{u} = (K'u_1, \dots, K'u_m)$ and $\hat{\lambda} = (K'\lambda_1, \dots, K'\lambda_k)$. It follows that $(\hat{x}, \hat{u}, \hat{\lambda})$ is a solution of system Sys_2 .

Indeed, Eqs. (1), (2) and (3) follow, respectively, from the fact that (x, u, λ) is a valid triplet and thus, $\sum_{s=1}^k \lambda_s = 1$, $u^t(b - Ax) = 0$ and $(\sum_{s=1}^k \lambda_s c^s - u^t A)x = 0$ (see system Sys_1). Inequality (4) follows because x must be a feasible solution of LP_{λ} and therefore it satisfies $Ax - b \geq 0$. Inequality (5) follows from (11). Equation (6) follows because we have proven above that Kx has integer coordinates in the range $[0, ub^P K]$. Finally, Eqs. (7) and (8) follow from (12).

Conversely, it follows by checking the inequalities that any solution $(x, u(x), \lambda(x))$ of Sys_2 defines a valid triplet $(x/K, u(x)/K', \lambda(x)/K')$. Therefore, by Lemma 2 and 3 x/K is a Pareto-optimal solution of MOLP. Finally, we have proven above that all Pareto-optimal extreme point solutions of MOLP are among the solutions of the system Sys_2 which concludes the proof. \square

The above transformation makes use of upper bounds to ensure that extreme points of all systems of rational inequalities that come from feasibility and optimality conditions of valid triplets are integer. In most cases, those bounds can be strengthened taking advantage of the particular structure of the problems giving rise to sharper bounds. In particular if the primal, the dual or both are integer polytopes one can take $K = 1$ or $K' = 1$, or both $K = K' = 1$; respectively. Eventually, we will need only to transform the range of the lambda variables to make them integer.

To finish this section, we emphasize the meaning of Theorem 5. We have proven that the entire set of Pareto-optimal extreme points is encoded in the set of solutions of one system of polynomial inequalities, projecting the first n components of its solutions (note that this set may include some extra Pareto-optimal solutions not being extreme points). The finiteness of the solution sets of these systems is crucial to develop an exact method to get the entire Pareto-optimal set, as we will show in the next section.

4 Semidefinite Programming versus Multiobjective Linear Programming

In this section we describe how to obtain the Pareto-optimal set of any MOLP by applying tools borrowed from the theory of moments and SDP. We use standard notation in the field (see e.g. [15]).

We denote by $\mathbb{R}[x, u, \lambda]$ the ring of real polynomials in the variables $x = (x_1, \dots, x_n)$, $u = (u_1, \dots, u_m)$, $\lambda = (\lambda_1, \dots, \lambda_k)$; and by $\mathbb{R}[x, u, \lambda]_d \subset \mathbb{R}[x, u, \lambda]$ the space of polynomials of degree at most $d \in \mathbb{N}$ (here \mathbb{N} denotes the set of nonnegative integers). We also denote by $\mathcal{B} = \{x^\alpha u^\beta \lambda^\gamma : (\alpha, \beta, \gamma) \in \mathbb{N}^{n \times m \times k}\}$ a canonical basis of monomials for $\mathbb{R}[x, u, \lambda]$, where $x^\alpha u^\beta \lambda^\gamma = x_1^{\alpha_1} \dots x_n^{\alpha_n} u_1^{\beta_1} \dots u_m^{\beta_m} \lambda_1^{\gamma_1} \dots \lambda_k^{\gamma_k}$, for any $(\alpha, \beta, \gamma) \in \mathbb{N}^{n \times m \times k}$.

For any real sequence, $\mathbf{y} = (y_{\alpha\beta\gamma}) \subset \mathbb{R}$ indexed in the canonical monomial basis \mathcal{B} , let $L_{\mathbf{y}} : \mathbb{R}[x, u, \lambda] \rightarrow \mathbb{R}$ be the linear functional defined, for any $f = \sum_{\alpha\beta\gamma \in \mathbb{N}^{n \times m \times k}} f_{\alpha\beta\gamma} x^\alpha u^\beta \lambda^\gamma \in \mathbb{R}[x, u, \lambda]$, as $L_{\mathbf{y}}(f) := \sum_{\alpha\beta\gamma \in \mathbb{N}^{n \times m \times k}} f_{\alpha\beta\gamma} y_{\alpha\beta\gamma}$.

The *moment* matrix $M_d(\mathbf{y})$ of order d associated with \mathbf{y} , has its rows and columns indexed by $(x^\alpha u^\beta \lambda^\gamma)$ and $M_d(\mathbf{y})(\alpha\beta\gamma, \alpha'\beta'\gamma') := L_{\mathbf{y}}(x^{(\alpha+\alpha')} u^{(\beta+\beta')} \lambda^{(\gamma+\gamma')}) = y_{(\alpha+\alpha')(\beta+\beta')(\gamma+\gamma')}$, for $|\alpha\beta\gamma|, |\alpha'\beta'\gamma'| \leq d$.

For $g := \sum_{\delta \in \mathbb{N}^{n \times m \times k}} g_\delta (xu\lambda)^\delta \in \mathbb{R}[x, u, \lambda]$, the *localizing* matrix $M_d(g, \mathbf{y})$ of order d associated with \mathbf{y} and g , has its rows and columns indexed by $((xu\lambda)^\delta)$ and $M_d(g, \mathbf{y})(\alpha\beta\gamma, \alpha'\beta'\gamma') := L_{\mathbf{y}}(x^{(\alpha+\alpha')} u^{(\beta+\beta')} \lambda^{(\gamma+\gamma')} g(xu\lambda)) = \sum_{\delta \in \mathbb{N}^{n \times m \times k}} g_\delta y_\delta + (\alpha + \alpha')(\beta + \beta')(\gamma + \gamma')$, for $|\alpha\beta\gamma|, |\alpha'\beta'\gamma'| \leq d$.

Let us consider $J = \langle h_0^0, h_1^0, h_2, p_1, \dots, p_n, q_1, \dots, q_m, t_1, \dots, t_k \rangle$ the zero-dimensional ideal in $\mathbb{R}[x, u, \lambda]$, generated by the polynomial equations defining the System Sys₂. Note that the ideal J to be zero dimensional is equivalent to suppose that there is a finite number of solutions of the set of polynomial equations defining the system. This condition is ensured by Theorem 4. This set of real solutions is denoted by $V(J)$ and it is usually called the variety of J .

Depending on their parity, let $2\zeta_j$ or $2\zeta_j - 1$ be the degree of p_j , $j = 1, \dots, n$; $2\eta_s$ or $2\eta_s - 1$ be the degree of q_s , $s = 1, \dots, m$ and $2\nu_r$ or $2\nu_r - 1$ be the degree of t_r , $r = 1, \dots, k$. Recall that these functions were defined associated with some of the constraints that appear in Sys₂. Finally, for any symmetric matrix P by $P \geq 0$ we refer to P being positive semidefinite.

With this notation, we are in position to present our next result.

Theorem 5 *The entire set of Pareto-optimal extreme point solutions of MOLP is encoded in the optimal solutions, $\mathbf{y} = (y_{\alpha\beta\gamma}) \subset \mathbb{R}$, of the semidefinite program $SDP - N^*$, for some $N^* \in \mathbb{N}$.*

$$\begin{aligned} \min \quad & y_0 := 1 && (SDP-N^*) \\ \text{s.t.} \quad & M_{N^*}(\mathbf{y}) \succeq 0, \\ & M_{N^*-1}(h_0^0 \mathbf{y}) = 0, \\ & M_{N^*-1}(h_1^0 \mathbf{y}) = 0, \end{aligned}$$

$$M_{N^*-1}(h_2\mathbf{y}) = 0,$$

$$M_{N^*-1}(g_s^0\mathbf{y}) \geq 0, \quad s = 1, \dots, m,$$

$$M_{N^*-1}(g_j\mathbf{y}) \geq 0, \quad j = 1, \dots, n,$$

$$M_{N^*-\zeta_j}(p_j\mathbf{y}) = 0, \quad j = 1, \dots, n, \tag{13}$$

$$M_{N^*-\eta_s}(q_s\mathbf{y}) = 0, \quad s = 1, \dots, n, \tag{14}$$

$$M_{N^*-\nu_r}(t_r\mathbf{y}) = 0, \quad r = 1, \dots, k. \tag{15}$$

Proof Theorem 4 proves that the entire set of Pareto-optimal extreme point solutions of MOLP can be obtained as the projection of the solutions of System Sys₂. This system is defined by a closed, compact semi-algebraic set with finitely many feasible solutions that by finiteness satisfies the Archimedean condition. Recall that the Archimedean condition is equivalent to impose that for some $L > 0$ the quadratic polynomial $L - \sum_{\ell=1}^n x_\ell^2 - \sum_{j=1}^m u_j^2 - \sum_{s=1}^n \lambda_s^2 \geq 0$ has a representation in the quadratic module induced by the inequalities defining the feasible region of the problem. Note that since in our case this set is finite we can augment the above quadratic polynomial as a redundant constraints being L large enough and the Archimedean condition trivially holds for Sys₂ (see [15]). In addition, the ideal J satisfies by Theorem 4 that the variety $V(J)$ is finite. Therefore, we satisfy the hypothesis and hence we can apply [15, Theorem 6.1 Part (b)] to conclude that there exists $N^* < +\infty$ so that the solutions of System Sys₂ can be obtained from the solutions $\mathbf{y} = (y_{\alpha\beta\gamma})_{|\alpha\beta\gamma| \leq 2N^*}$ of the SDP problem $SDP - N^*$. \square

We note in passing that the finiteness of N^* is explicit and it can be bounded above by some known constants which depend on the input data. The interested reader is referred to [21] and [26].

As a direct consequence of the equivalence between MOLP and solving a unique SDP (Theorem 5); and the fact that SDP can be solved, among other, by interior point algorithms we conclude the following remark.

Remark 6 All extreme Pareto-optimal solutions of MOLP can be obtained using interior point algorithms applied to the problem $SDP - N^*$.

It is well-known that solving SDP problems is polynomially doable. Therefore, one may conclude from the above reformulation that obtaining the entire Pareto-optimal set of MOLP is polynomial. However, we cannot conclude this result from Theorem 5. The reason being that the dimension of the SDP problem that needs to be solved is not polynomial in the size of the original problem because N^* is finite but its size may be exponential in n, m, k . Therefore, the above approach only gives a pseudo-polynomial approach to obtain the Pareto-optimal set of MOLP programs.

To conclude this section, we want to describe, based on the above results, an explicit methodology to enumerate the real solutions of the above closed semi-algebraic set Sys₂. Note that this is equivalent to enumerate all the optimal solutions of a polynomial optimization problem where the objective is a constant, see Theorem 5. Finally, the reader should observe that, according to Theorem 4, the projections onto the first components of these solutions give all the Pareto-optimal extreme point solutions of MOLP.

In order to do that we first transform w.l.o.g. Sys₂ into an algebraic set, in the standard manner, by simply adding non-positive slack variables $w \in \mathbb{R}^m, z \in \mathbb{R}^n$ to the constraints (4) and (5), respectively. Thus, from now on we assume that all the constraints in the SDP problem $SDP - N^*$ are in equality form.

Let us consider $\hat{J} = \langle h_0^0, h_1^0, h_2, g_1^0, \dots, g_m^0, g_1, \dots, g_n, p_1, \dots, p_n, q_1, \dots, q_m, t_1, \dots, t_k \rangle$ the zero-dimensional ideal in $\mathbb{R}[x, u, \lambda]$, generated by all the polynomial equations defining the System Sys₂.

Let us consider the quotient space $\mathbb{R}[x, u, \lambda]/\hat{J}$ whose elements are the cosets $[f] = \{f + q : q \in \hat{J}\}$ for any $f \in \mathbb{R}[x, u, \lambda]$. Since \hat{J} is zero-dimensional, $\mathbb{R}[x, u, \lambda]/\hat{J}$ is a finite dimensional \mathbb{R} -vector space with the usual addition and scalar product. Furthermore, $\mathbb{R}[x, u, \lambda]/\hat{J}$ is an algebra with multiplication $[f][g] = [fg]$. Given an arbitrary polynomial $h \in \mathbb{R}[x, u, \lambda]$ define the linear multiplication operator $m_h : \mathbb{R}[x, u, \lambda]/\hat{J} \rightarrow \mathbb{R}[x, u, \lambda]/\hat{J}$, $m_h([f]) = [fh]$ for any $f \in \mathbb{R}[x, u, \lambda]$. If \mathcal{B}_j is a basis of $\mathbb{R}[x, u, \lambda]/\hat{J}$, let \hat{M}_h be the multiplication matrix associated with the linear operator m_h expressed in the basis \mathcal{B}_j . Multiplication matrices are instrumental for obtaining the points in $V(\hat{J})$. Indeed, if $\mathcal{B}_j = \{b_1, \dots, b_N\}$ for each $v \in V(\hat{J})$ let $r_v := (b_\ell(v))_{1 \leq \ell \leq N} \in \mathbb{R}^N$ be the evaluation of the point v by the polynomials that define the basis \mathcal{B}_j then Theorem 6.2 in [15] states that for any polynomial $h \in \mathbb{R}[x, u, \lambda]$ the set $\{h(v) : v \in V(\hat{J})\}$ is the set of eigenvalues of the multiplication matrix \hat{M}_h and $\hat{M}_h r_v = h(v)r_v$ for all $v \in V(\hat{J})$. The reader may observe that for adequate choices of polynomials $h := x_\ell$, $\ell = 1, \dots, n$, $h := u_j$, $j = 1, \dots, m$ and $h := \lambda_s$, $s = 1, \dots, k$ one would get the coordinates of all the points $v = (x, u, \lambda) \in V(\hat{J})$.

Let $d = \max\{\max_{j=1 \dots n} \zeta_j, \max_{s=1 \dots m} \eta_s\}$ be the maximum half degree of the polynomials defining System Sys₂. Finally, let us denote by $R(SDP - N^*)$ the feasible region of Problem $SDP - N^*$.

The above concepts allows us to apply the moment matrix algorithm [15, Algorithm 6.1] to the variety $V(\hat{J})$. See [17] for further details.

Theorem 7 For N^* large enough, there exists $d \leq t \leq N^*$ such that:

$$\text{rank}M_t(\mathbf{y}) = \text{rank}M_{t-d}(\mathbf{y}) = |V(\hat{J})|, \quad \mathbf{y} = (y_{\alpha\beta\gamma})_{|\alpha\beta\gamma| \leq 2N^*} \in R(SDP - N^*).$$

Moreover, one can obtain the coordinates of all $(x, u, \lambda) \in V(\hat{J})$, as the eigenvalues of multiplication matrices \hat{M}_{x_ℓ} , \hat{M}_{u_j} , \hat{M}_{λ_s} for all $\ell = 1, \dots, n$, $j = 1, \dots, m$, $s = 1 \dots, k$.

Proof First of all, we observe that for any N , $SDP - N$ is a SDP relaxation of Sys₂ based on the Problem of the Moment approach [15]. Thus, for any N the solutions of $SDP - N$, namely $(y_{\alpha\beta\gamma})_{|\alpha\beta\gamma| \leq 2N}$, are moments of finite probability measures with support contained in the feasible solutions of System Sys₂ [15]. This system is defined by a closed, compact semi-algebraic set with finitely many feasible solutions (see Theorem 4) that by finiteness satisfies the Archimedean condition. Hence, we are under the hypothesis of Theorem [15, Theorem 6.1] (System Sys₂ satisfies Archimedean condition and it has a finite number of feasible solutions) therefore for sufficiently large N^* the relaxation $SDP - N^*$ is exact and $\text{rank}(M_{N^*}(\mathbf{y})) = \text{rank}(M_{N^*-d}(\mathbf{y}))$.

Next, we translate the condition that we look for measures which give positive probability to all points in $V(\hat{J})$, on the variables of Problem $SDP - N^*$. In this way we shall get the points in $V(\hat{J})$ from the solutions \mathbf{y} of that problem after some manipulation. This extraction can be done in our case, since $V(\hat{J})$ is finite. Note that we are under the hypothesis of [15, Theorem 6.5] since the range of the moment matrix $M_{N^*}(\mathbf{y})$ in any generic solution of $SDP - N^*$ is maximal. Indeed, among all the finite probability measures defined on the support of the feasible set, namely $V(\hat{J})$, those assigning positive probability to all the solutions of $V(\hat{J})$ give maximal range to the moment matrix $M_{N^*}(\mathbf{y})$. Recall that the range of a moment matrix is equal to the number of optimal solutions of the original problem

that are identified by the probability measure defined by the moment variables \mathbf{y} . Obviously, this number cannot be greater than the cardinality of $V(\hat{J})$ and it is attained in any generic solution. (The interested reader is referred to Section 6.3 in [15] for a non technical explanation).

Therefore, applying Theorem [15, Theorem 6.5] there exist $d \leq t \leq N^*$ (eventually t could be equal to N^*) such that the rank condition $\text{rank}M_t(\mathbf{y}) = \text{rank}M_{t-d}(\mathbf{y}) = |V(\hat{J})|$ holds.

Now, if the rank condition holds Theorem 6.5 in [15] states that the ideal $J_0 := \langle \text{Ker}M_t(\mathbf{y}) \rangle$ (the ideal generated by the kernel of the matrix $M_t(\mathbf{y})$) coincides with the vanishing ideal $I(V(\hat{J})) := \{h : h(v) = 0, \forall v \in V(\hat{J})\}$ of the variety $V(\hat{J})$ and any basis of the column space of $M_{t-1}(\mathbf{y})$ is a basis of $\mathbb{R}[x, u, \lambda]/J_0$.

Now, using that basis we can construct, in the vector space $\mathbb{R}[x, u, \lambda]/J_0$, the linear multiplication matrices $\hat{M}_{x_\ell}, \hat{M}_{u_j}, \hat{M}_{\lambda_s}$ for all ℓ, j, s , from the matrix $M_{t-1}(\mathbf{y})$. This construction gives us, according to Theorem 6.2 in [15], the coordinates of all the points $(x, u, \lambda) \in V(\hat{J})$ as the eigenvalues of the above mentioned multiplication matrices $\hat{M}_{x_\ell}, \hat{M}_{u_j}, \hat{M}_{\lambda_s}$ for all ℓ, j, s .

Moreover, obtaining the solutions in $V(\hat{J})$ can be effectively implemented by using the moment matrix algorithm as described in [15, Algorithm 6.1]. (The reader is referred to [17, 18] for further details on the moment matrix algorithm.) □

The importance of Theorem 7 stems from the fact that it provides a constructive method to obtain the Pareto-optimal solutions that are extreme points of the feasible region of MOLP. It states that there exists a finite relaxation order N^* such that the problem $SDP - N^*$ satisfies the rank condition and so all the Pareto-optimal extreme point solutions can be obtained from the moment matrices of $SDP - N^*$ using the extraction algorithm. This can be seen as projecting the solutions of $SDP - N^*$ onto the original x, u, λ variables of MOLP. This methodology is for instance implemented in the open source software Gloptipoly 3 [13].

The following example, that appears in [6], illustrates the methodology proposed in this paper.

Example 8 Consider problem MOLP with the data:

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 3 \\ 4 \\ -5 \\ -5 \end{pmatrix}.$$

Observe that the last two constraints refer to the upper bound constraints $x_1 \leq 5$ and $x_2 \leq 5$, so they are not considered as rows of the matrix A but as the sets of upper bounds in the polynomial constraints $p_j(x)$ in Theorem 4. Moreover, by the form of $DL P_\lambda$ we can use $ub_i^D = 1$ for $i = 1, 2, 3$ as valid upper bounds for the variables in the dual problems.

The System (Sys₂) is:

$$\begin{aligned}
 & \left. \begin{aligned}
 & h_1^0 = u_1(24 - 2x_1 - x_2) + u_2(18 - x_1 - x_2) + u_3(24 - x_1 - 2x_2) = 0 \\
 & h_2^1 = (-2u_1 - u_2 - u_3 + \lambda_1)x_1 + (-u_1 - u_2 - 2u_3 + \lambda_2)x_2 = 0 \\
 & g_1^0 = 2x_1 + x_2 - 24 \geq 0 \\
 & g_2^0 = x_1 + x_2 - 18 \geq 0 \\
 & g_3^0 = x_1 + 2x_2 - 24 \geq 0 \\
 & g_1^1 = -2u_1 - u_2 - u_3 + \lambda_1 \geq 0 \\
 & g_2^1 = -u_1 - u_2 - 2u_3 + \lambda_2 \geq 0 \\
 & p_1 = \prod_{\ell=1}^{30} (x_1 - \ell) = 0 \\
 & p_2 = \prod_{\ell=1}^{30} (x_2 - \ell) = 0 \\
 & q_1 = \prod_{\ell=1}^6 (u_1 - \ell) = 0 \\
 & q_2 = \prod_{\ell=1}^6 (u_2 - \ell) = 0 \\
 & q_3 = \prod_{\ell=1}^6 (u_3 - \ell) = 0 \\
 & t_1 = \prod_{\ell=1}^6 (\lambda_1 - \ell) = 0 \\
 & t_2 = \prod_{\ell=1}^6 (\lambda_2 - \ell) = 0
 \end{aligned} \right\} \text{(Sys}_2\text{)}
 \end{aligned}$$

We use Gloptipoly 3 [13] to formulate the semidefinite problems of Theorem 5 induced from the above systems and SEDUMI 1.3 [28] as the SDP solver. According to Theorem 5, we must solve one semidefinite problem for each relaxation order N^* . The greatest common divisor of the determinants of all the full rank submatrices of A is $K = 6$. Analogously, we get that $K' = 6$ and for $N^* = 4$, the rank condition of Theorem 7 is satisfied, i.e. $\text{rank}M_4(x, u, \lambda) = \text{rank}M_1(x, \mu, \lambda) = 6$, and we extract the following solutions of $SDP - N^*$:

	Solutions		
	x	u	λ
Sol. #1	(6, 12)	(2, 0, 0)	(4, 2)
Sol. #2	(0, 24)	(2, 0, 0)	(4, 2)
Sol. #3	(6, 12)	(0, 3, 0)	(3, 3)
Sol. #4	(12, 6)	(0, 3, 0)	(3, 3)
Sol. #5	(12, 6)	(0, 0, 3)	(3, 3)
Sol. #6	(24, 0)	(0, 0, 2)	(2, 4)

Note that the number of moments involved in the SDP problem that had to be solved was 6435. In this problem, the moment matrix $M_{N^*}(x, u, \lambda)$ has size 330×330 .

Thus, projecting the set of extracted solutions onto the x -coordinates and dividing by K , we get the set of extreme Pareto-optimal solutions of the problem, $X_E = \{(4, 0), (1, 2), (2, 1), (0, 4)\}$. These Pareto-optimal solutions and the complete Pareto-optimal set are shown in Fig. 1 (black dots and black segments, respectively).

5 Conclusions

Due to the similarities between standard linear programming and Multiobjective Linear Programming several authors wondered whether it would exist a non-active-set method valid

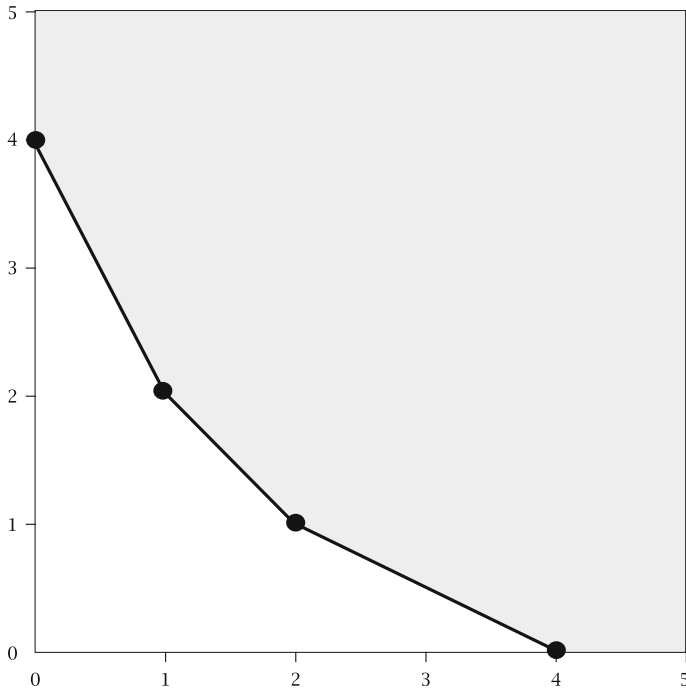


Fig. 1 Pareto-optimal set of Example 8

to find the entire set of Pareto-optimal solutions of MOLP. This paper answers positively this question presenting a SDP approach to find that set. Our approach is constructive and we give an explicit SDP problem the solutions of which encode all the Pareto-optimal extreme points of MOLP. Moreover, we show how all these points can be obtained by applying the so called moment matrix algorithm (see [15, 18]). However, although our construction is explicit, we do not claim that it is computationally competitive with other currently available methods. The main drawback is the size of the SDP problem to be considered which is not polynomial in the input size of MOLP. The importance of our results is mainly theoretical because they show, as expected, the close relationship between scalar and Multiobjective Linear Programming with regards to the solutions techniques. In addition, our results also show the power of some techniques developed in the field of polynomial optimization [15, 16] to be applied in apparently different areas such as Multiobjective Optimization.

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