Small-time controllability of nonlinear parabolic evolution equations with bilinear controls

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Outline

1. Introduction to bilinear control problems

2. Controllability in small time of nonlinear parabolic problems
2.1 Setting and local well-posedness
2.2 Global approximate controllability in small time
2.3 Local exact controllability
2.4 Global small time exact controllability

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1. Introduction to bilinear control problems

2. Controllability in small time of nonlinear parabolic problems

Different kinds of control systems

Boundary control:

$$\begin{cases} u' = Au + Bu \\ u = \mathbf{p}|_{\partial\Omega} \\ u(0) = 0 \end{cases}$$

Locally distributed control:

$$\begin{cases} u' = Au + Bu + \mathbf{p}\mathbb{1}_\omega \\ u = g|_{\partial\Omega} \\ u(0) = 0 \end{cases}$$

Multiplicative control:

$$\begin{cases} u' = Au + \mathbf{p}Bu \\ u = g|_{\partial\Omega} \\ u(0) = u_0 \end{cases}$$

Different kinds of control systems

The map $\Phi : \mathbf{p} \mapsto u$ is

Boundary control:

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$\downarrow$

linear

Locally distributed control:

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Bilinear control systems

Multiplicative control:

$$\begin{cases} u' = Au + \mathbf{p}Bu \\ u = g|_{\partial\Omega} \\ u(0) = u_0 \end{cases} \qquad \text{(BCS)}$$

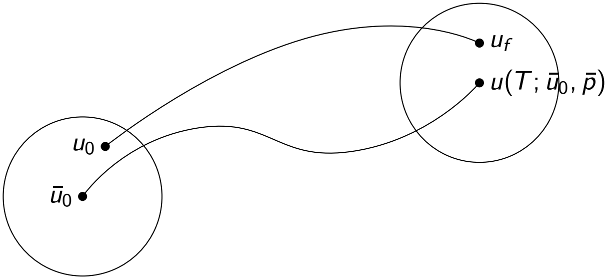
Theorem (Ball, Marsden, Slemrod 1982)

Let X be a Banach space with $\dim(X)=+\infty$. Let A generate a C^0-semigroup of bounded linear operators on X and $B : X \to X$ be a bounded linear operator. Let $u_0 \in X$ be fixed, and let $u(t; p, u_0)$ denote the unique solution of (BCS) for $p \in L^1_{loc}([0, +\infty), \mathbb{R})$. The set of states accessible from u_0 defined by

$$S(u_0) = \{u(t; p, u_0); t \geq 0, p \in L^r_{loc}([0, +\infty), \mathbb{R}), r > 1\}$$

is contained in a countable union of compact subsets of X and, in particular, has a dense complement.

Bilinear control systems



Bilinear control problem (BCP):

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Literature on exact bilinear controllability of hyperbolic pbms

Exact controllability of bilinear **hyperbolic** equations (nonexhaustive list):

- K. Beauchard, C. Laurent. "Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control." J. de Math. Pures et Appl. (2010)

  $\rightsquigarrow$ controllability in $H^3_{(0)}(0,1)$
- K. Beauchard "Local controllability and non-controllability for a 1D wave equation with bilinear control." J. of Diff. Eq. (2011)

  $\rightsquigarrow$ controllability in $H^3_{(0)}(0,1) \times H^2_{(0)}(0,1)$
- M. Morancey. "Simultaneous local exact controllability of 1D bilinear Schrödinger equations." Ann. de l'Inst. Henri Poincare (C) Non Linear Analysis. (2014)

  $\rightsquigarrow$ controllability in $(H^3_{(0)}(0,1))^N$
- A. Duca. "Global exact controllability of bilinear quantum systems on compact graphs and energetic controllability." SIAM J. on Contr. and Opt. (2020)

  $\rightsquigarrow$ controllability in $H^{2+d}_{\mathcal{G}}$
- P. Cannarsa, P. Martinez, C Urbani. "Bilinear control of a degenerate hyperbolic equation." SIAM J. of Math. An., vol. 55, n. 6, pp 6517–6553 (2023)

  $\rightsquigarrow$ controllability in $H^3_{(\alpha)}(0,1) \times H^2_{(\alpha)}(0,1)$

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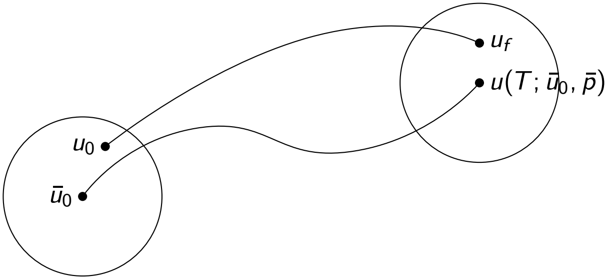

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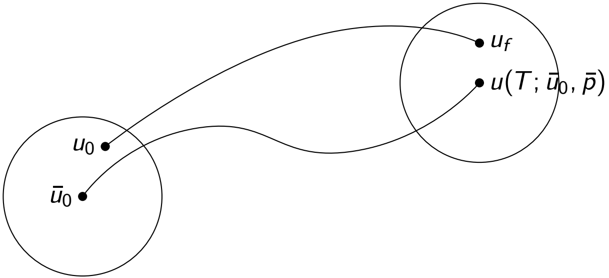

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Setting of the problem

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$$\begin{cases} \partial_t \psi(t,x) = \Delta\psi(t,x) - \kappa\psi^{p+1}(t,x) + \langle u(t), Q(x)\rangle \psi(t,x), \quad x \in \mathbb{T}^d,\ t > 0, \\ \psi(0,x) = \psi_0(x), \end{cases} \tag{NHE}$$

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- $p \in \mathbb{N}$, $\kappa \in \mathbb{R}$
- $\boldsymbol{Q} = (Q_1, \dots, Q_q, \mu_1, \mu_2) : \mathbb{T}^d \to \mathbb{R}^{q+2}$ potentials, $q \in \mathbb{N}$, $q \geq 2d+1$,

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More precisely...

Main results

Proposition (Duca, Pozzoli, Urbani 2025)

(assumptions)

(i) Let $\psi_0, \psi_1 \in H^s(\mathbb{T}^d, \mathbb{R})$ be such that $\mathrm{sign}(\psi_0) = \mathrm{sign}(\psi_1)$. For any $\epsilon > 0$ and $T > 0$, there exist $\tau \in (0, T]$ and $u \in L^2((0,\tau), \mathbb{R}^{q+2})$ such that the solution $\psi(t; \psi_0, u)$ of (NHE) satisfies

$$\|\psi(\tau; \psi_0, u) - \psi_1\|_{L^2} < \epsilon$$

(ii) Let $\psi_0, \psi_1 \in H^s(\mathbb{T}^d, \mathbb{R})$ be such that $\psi_0, \psi_1 > 0$ (or $\psi_0, \psi_1 < 0$). For any $\epsilon > 0$ and $T > 0$, there exists $u \in L^2((0, T), \mathbb{R}^{q+2})$ such that the solution $\psi(t; \psi_0, u)$ of (NHE)

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Theorem (Duca, Pozzoli, Urbani 2025)

(assumptions), then (NHE-1D) is locally exactly controllable to the ground state solution c_0 in any positive time: for any $T > 0$ there exists $R_T > 0$ such that, for any

$$\psi_0 \in \{\psi \in H^3(\mathbb{T}, \mathbb{R}) : \|\psi - c_0\|_{H^1} < R_T\},$$

there exists $u \in H^1((0, T), \mathbb{R}^{q+2})$ such that $\psi(T; \psi_0, u) = c_0$.

Local well-posedness

Proposition (D-P-U 2025)

Let $s > d/2$ and $Q \in H^s(\mathbb{T}^d, \mathbb{R}^{q+2})$. For any $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$ and $u \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^{q+2})$ there exists a maximal time $\mathcal{T} = \mathcal{T}(\psi_0, u) > 0$ and a unique mild solution $\psi \in C^0([0,T], H^s(\mathbb{T}^d, \mathbb{R}))$, $\forall\, T < \mathcal{T}$, of (NHE) represented by

$$\psi(t; \psi_0, u) = e^{t\Delta}\psi_0 + \int_0^t e^{(t-s)\Delta}\left(\langle u(s), Q(x)\rangle \psi(s,x) - \kappa\psi(s,x)^{p+1}\right) ds.$$

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i. if $\psi_0, \phi_0 \in B_{H^s(\mathbb{T}^d,\mathbb{R})}(0, R)$, with $R > 0$, and $u, v \in L^2_{loc}(\mathbb{R}^+, \mathbb{R}^{q+2})$, then for any $0 \le T \le \min\{\mathcal{T}(\psi_0, u), \mathcal{T}(\phi_0, v)\}$, there exists $C = C(u, v)$ such that
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ii. set $K = \|\psi\|_{C([0,T],H^s)} + \|\psi_0\|_{H^s} + \|u\|_{L^2}$. There exists $\delta = \delta(\mathcal{T}(\psi_0,u), K) > 0$ such that, for any $\hat{\psi}_0 \in H^s(\mathbb{T}^d, \mathbb{R})$ and $\hat{u} \in L^2((0,T), \mathbb{R}^{q+2})$ satisfying
$$\|\hat{\psi}_0 - \psi_0\|_{H^s} + \|\hat{u} - u\|_{L^2} < \delta,$$
problem (NHE) admits a unique mild solution $\hat{\psi} \in C\left([0,T], H^s(\mathbb{T}^d,\mathbb{R})\right)$ with initial condition $\hat{\psi}_0$ and control $\hat{u}$.

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Small-time limit of conjugated dynamics

Define the non-linear operator

$$\mathbb{B}(\varphi)(x) = \sum_{j=1}^{d} \left(\partial_{x_j}\varphi(x)\right)^2, \qquad \forall\, \varphi \in C^1(\mathbb{T}^d, \mathbb{R}).$$

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Corollary (D-P-U 2025)

Let $s > d/2$ and $1 \in \mathrm{span}\{Q_1, \dots, Q_q\}$. Let $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$. For any $\epsilon, T > 0$ there exists a constant control $u \in \mathbb{R}^{q+2}$ such that the solution $\psi(t; \psi_0, u)$ of (NHE) is defined in $[0, T]$ and

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Apply the limit of conjugate dynamics with $\varphi = 0$ and $-c = \sum_{j=1}^{q} u_j Q_j$ small enough.

An intermediate controllability result

Given $Q_1, \ldots, Q_q \in C^\infty(\mathbb{T}^d, \mathbb{R})$, $q \in \mathbb{N}^*$, define the vector space

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Define $\mathcal{H}_j$, $j \in \mathbb{N}^*$, as the largest vector space whose elements ψ can be written as

$$\psi = \varphi_0 + \sum_{k=1}^{n} \mathbb{B}(\varphi_k), \qquad \varphi_0, \dots, \varphi_n \in \mathcal{H}_{j-1}, \quad n \in \mathbb{N},$$

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An intermediate controllability result

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Proposition (D-P-U 2025)

Let $s > d/2$ and $(Q_1, ..., Q_q) \in C^\infty(\mathbb{T}^d, \mathbb{R}^q)$ be such that $1 \in \mathcal{H}_0$. Assume that $\mathcal{H}_\infty$ is dense in $H^s(\mathbb{T}^d, \mathbb{R})$. Let $\psi_0 \in H^s(\mathbb{T}^d, \mathbb{R})$ and $\varphi \in H^s(\mathbb{T}^d, \mathbb{R})$. For any $\epsilon, T > 0$, there exist $\tau \in [0, T)$ and $(u_1, ..., u_q) \in L^2((0, \tau), \mathbb{R}^q)$ such that the solution $\psi(t; \psi_0, u)$ of (NHE) with control $u = (u_1, ..., u_q, 0, 0)$ is defined in $[0, \tau]$ and

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Small time global approximate controllability

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Let $s > d/2$ and let $(Q_1, \dots, Q_q) \in C^\infty(\mathbb{T}^d, \mathbb{R}^q)$ be such that $1 \in \mathcal{H}_0$ and $\mathcal{H}_\infty$ is dense in $H^s(\mathbb{T}^d, \mathbb{R})$.

(i) Let $\psi_0, \psi_1 \in H^s(\mathbb{T}^d, \mathbb{R})$ be such that $\mathrm{sign}(\psi_0) = \mathrm{sign}(\psi_1)$. For any $\epsilon > 0$ and $T > 0$, there exist $\tau \in (0, T]$ and $(u_1, \dots, u_q) \in L^2((0, \tau), \mathbb{R}^q)$ for which the solution $\psi(t; \psi_0, u)$ of (NHE) with control $u = (u_1, \dots, u_q, 0, 0)$ is defined in $[0, \tau]$ and satisfies

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Idea of the proof of point (i):

- define $Z = \{x \in \mathbb{T}^d \,:\, \psi_0(x) = \psi_1(x) = 0\}$ and $Z_\eta = \{x \in \mathbb{T}^d \,:\, \mathrm{dist}(x, Z) < \eta\}$

Small time global approximate controllability

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- define $\phi_\eta = \mathbb{1}_{Z_\eta^c} \log(\psi_1/\psi_0) \in \boldsymbol{L^2(\mathbb{T}^d)}$
- observe that $\|e^{\phi_\eta}\psi_0 - \psi_1\|_{L^2(\mathbb{T}^d)} \le \|\psi_0 - \psi_1\|_{L^2(Z_\eta)} < \frac{\varepsilon}{3}$ for η small enough

Small time global approximate controllability

- observe that $H^1(\mathbb{T}^d)$ is dense in $L^2(\mathbb{T}^d) \Longrightarrow \exists\, \tilde{\phi}_\eta \in H^1(\mathbb{T}^d)$ such that $\|e^{\tilde{\phi}_\eta}\psi_0 - e^{\phi_\eta}\psi_0\|_{L^2(\mathbb{T}^d)} < \frac{\varepsilon}{3}$

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Where does this method come from?

Small time global approximate controllability

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□

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- Jurdjevic, Kupka "*Polynomial control systems*", 1985 **(finite dimensional control systems)**
- Agrachev, Sarychev "*Navier-Stokes equations: controllability by means of low modes forcing*", 2005 **(infinite dimensional control systems)**
- Agrachev, Sarychev "*Controllability of 2D Euler and Navier-Stokes equations by degenerate forcing*", 2006 **(infinite dimensional control systems)**
- Shirikyan "*Approximate controllability of three-dimensional Navier-Stokes equations*", 2006
- Shirikyan "*Approximate controllability of the viscous Burgers equation on the real line*", 2014
- Glatt-Holtz, Herzog, Mattingly "*Scaling and saturation in infinite-dimensional control problems with applications to stochastic partial differential equations*", 2018
- Nersesyan, "*Approximate controllability of nonlinear parabolic PDEs in arbitrary space dimension*", 2021
- Duca, Nersesyan "*Bilinear control and growth of Sobolev norms for the nonlinear Schrödinger equation*", 2025 **(bilinear controls)**

Small time global approximate controllability

...going back to the density of $\mathcal{H}_\infty$:

Small time global approximate controllability

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Theorem (Duca, Nersesyan 2025)

Assume that
$$\{Q_1, \ldots, Q_q\} = \{1, \sin\langle k, x\rangle), \cos\langle k, x\rangle\}_{k\in L},$$
for some $L \subset \mathbb{Z}^d$. Then, $\mathcal{H}_\infty$ is dense in $H^s(\mathbb{T}^d, \mathbb{R}), s \geq 0$, if and only if

- L is a generator,
- for any $l, m \in L$, there exists $\{n_j\}_{j=1}^r \subset L$ such that $l \not\perp n_1, n_j \not\perp n_{j+1}, j = 1, \ldots, r-1$, and $n_r \not\perp m$.

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In our result we assume that $Q_1, \dots, Q_q \in C^\infty(\mathbb{T}^d, \mathbb{R})$ and

$$\left\{1, \cos\langle k, x\rangle, \ \sin\langle k, x\rangle\right\}_{k \in L} \subset \mathcal{H}_0,$$

with

$$L = \{(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, \dots, 1, 0), (1, \dots, 1)\} \subset \mathbb{Z}^d.$$

Literature

- A. Y. Khapalov, *Global non-negative controllability of the semilinear parabolic equation governed by bilinear control*, ESAIM Control Optim. Calc. Var., 7, 269–283, 2002
- A. Y. Khapalov, *Controllability of partial differential equations governed by multiplicative controls*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1995, 2010
- P. Cannarsa and G. Floridia, *Approximate multiplicative controllability for degenerate parabolic problems with Robin boundary conditions*, Commun. Appl. Ind. Math. 2(2), e-376, 16, 2011
- G. Floridia, *Approximate controllability for nonlinear degenerate parabolic problems with bilinear control*, J. Differ. Equ. 257(9):3382–3422, 2014
- P. Cannarsa, G. Floridia and A. Y. Khapalov, *Multiplicative controllability for semilinear reaction-diffusion equations with target states admitting finitely many changes of sign*, Journal de Mathématiques Pures et Appliquées, 108(4), 25–458, 2017
- G. Floridia, C. Nitsch and C. Trombetti, *Multiplicative controllability for nonlinear degenerate parabolic equations between sign-changing states*, ESAIM Control Optim. Calc. Var., 26(18), 34, 2020

Literature

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Our approximate controllability achieved

- with a scalar input control $u(t)$

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Literature

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- P. Cannarsa and G. Floridia, *Approximate multiplicative controllability for degenerate parabolic problems with Robin boundary conditions*, Commun. Appl. Ind. Math. 2(2), e-376, 16, 2011
- G. Floridia, *Approximate controllability for nonlinear degenerate parabolic problems with bilinear control*, J. Differ. Equ. 257(9):3382–3422, 2014
- P. Cannarsa, G. Floridia and A. Y. Khapalov, *Multiplicative controllability for semilinear reaction-diffusion equations with target states admitting finitely many changes of sign*, Journal de Mathématiques Pures et Appliquées, 108(4), 25–458, 2017
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Our approximate controllability achieved

- with a scalar input control $u(t)$
- in arbitrarily small time
- in arbitrary spatial dimensions, $\mathbb{T}^d$, $d \in \mathbb{N}^*$

Literature

- A. Y. Khapalov, *Global non-negative controllability of the semilinear parabolic equation governed by bilinear control*, ESAIM Control Optim. Calc. Var., 7, 269–283, 2002
- A. Y. Khapalov, *Controllability of partial differential equations governed by multiplicative controls*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1995, 2010
- P. Cannarsa and G. Floridia, *Approximate multiplicative controllability for degenerate parabolic problems with Robin boundary conditions*, Commun. Appl. Ind. Math. 2(2), e-376, 16, 2011
- G. Floridia, *Approximate controllability for nonlinear degenerate parabolic problems with bilinear control*, J. Differ. Equ. 257(9):3382–3422, 2014
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- in presence of a (potentially high power) polynomial nonlinearity

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Properties of the operator and assumptions

➔ 1-dimensional case: $\mathbb{T}$

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Example:

$$\mu_1(x) = x^3 (2\pi - x)^3, \qquad \mu_2(x) = x^3 (x - \pi)^3 (x - 2\pi)^3$$

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We need the solution to be globally (in time) defined:

Proposition (D-P-U 2025)

Let $p \in 2\mathbb{N}$, $\psi_0 \in H^3(\mathbb{T}, \mathbb{R})$, $Q \in H^3(\mathbb{T}, \mathbb{R}^{q+2})$, $u \in H^1_{loc}((0, +\infty), \mathbb{R}^{q+2})$ and $\kappa \geq 0$. Then, for any $T > 0$ there exists a unique mild solution $\psi \in C^0([0, T], H^3(\mathbb{T}, \mathbb{R}))$ of (NHE-1D).

Local controllability to the ground state solution

Theorem (D-P-U 2025)

Let $\kappa \geq 0$ and $p \in 2\mathbb{N}$. Suppose that Assumptions on Q is satisfied. Then, (NHE-1D) is locally exactly controllable to the ground state solution c_0, in any positive time. In other words, for any $T > 0$ there exists $R_T > 0$ such that, for any

$$\psi_0 \in \{\psi \in H^3(\mathbb{T}, \mathbb{R}) : \|\psi - c_0\|_{H^1} < R_T\},$$

there exists $(u_1, u_2) \in H^1((0, T), \mathbb{R}^2)$ such that $\psi(T; \psi_0, u) = c_0$, where $u = (\frac{\kappa}{c_0^p}, 0, ..., 0, u_1, u_2)$.

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Steps of the proof:

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- linearization of the problem
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- proof that the distance of the solution w.r.t. the target is zero a time T

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Global small time controllability to the ground state solution

Theorem (D-P-U 2025)

Let $d = 1$, $\kappa \geq 0$ and $p \in 2\mathbb{N}$. Suppose that $Q_1, \dots, Q_q \in C^\infty(\mathbb{T}, \mathbb{R})$ and

$$\{1, \cos x,\ \sin x\} \subset \mathcal{H}_0.$$

Assume moreover that $Q_1 = 1$, $\mu_1, \mu_2 \in H^3(\mathbb{T}, \mathbb{R})$ and

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Then, (NHE-1D) is exactly controllable to the ground state solution c_0 in any positive time from any positive state. More precisely, for any $T > 0$ and $\psi_0 \in H^3(\mathbb{T}, \mathbb{R})$ such that $\psi_0 > 0$, there exists $u \in L^2((0, T), \mathbb{R}^{q+2})$, such that

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Global small time controllability to the ground state solution

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Example of a suitable potential:

$$Q(x) = (1, \cos x, \sin x, x^3(2\pi - x)^3, x^3(x - \pi)^3(x - 2\pi)^3).$$

References

- **Duca, Pozzoli, Urbani "On the small-time bilinear control of a nonlinear heat equation: global approximate controllability and exact controllability to trajectories", Journal de Mathématique Pures et Appliquées, vol. 203, pp. 103758 (2025)**
- Alabau-Boussouira, Cannarsa, Urbani "Bilinear control of evolution equations with unbounded lower order terms. Application to the Fokker-Planck equation", Comptes Rendus Math., vol. 362, pp. 511–545 (2024)
- Alabau-Boussouira, Cannarsa, Urbani "Exact controllability to eigensolutions for evolution equations of parabolic type via bilinear control, Nonlinear Differential Equations and Applications, vol. 29(38) (2022)
- Cannarsa, Duca, Urbani "Exact controllability to eigensolutions of the bilinear heat equation on compact networks", Discrete and Continuous Dynamical Systems - Series S, vol. 15(6), pages 1377–1401 (2022)
- Alabau-Boussouira, Cannarsa, Urbani "Superexponential stabilizability of evolution equations of parabolic type via bilinear control", Journal of Evolution Equations, vol. 20, Springer (2021)
- Cannarsa, Urbani "Superexponential stabilizability of degenerate parabolic equations via bilinear control", Inverse Problems and Related Topics, vol. 310, pp. 31–45, Springer Singapore (2020)
- Alabau-Boussouira , Urbani "A constructive algorithm for building mixing coupling real valued potentials with time control. Applications to exact controllability and stabilization in bilinear control", preprint (2019)


THANK YOU! GRAZIE!
GRACIAS! MERCI! DANKE!