Global stabilization of a rigid body moving in a compressible viscous fluid

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Contents

Examples: Fluid-Structure interaction

- It is the interaction of some structure with an internal or surrounding fluid flow.
- Structures: Rigid/Elastic.
- Naval and Aerospace engineering: Airflow around an Aircraft, Submarine.
- Biology: Blood flow in arteries, Swimming of fish, micro-organism motion.
- Mathematical challenges: coupled PDE model, presence of strong nonlinearities and free boundaries due to motion of the structure.

Interesting Questions

Analysis on the interaction between Fluid and Structure:

- Existence and uniqueness of solutions (weak and strong).
- Contact issues: structure-boundary or structure-structure.
- Size of the solid goes to zero.
- Long time behaviour.
- Control and Stabilization.

Immersed structure: solids inside fluid

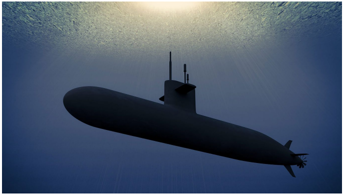

Figure: Submarine

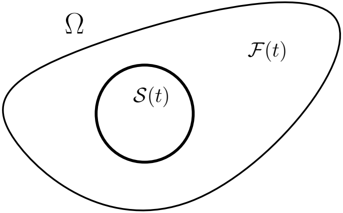

Figure: Rigid body inside a fluid domain

Problem Description

- Ω: Bounded domain in $\mathbb{R}^3$.
- $\mathcal{S}(t)$: the closed rigid ball of radius 1 and of center $h(t)$.
- Fluid domain: $\mathcal{F}(t) = \Omega \setminus \overline{\mathcal{S}(t)}$.
- h_1: fixed anchor point.
- The center of the ball is connected to the fixed anchor point h_1 by a spring and a mechanical damper.
- Initially the ball is away from boundary:

$$\operatorname{dist}(\mathcal{S}(0), \partial\Omega) \geqslant \nu > 0.$$

Mathematical set up: fluid equation

- ρ: density, u: velocity, p: pressure of the fluid.
- The fluid flow is modeled by the compressible Navier-Stokes system $(t > 0,\ x \in \mathcal{F}(t))$:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0,$$

$$\rho \frac{\partial u}{\partial t} + \rho \,(u \cdot \nabla)\, u - \operatorname{div} \sigma(u, p) = 0.$$

- The Cauchy stress tensor is defined as:

$$\sigma(u, p) = 2\mu \mathbb{D}(u) + \alpha \operatorname{div} u \mathbb{I}_3 - p \mathbb{I}_3,$$

- $\mathbb{D}(u) = \frac{1}{2}\left(\nabla u + \nabla u^{\top}\right)$ denotes the symmetric part of the velocity gradient.
- λ, μ are the viscosity coefficients satisfying $\mu > 0, \quad \lambda + \mu \geqslant 0.$
- The flow is isentropic - independent of temperature.
- It is in the barotropic regime where the relation between p and ρ is given by $p = a\rho^{\gamma}$, with $a > 0$ and the adiabatic constant $\gamma > \frac{3}{2}$.

Equation: Rigid ball

- Let $\ell = h'$ and ω be the linear and angular velocities of the rigid body.

- The balance of linear and angular momentum:

$$m\ell' = -\int_{\partial\mathcal{S}(t)} \sigma(u,p)N \, d\Gamma + w,$$

$$J\omega' = -\int_{\partial\mathcal{S}(t)} (x - h(t)) \times \sigma(u,p)N \, d\Gamma,$$

where w is the external control force, $N(t,x)$ is the unit normal to $\partial\mathcal{S}(t)$ at the point $x \in \partial\mathcal{S}(t)$, directed to the interior of the ball.

- If $\rho_{\mathcal{S}}$ is the mass density of the ball, then the formulae for m and J are

$$m = \frac{4}{3}\pi\rho_{\mathcal{S}}, \quad J = \frac{2m}{5}\mathbb{I}_3.$$

Initial and boundary conditions

We impose the no-slip boundary conditions:

- $u(t,x) = 0 \quad t > 0,\ x \in \partial\Omega.$
- Continuity of velocity at the interface

$$u(t,x) = \ell(t) + \omega(t) \times (x - h(t)) \quad t > 0, x \in \partial\mathcal{S}(t).$$

- The initial conditions:

$$\rho(0,\cdot) = \rho_0(\cdot), \quad u(0,\cdot) = u_0(\cdot) \quad \text{in } \mathcal{F}(0),$$
$$h(0) = h_0, \quad \ell(0) = \ell_0, \quad \omega(0) = \omega_0.$$

Feedback Law

The control force w is given by a feedback law:

$$w(t) = k_p(h_1 - h(t)) - k_d\ell(t).$$

- This type of control is known as Proportional-Derivative (PD) controller.
- The spring-damper is connected from the center of mass of the ball to the fixed anchor point h_1 and it is attracting the ball towards the point h_1.
- We want to show $\lim_{t\to\infty} h(t) = h_1$, whereas the velocities of the fluid and of the rigid ball go to 0:

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} h'(t) = 0, \quad \lim_{t\to\infty} \omega(t) = 0.$$

Revisit some results

- T. Takahashi, M. Tucsnak, G. Weiss: 3d case rigid ball-viscous incompressible fluid and obtain a stabilization result by using a spring-damper control.

- A. Roy, T. Takahashi: Rigid ball-compressible fluid. Stabilization result is obtained under a smallness condition on the initial velocities and on the distance between the initial position of the center of the ball and h_1: Local stabilization.

→ Several compatibility conditions (Hilbert space setup)

- Aim: a stabilization result but without the requirement that the initial position of the center of the ball is close to h_1: Global stabilization.

→ Remedy: $L^p - L^q$ setup.

Strategy: Existence

- The domains $\mathcal{F}(t)$, $\mathcal{S}(t)$ are evolving w.r.t time: the domain of the fluid equation is one of the unknowns.

- Introducing Lagrangian variables:

$\rightarrow$ Rewrite the coupled system in a fixed cylindrical domain.

$\rightarrow$ Tackle the term $u \cdot \nabla \rho$ in the density equation.

- Associate a linear problem with the nonlinear one, involving non-homogeneous source terms.

- Establish the maximal L^p regularity property for this linear problem.

- Apriori estimate taking into account the PD controller.

Assumptions

- Conditions on initial data and on (p, q):

$$2 < p < \infty, \qquad 3 < q < \infty, \qquad \frac{1}{p} + \frac{1}{2q} \neq \frac{1}{2},$$
$$h_0 \in \Omega^0, \quad h_1 \in \Omega^0, \quad \ell_0 \in \mathbb{R}^3, \quad \omega_0 \in \mathbb{R}^3,$$
$$\rho_0 \in W^{1,q}(\mathcal{F}(0)), \quad u_0 \in B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))^3, \quad \min_{\mathcal{F}(0)} \rho_0 > 0.$$

- Compatibility conditions

$$u_0 = 0 \text{ on } \partial\Omega \text{ and } u_0(y) = \ell_0 + \omega_0 \times (y - h_0),\ y \in \partial\mathcal{S}(0).$$

- Let us define

$$\Omega^0 := \{x \in \Omega \ ; \ \operatorname{dist}(x, \partial\Omega) > 1\}.$$

- Let $k \in \mathbb{N}$. For every $0 < s < k$, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, define the Besov spaces by real interpolation of Sobolev spaces

$$B^s_{q,p}(\mathcal{F}) = (L^q(\mathcal{F}), W^{k,q}(\mathcal{F}))_{s/k,p}\ .$$

Functional Spaces

- If $T \in (0, \infty]$, we set

$$W^{1,2}_{p,q}((0,T);\mathcal{F}) = L^p(0,T;W^{2,q}(\mathcal{F})) \cap W^{1,p}(0,T;L^q(\mathcal{F})).$$

$$\|\cdot\|_{W^{1,2}_{p,q}((0,T);\mathcal{F})} := \|\cdot\|_{L^p(0,T;W^{2,q}(\mathcal{F}))} + \|\cdot\|_{W^{1,p}(0,T;L^q(\mathcal{F}))} + \|\cdot\|_{C^0_b([0,T);B^{2(1-1/p)}_{q,p}(\mathcal{F}))}.$$

- Functional spaces with time decay: $\beta \in \mathbb{R}$, $p \in [1, \infty]$ and $\mathbb{X}$ a Banach space,

$$L^p_\beta(0,\infty;\mathbb{X}) =: \left\{ f \mid t \to e^{\beta t} f(t) \in L^p(0,\infty;\mathbb{X}) \right\}.$$

- Let $\Lambda(t,\cdot)$ be a C^1-diffeomorphism from $\mathcal{F}(0)$ onto $\mathcal{F}(t)$. For all functions $v(t,\cdot) : \mathcal{F}(t) \to \mathbb{R}$, we denote $\widehat{v}(t,y) = v(t,\Lambda(t,y))$. Then for any $1 < p, q < \infty$ we define

$$L^p(0,T;L^q(\mathcal{F}(\cdot)) = \left\{ v \mid \widehat{v} \in L^p(0,T;L^q(\mathcal{F}(0))) \right\}.$$

Existence result

Let $\overline{\rho} > 0$ be a given constant. Assume that Ω^0 is non empty, connected and $h_1 \in \Omega^0$, w is given by the feedback law with $k_p > 0$, $k_d > 0$. Then there exist $\beta > 0$ and $\delta > 0$, such that, for any $(\rho_0, u_0, h_0, \ell_0, \omega_0)$ satisfying the above conditions, with

$$\overline{\rho} = \frac{1}{|\mathcal{F}(0)|} \int_{\mathcal{F}(0)} \rho_0 \, \mathrm{d}x,$$

$$\|\rho_0 - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(0))} + \|u_0\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))^3} + \|h_0 - h_1\|_{\mathbb{R}^3} + \|\ell_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3} \leqslant \delta,$$

the system admits a unique strong solution $(\rho, u, h, \ell, \omega)$ satisfying

$$\rho \in C^0_b([0,\infty); W^{1,q}(\mathcal{F}(\cdot))), \ \nabla\rho \in W^{1,p}_\beta(0,\infty; L^q(\mathcal{F}(\cdot))),$$
$$u \in W^{1,2}_{p,q,\beta}((0,\infty); \mathcal{F}(\cdot)),$$
$$h - h_1 \in W^{2,p}_\beta(0,\infty; \mathbb{R}^3), \quad \ell \in W^{1,p}_\beta(0,\infty; \mathbb{R}^3), \quad \omega \in W^{1,p}_\beta(0,\infty; \mathbb{R},^3).$$

Moreover, $\rho(t,x) \geqslant \dfrac{\overline{\rho}}{2}$ for all $t \in (0,\infty)$, $x \in \overline{\mathcal{F}(t)}$ and $\mathrm{dist}(\mathcal{S}(t), \partial\Omega) \geqslant \nu/2$ for all $t \in [0,\infty)$.

Result on long time behaviour

Under the assumptions and notation in the above Theorem, we have

$$\|\rho(t,\cdot)-\overline{\rho}\|_{W^{1,q}(\mathcal{F}(t))}+\|u(t,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(t))^3}+\|h(t)-h_1\|_{\mathbb{R}^3}$$
$$+\|\ell(t)\|_{\mathbb{R}^3}+\|\omega(t)\|_{\mathbb{R}^3}\leqslant C\delta e^{-\beta t},$$

where the constant C is independent of $t>0$. In particular,

$$\lim_{t\to\infty}\|\rho(t,\cdot)-\overline{\rho}\|_{W^{1,q}(\mathcal{F}(t))}=0,\quad \lim_{t\to\infty}\|u(t,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(t))^3}=0,$$
$$\lim_{t\to\infty}h(t)=h_1,\quad \lim_{t\to\infty}\ell(t)=0,\qquad \lim_{t\to\infty}\omega(t)=0.$$

Linearized system

Linearise the system around $(\overline{\rho}, 0, 0, 0, 0)$.

$$\begin{aligned}
\partial_t \rho + \overline{\rho} \operatorname{div} u &= f_1 \text{ in } (0,\infty) \times \mathcal{F}, \\
\partial_t u - \operatorname{div} \sigma_l(\rho, u) &= f_2 \text{ in } (0,\infty) \times \mathcal{F}, \\
u &= 0 \text{ on } (0,\infty) \times \partial\Omega, \\
u &= \ell + \omega \times y \text{ on } (0,\infty) \times \partial\mathcal{S}, \\
\frac{\mathrm{d}}{\mathrm{dt}} h &= \ell \text{ in } (0,\infty), \\
\frac{\mathrm{d}}{\mathrm{dt}} \ell + m^{-1} k_p h + m^{-1} k_d \ell &= -m^{-1} \int_{\partial\mathcal{S}} \sigma_l(\rho, u) n \, \mathrm{d}\Gamma + \mathrm{g}_1 \text{ in } (0,\infty), \\
\frac{\mathrm{d}}{\mathrm{dt}} \omega &= -J(0)^{-1} \int_{\partial\mathcal{S}} y \times \sigma_l(\rho, u) n \, \mathrm{d}\Gamma + \mathrm{g}_2 \text{ in } (0,\infty), \\
\rho(0) = \rho_0, \quad u(0) &= u_0, \text{ in } \mathcal{F}, \\
h(0) = h_0 - h_1, \quad \ell(0) &= \ell_0, \quad \omega(0) = \omega_0,
\end{aligned}$$

where

$$\sigma_l(\rho, u) = \frac{2\mu}{\overline{\rho}} \mathbb{D} u + \frac{1}{\overline{\rho}} \left(\lambda \operatorname{div} u - a\gamma(\overline{\rho})^{\gamma-1} \rho \right) I_3, \quad \mathbb{D}(u) = \frac{1}{2}(\nabla u + \nabla u^\top).$$

Fluid-structure operator

Given $(\ell, \omega) \in \mathbb{C}^3 \times \mathbb{C}^3$, let us consider the following problem

$$\begin{cases} -\frac{\mu}{\overline{\rho}}\Delta W - \frac{\alpha+\mu}{\overline{\rho}}\nabla \mathrm{div} W = 0 \text{ in } \mathcal{F}, \\ W = \ell + \omega \times y \text{ on } \partial\mathcal{S}, \quad W = 0 \text{ on } \partial\Omega. \end{cases}$$

We can define the Dirichlet operator

$$D_{\mathsf{s}} \in \mathcal{L}(\mathbb{C}^3 \times \mathbb{C}^3; W^{2,q}(\mathcal{F})^3), \qquad D_{\mathsf{s}}(\ell, \omega) = W.$$

For $q \in (1, \infty)$, let us set

$$\mathcal{X} = W^{1,q}(\mathcal{F}) \times L^q(\mathcal{F})^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3.$$

Fluid-structure operator $\mathcal{A}_{\mathsf{FS}} : \mathcal{D}(\mathcal{A}_{\mathsf{FS}}) \to \mathcal{X}$ defined by

$$\mathcal{D}(\mathcal{A}_{\mathsf{FS}}) = \Big\{ (\rho, u, h, \ell, \omega) \in W^{1,q}(\mathcal{F}) \times W^{2,q}(\mathcal{F})^3 \times \mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^3 ; \\ u - D_{\mathsf{s}}(\ell, \omega) \in \mathcal{D}(A_{\mathsf{u}}) \Big\},$$

where

$$\mathcal{D}(A_{\mathsf{u}}) = W^{2,q}(\mathcal{F}) \cap W^{1,q}_0(\mathcal{F}), \quad A_{\mathsf{u}} = \frac{\mu}{\overline{\rho}}\Delta + \frac{\alpha+\mu}{\overline{\rho}}\nabla \mathrm{div}.$$

$$\mathcal{A}_{\mathsf{FS}} = \mathcal{A}^0_{\mathsf{FS}} + \mathcal{B}_{\mathsf{FS}},$$

with

$$\mathcal{A}^0_{\mathsf{FS}} \begin{bmatrix} \rho \\ u \\ h \\ \ell \\ \omega \end{bmatrix} = \begin{bmatrix} -\overline{\rho}\mathrm{div}\ u \\ A_{\mathsf{u}}\left(u - D_{\mathsf{s}}(\ell,\omega)\right) \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathcal{B}_{\mathsf{FS}} \begin{bmatrix} \rho \\ u \\ h \\ \ell \\ \omega \end{bmatrix} = \begin{bmatrix} 0 \\ -a\gamma(\overline{\rho})^{\gamma-1} \\ \ell \\ -m^{-1}\displaystyle\int_{\partial\mathcal{S}} \sigma_l(\rho,u)n\ \mathrm{d}\Gamma - \mathrm{m}^{-1}\mathrm{k_p h} - \mathrm{m}^{-1}\mathrm{k_d}\ell \\ -J(0)^{-1}\displaystyle\int_{\partial\mathcal{S}} y\times\sigma_l(\rho,u)n\ \mathrm{d}\Gamma \end{bmatrix}.$$

Global Stabilization: h_0 is not close to h_1

Under the assumptions of above Theorem, there exists $\delta > 0$, depending only on Ω, k_p, k_d such that for any $(\rho_0, u_0, h_0, \ell_0, \omega_0)$ satisfying the assumptions and

$$\|\rho_0 - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(0))} + \|u_0\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(0))^3} + \|\ell_0\|_{\mathbb{R}^3} + \|\omega_0\|_{\mathbb{R}^3} \leqslant \frac{\delta}{2},$$

there exists a piecewise constant function $s : [0, \infty) \to \Omega$ satisfying $\operatorname{dist}(s(t), \partial\Omega) > 1$, for all $t \geqslant 0$, such that the strong solution of the system with switching feedback law

$$w(t) = k_p(s(t) - h(t)) - k_d h'(t), \quad t \geqslant 0,$$

satisfies the stability properties

$$\lim_{t\to\infty} \|\rho(t,\cdot) - \overline{\rho}\|_{W^{1,q}(\mathcal{F}(t))} = 0, \quad \lim_{t\to\infty} \|u(t,\cdot)\|_{B^{2(1-1/p)}_{q,p}(\mathcal{F}(t))^3} = 0,$$
$$\lim_{t\to\infty} h(t) = h_1, \quad \lim_{t\to\infty} h'(t) = 0, \quad \lim_{t\to\infty} \omega(t) = 0.$$

References

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Thank You