The moment method applied to the N-dimensional heat equation

Manuel González-Burgos,
UNIVERSIDAD DE SEVILLA

COPI2A: Control, Problemas Inversos y Aprendizaje Automático
Castro Urdiales, December 2025

Objective

To revisit the moment method and apply this method to obtain the distributed null controllability for the N-dimensional heat equation. We also want to analyze the so-called control cost.

Contents

1. A distributed controllability problem for the N-dimensional heat equation

[AMMAR KHODJA, BENABDALLAH, G.-B., MORANCEY, DE TERESA], arXiv:2406.05104.

1. A controllability problem for the heat equation

Fix $N \geq 2$, $\boxed{\Omega \subset \mathbb{R}^N}$, a regular bounded domain, $\boxed{\omega \subset \Omega}$, an arbitrary nonempty open set, and $T > 0$.

Null controllability problem

Given $y_0 \in L^2(\Omega)$, find a control $v \in L^2(\omega \times (0, T))$ such that the solution $y \in C^0([0, T]; L^2(\Omega))$ of

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T := \Omega \times (0, T), \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0, T), \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

satisfies $y(\cdot, T) = 0$ in Ω (**Null Controllability** of system (1) at time $T > 0$).

1_ω is the characteristic function on ω.

1. A controllability problem for the heat equation

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

If system (1) is **null controllable** at time $T > 0$:

Control cost

We define the **control cost** for system (1) at time T as

$$\boxed{\mathcal{K}(T, \omega) = \sup_{\|y_0\|_{L^2(\Omega)}=1} \left(\inf_{v \in \mathcal{Z}_T(y_0, \Omega)} \|v\|_{L^2(Q_T)} \right)}, \quad \forall T > 0,$$

where

$$\mathcal{Z}_T(y_0, \omega) := \{v \in L^2(Q_T) : y(\cdot, T) = 0\} \neq \emptyset.$$

In particular, we want to analyze

$$\mathcal{C}(\omega) = \limsup_{T \to 0} \left(T \log(\mathcal{K}(T, \omega)) \right) \iff \mathcal{K}(T, \omega) \sim \exp\left(\frac{\mathcal{C}(\omega)}{T} \right).$$

1. A controllability problem for the heat equation

$$
(1) \qquad \begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}
$$

Previous results, several techniques

1. $\boxed{N = 1}$; [FATTORINI-RUSSELL] (1971). Moment method for an equivalent boundary controllability problem (more details later).
2. $\boxed{N \geq 2}$; [LEBEAU-ROBBIANO] (1996). Spectral inequalities for the eigenfunctions of the Dirichlet-Laplace operator in Ω (more details later).
3. $\boxed{N \geq 2}$; [FURSIKOV-IMANUVILOV] (1996). Global Carleman inequalities for the adjoint problem to (1).

Objective:

Combine the moment method for the heat equation in a N-dimensional framework and the strategy of Lebeau-Robbiano in order to solve the problem.

$$(1) \quad \begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

How to attack the null controllability problem of (1)?

Moment method

Main tool:

Eigenvalues and eigenvectors of the Dirichlet-Laplace operator on $\boxed{\Omega \subset \mathbb{R}^N}$:

1. $\{\mu_m\}_{m\geq 1} \subset (0, \infty)$ is the sequence of eigenvalues;
2. $\{\psi_m\}_{m\geq 1}$ is the associated sequence of normalized eigenfunctions in $L^2(\Omega)$.

1. A controllability problem for the heat equation

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases}$$

The idea of the **moment method** is simple: Given $y_0 \in L^2(\Omega)$ and $v \in L^2(\omega \times (0,T))$, then we can explicitly solve problem (1) using the basis of eigenfunctions $\{\psi_m\}_{m\geq 1}$. Thus,

Property: Moment problem

There exists a control $v \in L^2(\omega \times (0,T))$ s.t. $y(\cdot,T) = 0$ in Ω (y is the solution of (1)) if and only if there exists $\boxed{v \in L^2(\omega \times (0,T))}$ satisfying

$$\boxed{\int_0^T \int_\omega e^{-\mu_m(T-t)} \psi_m(x) \, v(x,t) \, dx \, dt = -e^{-\mu_m T} y_{0,m}}, \quad \forall m \geq 1,$$

where $y_{0,m}$ are the Fourier coefficients of y_0 associated to the orthonormal basis $\{\psi_m\}_{m\geq 1}$.

1. A controllability problem for the heat equation

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

Moment problem:

Given $y_0 \in L^2(\Omega)$, find $\boxed{v \in L^2(\omega \times (0,T))}$ s. t.

$$(2) \qquad \int_0^T \int_\omega e^{-\mu_m (T-t)} \psi_m(x)\, v(x,t)\, dx\, dt = -e^{-\mu_m T} y_{0,m}, \quad \forall m \geq 1,$$

($y_{0,m}$ Fourier coefficients of y_0).

Biorthogonal families in $L^2(\omega \times (0,T))$

Let us introduce the function defined by:

$$F_m(x,t) := e^{-\mu_m t}\psi_m(x), \quad \forall m \geq 1, \quad (x,t) \in \Omega \times (0,T).$$

$\{Q_m\}_{m\geq 1}$ is a biorthogonal family in $L^2(\omega \times (0,T))$ to $\{F_m\}_{m\geq 1}$ if

$$\int_0^T \int_\omega F_m(x,t) Q_n(x,t)\, dx\, dt = \delta_{mn}, \quad \forall m,n \geq 1.$$

Then, a formal solution of the moment problem (2) is given by

$$v(x,t) = -\sum_{m\geq 1} e^{-\mu_m T} y_{0,m}\, Q_m(x, T-t), \quad \forall (x,t) \in \omega \times (0,T).$$

1. Existence of biorthogonal sequences to $\{F_m\}_{m\geq 1}$ in $L^2(\omega \times (0,T))$?
2. Convergence in $L^2(\omega \times (0,T))$ of the previous series?

$$F_m(x,t) := e^{-\mu_m t}\psi_m(x), \quad \forall m \geq 1, \quad (x,t) \in \Omega \times (0,T).$$

Remark

The existence of a biorthogonal family to $\{F_m\}_{m\geq 1}$ in $L^2(\omega \times (0,T))$ is equivalent to the property (**minimality**):

$$F_m \notin \overline{\text{span}\left\{F_n|_{\omega\times(0,T)} : n \neq m\right\}}^{\|\cdot\|_{L^2(\omega\times(0,T))}}, \quad m,n \geq 1.$$

A simpler case

Let us first analyze the simplest case: $\omega \equiv \Omega$ (global control).
In this case we will use that $\{\psi_m\}_{m\geq 1}$ is an orthonormal basis of $L^2(\Omega)$.

2. The case $\omega \equiv \Omega$: Global controls

2. The case $\omega \equiv \Omega$: Global controls

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = v & \text{in } Q_T := \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(\cdot,0) = y_0 & \text{in } \Omega, \end{cases}$$

with $v \in L^2(Q_T)$. The null controllability of system (1) is equivalent to: find $v \in L^2(Q_T)$ such that

$$(2) \qquad \int_0^T \int_\Omega e^{-\mu_m(T-t)}\, \psi_m(x)\, v(x,t)\, dx\, dt = -e^{-\mu_m T} y_{0,m}, \quad \forall m \geq 1,$$

($y_{0,m}$ Fourier coefficients of y_0).

Objective:

Find a biorthogonal family in $\boxed{L^2(Q_T)}$ of $\{F_m\}_{m\geq 1}$ ($F_m(x,t) := e^{-\mu_m t}\psi_m(x)$).

Existence + estimate: the main result when $\omega \equiv \Omega$

Theorem

Assume $\mathcal{B} = \{\psi_m\}_{m\geq 1}$ is an orthonormal basis of $L^2(\Omega)$. Then, $\forall \varepsilon \in [0,T)$, $\exists$ a sequence $\{q_m^\varepsilon\}_{m\geq 1}$ biorthogonal in $L^2(Q_T)$ to $\{F_m\}_{m\geq 1}$ satisfying

$$\begin{cases} \text{Supp } (q_m^\varepsilon) \subset [\varepsilon, T] \times \overline{\Omega}, \\ \|q_m^\varepsilon\|^2_{L^2(Q_T)} \leq 2e\left(\dfrac{1}{T-\varepsilon} + \mu_m\right) e^{2\varepsilon\mu_m}, \quad \forall m \geq 1. \end{cases}$$

Proof: For simplicity, let us assume that $\varepsilon = 0$. Then,

$$q_m^0(x,t) = \frac{1}{C_m^T} e^{-\mu_m t}\psi_m(x), \ (x,t) \in Q_T, \quad \text{with } C_m^T = \int_0^T e^{-2\mu_m t}\, dt, \quad m \geq 1,$$

and we can prove

$$\frac{1}{C_m^T} \leq 2e\left(\frac{1}{T} + \mu_m\right).$$

2. The case $\omega \equiv \Omega$: Global controls

(1) $$\begin{cases} \partial_t y - \Delta y = v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

The null controllability of system (1) is equivalent to: find $v \in L^2(Q_T)$ such that

(2) $$\int_0^T \int_\Omega e^{-\mu_m (T-t)} \psi_m(x) \, v(x,t) \, dx \, dt = -e^{-\mu_m T} y_{0,m}, \quad \forall m \geq 1,$$

($y_{0,m}$ Fourier coefficients of y_0) and a candidate is

$$v(x,t) = -\sum_{m \geq 1} e^{-\mu_m T} y_{0,m} \, q_m^0(x, T-t), \quad \forall (x,t) \in Q_T.$$

Question

Is $v \in L^2(Q_T)$??

2. The case $\omega \equiv \Omega$: Global controls

Let us compute $\|v\|_{L^2(Q_T)}$. One has,

$$\begin{cases} \|v\|_{L^2(Q_T)} \leq \left(\sum_{m\geq 1} e^{-2\mu_m T} \left\| q_m^0 \right\|^2_{L^2(Q_T)} \right)^{1/2} \|y_0\|_{L^2(\Omega)} \\ \qquad \leq \sqrt{2e} \left(\sum_{m\geq 1} \left(\frac{1}{T} + \mu_m \right) e^{-2\mu_m T} \right)^{1/2} \|y_0\|_{L^2(\Omega)} := S(T) \|y_0\|_{L^2(\Omega)} . \end{cases}$$

Finally, using Weyl's law: $\boxed{\mathcal{N}(r) := \sharp \{ m \geq 1 : \mu_m \leq r \} \leq \kappa r^{N/2}}$, $\forall r > 0$, we deduce,

$$S(T) \leq \frac{\mathcal{C}}{T^{\frac{N+2}{4}}},$$

where $\mathcal{C} > 0$ is a constant only depending on N and Ω.

We have solved the null controllability problem for system (1) when $\omega = \Omega$.

2. The case $\omega \equiv \Omega$: Global controls

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

Remark

In particular ($\omega \equiv \Omega$), the set

$$\mathcal{Z}_T(y_0, \omega) := \{v \in L^2(Q_T) : y(\cdot, T) = 0\} \neq \emptyset,$$

and the **control cost** for system (1) at time T satisfies

$$\boxed{\mathcal{K}(T, \omega) = \sup_{\|y_0\|_{L^2(\Omega)}=1} \left(\inf_{v \in \mathcal{Z}_T(y_0, \Omega)} \|v\|_{L^2(Q_T)} \right) \leq \frac{\mathcal{C}}{T^{\frac{N+2}{4}}}}, \quad \forall T > 0.$$

2. The case $\omega \equiv \Omega$: Global controls

$$(1) \quad \begin{cases} \partial_t y - \Delta y = v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

We have proved:

Theorem

Assume $\omega = \Omega$. Then, system (1) is null controllable at time $T > 0$ and

$$\mathcal{K}(T, \omega) \leq \frac{\mathcal{C}}{T^{\frac{N+2}{4}}},$$

where $\mathcal{C} > 0$ is a constant only depending on N and Ω.

Important:

We have used that $\mathcal{B} = \{\psi_m\}_{m \geq 1}$ is an orthonormal basis of $L^2(\Omega)$. This property is lost when ω is a proper subset of Ω: $\omega \subset \Omega$ and $\boxed{\omega \not\equiv \Omega}$.

3. Biorthogonal families in $L^2(\omega \times (0,T))$

[Ammar Khodja, Benabdallah, G.-B., Morancey, de Teresa], arXiv:2406.05104.

3. Biorthogonal families in $L^2(\omega \times (0,T))$

Recall

1. $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\partial\Omega$ regular enough ($N \geq 2$);
2. Eigenvalues and eigenvectors of the Dirichlet-Laplace operator on $\boxed{\Omega \subset \mathbb{R}^N}$: $\boxed{\{\mu_m\}_{m\geq 1}}$ is the sequence of eigenvalues; $\boxed{\{\psi_m\}_{m\geq 1}}$ is the associated sequence of normalized eigenfunctions in $L^2(\Omega)$;
3. $\boxed{F_m(x,t) := e^{-\mu_m t}\psi_m(x)}$, $\forall m \geq 1$, $(x,t) \in \Omega \times (0,T)$.

Objective:

Prove the existence of a biorthogonal family $\{Q_m\}_{m\geq 1}$ to $\{F_m\}_{m\geq 1}$ in $\boxed{L^2(\omega \times (0,T))}$: $\int_0^T \int_\omega F_m(x,t) Q_n(x,t)\, dx\, dt = \delta_{mn}, \quad \forall m, n \geq 1$, and estimate the norm.

Main difficulty: $\omega \subset \Omega$ and $\boxed{\omega \neq \Omega}$.

We have also proved:

Theorem

Assume $\mathcal{B} = \{\psi_m\}_{m\geq 1}$ is an orthonormal basis of $L^2(\Omega)$. Then, $\forall \varepsilon \in [0,T)$, $\exists$ a sequence $\{q_m^\varepsilon\}_{m\geq 1}$ biorthogonal in $\boxed{L^2(\Omega \times (0,T))}$ to $\{F_m\}_{m\geq 1}$ satisfying

$$\begin{cases} \text{Supp } (q_m^\varepsilon) \subset [\varepsilon, T] \times \overline{\Omega}, \\ \|q_m^\varepsilon\|^2_{L^2(Q_T)} \leq 2e \left(\dfrac{1}{T-\varepsilon} + \mu_m \right) e^{2\varepsilon\mu_m}, \quad \forall m \geq 1. \end{cases}$$

3. Biorthogonal families in $L^2(\omega \times (0,T))$

Steps:

1. Main argument: The restriction operator

$$\mathcal{R}_\omega : \psi \in L^2(\Omega \times (0,T)) \mapsto \psi|_\omega \in L^2(\omega \times (0,T))$$

from the closed subspace of $L^2_\rho(\Omega \times (0,T))$ spanned by $\{F_m\}_{m\geq 1}$ (for an appropriate weight function ρ) into the subspace of $L^2(\omega \times (0,T))$ spanned by $\{F_m\}_{m\geq 1}$ is a **bi-continuous bijection**.

2. As a consequence, we will construct a biorthogonal family to $\{F_m\}_{m\geq 1}$ in $L^2(\omega \times (0,T))$.

3. Biorthogonal families in $L^2(\omega \times (0,T))$

1. First step. The restriction operator

Proposition ("Spectral inequality")

Fix $\Omega \subset \mathbb{R}^N$, a bounded domain regular enough, and $\omega \subset \Omega$, an arbitrary nonempty open subset. Then, there exists a constant $\beta > 0$ (only depending on ω and Ω) s. t. $\boxed{\forall \{c_m\}_{m\geq 1} \subset \mathbb{R}}$ *and* $\boxed{\forall \lambda \in (1,\infty)}$, *one has*

$$\int_\Omega \left| \sum_{\sqrt{\mu_m}\leq\lambda} c_m \psi_m(x) \right|^2 dx \equiv \sum_{\sqrt{\mu_m}\leq\lambda} |c_m|^2 \leq e^{\beta\lambda} \int_\omega \left| \sum_{\sqrt{\mu_m}\leq\lambda} c_m \psi_m(x) \right|^2 dx,$$

where $\{\psi_m\}_{m\in\mathbb{N}}$ is the sequence of normalized eigenfunctions of the Dirichlet-Laplace operator in $\Omega \subset \mathbb{R}^N$.

For a proof:
[Jerison, Lebeau], Univ. Chicago Press, Chicago, Il., 1990.

For $M \geq 1$, we define:

$$P_M(x,t) := \sum_{\sqrt{\mu_m} \leq M} a_m^{(M)} F_m(x,t) = \sum_{\sqrt{\mu_m} \leq M} a_m^{(M)} e^{-\mu_m t} \psi_m(x)$$

where $a_m^{(M)} \in \mathbb{R}$, $1 \leq m \leq M$. Then,

Lemma

There exists $\alpha_0(T,\beta) > 0$ *satisfying*

$$\alpha_0(T,\beta) > 2\beta \textit{ and } \lim_{T \to 0^+} \alpha_0(T,\beta) = 2\beta,$$

such that for any $M \geq 1$*, any* P_M *and any* $\alpha \geq \alpha_0(T,\beta)$*, one has*

$$\int_0^T \int_\Omega e^{-\frac{\alpha\beta}{t}} |P_M(x,t)|^2 \, dx \, dt \leq 6 \int_0^T \int_\omega |P_M(x,t)|^2 \, dx \, dt.$$

Main tools:

1. Spectral Inequality: $\boxed{\forall \lambda \in (1,\infty)}$, one has

$$e^{-\beta\lambda} \int_\Omega \left| \sum_{\sqrt{\mu_m} \le \lambda} c_m \psi_m(x) \right|^2 dx \le \int_\omega \left| \sum_{\sqrt{\mu_m} \le \lambda} c_m \psi_m(x) \right|^2 dx.$$

And

$$\boxed{P_M(x,t) := \sum_{\sqrt{\mu_m} \le M} a_m^{(M)} F_m(x,t) = \sum_{\sqrt{\mu_m} \le M} a_m^{(M)} e^{-\mu_m t} \psi_m(x)}$$

2. Dissipation on the intervals $(0, \alpha/M)$ and $(\alpha/M, T)$.

Conclusion: We have proved:

Lemma

There exists $\alpha_0(T,\beta) > 0$ *satisfying*

$$\alpha_0(T,\beta) > 2\beta \text{ and } \lim_{T\to 0^+} \alpha_0(T,\beta) = 2\beta,$$

such that for any $M \geq 1$*, any* P_M *and any* $\alpha \geq \alpha_0(T,\beta)$*, one has*

$$\int_0^T \int_\Omega e^{-\frac{\alpha\beta}{t}} |P_M(x,t)|^2 \, dx \, dt \leq 6 \int_0^T \int_\omega |P_M(x,t)|^2 \, dx \, dt.$$

$$P_M(x,t) := \sum_{\sqrt{\mu_m}\leq M} a_m^{(M)} F_m(x,t) = \sum_{\sqrt{\mu_m}\leq M} a_m^{(M)} e^{-\mu_m t} \psi_m(x)$$

where $a_m^{(M)} \in \mathbb{R}$, $1 \leq m \leq M$. Then, Let us define ($\alpha \geq \alpha_0$)

$$\eta_\alpha(x) = \begin{cases} 0 & \text{if } x \in \omega \\ \alpha\beta & \text{if } x \in \Omega \setminus \omega, \end{cases}$$

$$(f,g)_{\eta_\alpha} := \int_0^T \int_\Omega e^{-\frac{\eta_\alpha(x)}{t}} f(x,t) g(x,t)\, dx\, dt, \quad \|f\|^2_{\eta_\alpha} := (f,f)_{\eta_\alpha}.$$

and the Hilbert spaces:

$$L^2_{\eta_\alpha}(\Omega \times (0,T)) := \left\{ f : \Omega \times (0,T) \to \mathbb{R} : \|f\|_{\eta_\alpha} < \infty \right\},$$

and

$$\begin{cases} E_{\eta_\alpha} = \overline{\text{span } \{F_m : m \geq 1\}}^{L^2_{\eta_\alpha}(\Omega \times (0,T))}, \\ E = \overline{\text{span } \{F_m|_\omega : m \geq 1\}}^{L^2(\omega \times (0,T))}, \end{cases}$$

Define the restriction operator

$$\mathcal{R}_\omega : \varphi \in L^2_{\eta_\alpha}(\Omega \times (0,T)) \mapsto \mathcal{R}_\omega(\varphi) = \varphi|_\omega \in L^2(\omega \times (0,T))$$

which satisfies $\mathcal{R}_\omega \in \mathcal{L}\left(L^2_{\eta_\alpha}(\Omega \times (0,T)), L^2(\omega \times (0,T))\right)$.

Theorem

If $\alpha \geq \alpha_0(T,\beta)$, the operator $\mathcal{R}_\omega$ satisfies

$$\|\varphi\|^2_{\eta_\alpha} \leq 7 \|\mathcal{R}_\omega(\varphi)\|^2_{L^2(\omega\times(0,T))} \leq 7 \|\varphi\|^2_{\eta_\alpha}, \quad \forall \varphi \in E_{\eta_\alpha}.$$

Moreover, $\mathcal{R}_\omega(E_{\eta_\alpha}) = E$ and, therefore, $\mathcal{R}_\omega \in \mathcal{L}(E_{\eta_\alpha}, E)$ is an ***isomorphism***.

Recall

$$\alpha_0(T,\beta) > 2\beta \text{ and } \lim_{T\to 0^+} \alpha_0(T,\beta) = 2\beta.$$

3. Biorthogonal families in $L^2(\omega \times (0, T))$

2. Conclusion

Theorem

The family $\{F_m\}_{m\geq 1}$ is minimal in $L^2(\omega \times (0,T))$. Moreover, there exists a biorthogonal family $\{Q_m\}_{m\geq 1} \subset E$ to $\{F_m\}_{m\geq 1}$ in $L^2(\omega \times (0,T))$ such that

$$(3) \qquad \|Q_m\|^2_{L^2(\omega\times(0,T))} \leq 14e\left(\frac{1}{T-\varepsilon} + \mu_m\right) e^{\frac{\alpha_0\beta}{\varepsilon}+2\varepsilon\mu_m}, \quad \forall m \geq 1,$$

for any $\varepsilon \in (0,T)$, with $\alpha_0 = \alpha_0(T,\beta) > 2\beta$ and $\lim_{T\to 0^+} \alpha_0(T,\beta) = 2\beta$.

Proof: The proof combines:

1. Theorem 3.1: existence of a sequence $\{q_m^\varepsilon\}_{m,k\geq 1}$ biorthogonal to $\{F_m\}_{m\geq 1}$ in $\boxed{L^2(\Omega \times (0,T))}$ s.t. $q_m^\varepsilon \equiv 0$ on $(0,\varepsilon)$.
2. The restriction operator $\mathcal{R}_\omega \in \mathcal{L}(E_{\eta_\alpha}, E)$ is an **isomorphism**: $\exists \{Q_m^\varepsilon\}_{m\geq 1} \subset E$ biorthogonal to $\{F_m\}_{m\geq 1}$ in $L^2(\omega \times (0,T))$ and one has (3).
3. In fact, $\{Q_m^\varepsilon\}_{m\geq 1} \subset E$ and the family does not depend on ε.

4. A consequence on the control cost for the N-dimensional heat equation

4. Control cost for the N-dimensional heat equation

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega. \end{cases}$$

Recall that null controllability of system (1) is equivalent to: find $v \in L^2(Q_T)$ such that

$$(2) \qquad \int_0^T \int_\omega e^{-\mu_m (T-t)} \psi_m(x) \, v(x,t) \, dx \, dt = -e^{-\mu_m T} y_{0,m}, \quad \forall m \geq 1,$$

($y_{0,m}$ Fourier coefficients of y_0) and a solution is given by

$$v(x,t) = -\sum_{m \geq 1} e^{-\mu_m T} y_{0,m} \, Q_m(x, T-t), \quad \forall (x,t) \in Q_T,$$

with $\|Q_m\|^2_{L^2(\omega \times (0,T))} \leq 14e \left(\frac{1}{T - \varepsilon} + \mu_m \right) e^{\frac{\alpha_0 \beta}{\varepsilon} + 2\varepsilon \mu_m}$, $\forall m \geq 1$, $\varepsilon \in (0, T)$

and

$$\alpha_0(T, \beta) > 2\beta \text{ and } \lim_{T \to 0^+} \alpha_0(T, \beta) = 2\beta.$$

4. Control cost for the N-dimensional heat equation

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

Let us compute $\|v\|^2_{L^2(Q_T)}$. One has,

$$\begin{cases} \|v\|_{L^2(Q_T)} \leq \sqrt{14e} \left(\sum_{m \geq 1} \left(\frac{1}{T - \varepsilon} + \mu_m \right) e^{-2\mu_m (T - \varepsilon)} \right)^{1/2} e^{\frac{\alpha_0 \beta}{2\varepsilon}} \|y_0\|_{L^2(\Omega)} \\ \qquad := S(\varepsilon, T) e^{\frac{\alpha_0 \beta}{2\varepsilon}} \|y_0\|_{L^2(\Omega)} . \end{cases}$$

As before,

$$S(\varepsilon, T) \leq \frac{\mathcal{C}}{(T - \varepsilon)^{\frac{N+2}{4}}}, \qquad \mathcal{C} = \mathcal{C}(\Omega, N) > 0.$$

Then,

$$\|v\|_{L^2(Q_T)} \leq \frac{\mathcal{C}}{(T - \varepsilon)^{\frac{N+2}{4}}} e^{\frac{\alpha_0 \beta}{2\varepsilon}} \|y_0\|_{L^2(\Omega)} , \qquad \forall \varepsilon \in (0, T).$$

4. Control cost for the N-dimensional heat equation

$$(1) \qquad \begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

We can minimize with respect to $\varepsilon \in (0, T)$

$$\|v\|_{L^2(Q_T)} \le \frac{\mathcal{C}}{(T-\varepsilon)^{\frac{N+2}{4}}} e^{\frac{\alpha_0 \beta}{2\varepsilon}} \|y_0\|_{L^2(\Omega)}, \quad \forall \varepsilon \in (0, T).$$

Using $\alpha_0(T, \beta) > 2\beta$ and $\lim_{T\to 0^+} \alpha_0(T, \beta) = 2\beta$, we deduce

Theorem

System (1) is null controllable at time $T > 0$ and, the control cost satisfies

$$\mathcal{C}(\omega) = \limsup_{T\to 0} \left(T \log(\mathcal{K}(T, \omega)) \right) \le \beta^2.$$

where $\mathcal{C} > 0$ is a constant only depending on N and Ω.

4. Control cost for the N-dimensional heat equation

$$(1) \quad \begin{cases} \partial_t y - \Delta y = 1_\omega v & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot, 0) = y_0 & \text{in } \Omega, \end{cases}$$

Conclusion

We have adapted the Lebeau-Robbiano strategy in order to obtain an appropriate biorthogonal family to $\{F_m\}_{m\geq 1}$ in $L^2(\omega \times (0,T))$. As a consequence,

1. We have proved the null controllability of system (1) at time $T > 0$.
2. We have obtained the behavior of the control cost $\mathcal{K}(T,\omega)$ $(\omega \neq \Omega)$:

$$\mathcal{K}(T,\omega) \sim \exp\left(\frac{\beta^2}{T}\right), \quad \text{as } T \to 0.$$

Remark

In [MILLER] (2010), the author adapted the Lebeau-Robbiano strategy and proved that, for all $T > 0$, the **observability inequality**

$$\left\| e^{-TA} y_0 \right\|_{L^2(\Omega)}^2 \leq \mathcal{K}(T,\omega)^2 \int_0^T \int_\omega \left| e^{-tA} y_0(x) \right|^2 \, dx \, dt, \quad \forall y_0 \in L^2(\Omega),$$

holds for a positive constant $\mathcal{K}(T,\omega)$ satisfying

$$\limsup_{T \to 0} \left(T \ln(\mathcal{K}(T,\omega)) \right) \leq \beta^2.$$

In the previous inequality $\left\{ e^{-tA} \right\}_{t \geq 0}$ denotes the semigroup generated by the Dirichlet-Laplace operator in $L^2(\Omega)$.

4. Control cost for the N-dimensional heat equation

The strategy presented in this talk can be applied to more complicated N-dimensional parabolic problems, providing controllability results where minimum control times appear.

For more details:

[Ammar Khodja, Benabdallah, G.-B., Morancey, de Teresa], arXiv:2406.05104.

Thank you for your attention!!