Estimate for the Stokes system: Optimality issues

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$$\begin{cases} \partial_t U - \Delta U + \nabla q = F \\ \operatorname{div} U = h \end{cases}$$ on $\mathcal{M}$ smooth open set of $\mathbb{R}^d$

+ initial conditions

+ Boundary conditions. general precise conditions given below (Lopatinskiĭ-Šapiro type)

typical exemples: . Dirichlet $U|_\partial = G$

. Navier: $\nu \cdot U = G_\nu$ $\quad ((\nabla U + {}^t\nabla U)\cdot \nu)_{tan} = G_{tan}$

. Neumann $({}^t\nabla U + n \nabla U)\nu - q\nu = G$

Observability

$T > 0$ $\quad \omega \subset \mathcal{M}$ open subset

$$\int_0^T \left(\|u\|^2 + \|q\|^2 \right) dt \lesssim \int_0^T \left(\|u\|^2_{L^2(\omega)} + \|q\|^2_{L^2(\omega)} \right) dt$$

$$+ \int_0^T \left(\|F\|^2 + \|h\|^2 + |G|^2 \right) dt$$

boundary value.

Ref : homogeneous Dirichlet + flat Laplace

- Fernandez-Cara, Guerrero, Imanuvilov, Puel 04
- Buffe, Takahashi 25

Application observability / null controllability of Oseen system

NS (local result)

- Navier : Guerrero 2006

Fernandez-Cara, Gonzalez-Burgos, Imanuvilov, Puel (2006–2008)

One strategy $\partial_t U - \Delta U = F - \nabla q$ + homog. Dirichlet

Heat-type Carleman estimate for U.

Apply divergence and use $\operatorname{div} U = 0$

$\Delta q = \operatorname{div} F$ No boundary condition.

Multiplier + microlocal argument $\longrightarrow$ estimation of $q|_{\partial_y}$

Elliptic Carleman estimate for q + time integration

$\longrightarrow$ combine estimations.

Carleman estimates – elliptic case

Estimations of the form

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2} + \tau^{1/2}\|e^{\tau\varphi}\nabla u\|_{L^2} \leq C\,\|e^{\tau\varphi}Au\|$$

- A order 2, elliptic
- $\varphi = \varphi(x)$ weight function.
- $u \in \mathcal{C}_c^\infty$, $\tau > 0$ large
- if u defined at a boundary

boundary condition (order k)

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2} + \tau^{1/2}\|e^{\tau\varphi}\nabla u\|_{L^2} \leq C\left(\|e^{\tau\varphi}Au\|_{L^2} + |e^{\tau\varphi}Bu|_{\frac{3}{2}-k}\right)$$

$$\underbrace{+\ \tau^{3/2}|e^{\tau\varphi}u|_\partial|_{L^2} + \tau^{1/2}|e^{\tau\varphi}\nabla u|_\partial|_{L^2}}_{\text{optimal ?}} \qquad + \text{ Observation term}$$

Carleman estimates – Parabolic version (Fursikov–Imanuvilov)

$$\left\| \tilde{\tau}^{3/2} e^{\tau\eta\phi} u \right\|_{L^2} + \left\| \tilde{\tau}^{1/2} e^{\tau\eta\phi} \nabla u \right\|_{L^2} \leq C \left\| e^{\tau\eta\phi} P u \right\|_{L^2} \quad \Big\} \text{ Optimal ?}$$

- $P = \partial_t - \Delta$ • $u \in C_c^\infty$, $\tau > 0$ large
- $\psi = \psi(x)$ weight function. $\eta = \dfrac{1}{t(T-t)}$
- $\phi = \psi - K < 0$
- $\tilde{\tau} = \tau \eta \varphi$

Back to Stokes — Away from the boundary

$$\partial_t U - \Delta U + \nabla q = F \qquad \mathrm{div}_g U = h$$

$$\gamma^{\frac{1}{2}} \left(\| \tilde{\tau}^{\frac{1}{2}} e^{\tau\eta\phi} q \|_{t,x} + \| \tilde{\tau}^{-\frac{1}{2}} e^{\tau\eta\phi} \nabla q \|_{t,x} \right) \quad \longleftarrow \text{velocity}$$

$$+ \gamma' \left(\| \tilde{\tau} e^{\tau\eta\phi} U \|_{t,x} + \| e^{\tau\eta\phi} \nabla U \|_{t,x} \right) \quad \longleftarrow \text{pressure}$$

$$\lesssim \| e^{\tau\eta\phi} F \|_{t,x} + \| e^{\tau\eta\phi} \partial_t h \|_{-1} + \gamma'^{\frac{1}{2}} \| \tilde{\tau}^{-\frac{1}{2}} e^{\tau\eta\phi} \Delta h \|_{-1}$$

- pressure q : $\frac{1}{2}$ derivative loss
- velocity U full derivative loss

$\Big\}$ Optimal ?

At a boundary

Theorem (LR – Robbiano)

$$\gamma^{\frac12}\left(\|\tilde\tau^{\frac12} e^{\tau\eta\phi} q\|_{t,x} + \|\tilde\tau^{-\frac12} e^{\tau\eta\phi}\nabla q\|_{t,x}\right) + \gamma\left(\|\tilde\tau e^{\tau\eta\phi} U\|_{t,x} + \|e^{\tau\eta\phi}\nabla U\|_{t,x}\right)$$

$$+ \left|\tilde\tau^{\frac32} e^{\tau\eta\phi} U|_{\partial M}\right|_{t,x} + \left|\tilde\tau^{\frac12} e^{\tau\eta\phi}\nabla U|_{\partial M}\right|_{t,x} + \left|\tilde\tau^{\frac12} e^{\tau\eta\phi} q|_{\partial M}\right|_{t,x}$$

$$\lesssim \sum_j \left| B_j u|_{\partial M}\right|_{t,x} + \text{source terms}$$

Theorem (LR – Robbiano)

$$\gamma^{\frac{1}{2}}\left(\|\tilde{\tau}^{\frac{1}{2}}e^{\tau\eta\phi}q\|_{t,x}+\|\tilde{\tau}^{-\frac{1}{2}}e^{\tau\eta\phi}\nabla q\|_{t,x}\right)+\gamma\left(\|\tilde{\tau}e^{\tau\eta\phi}U\|_{t,x}+\|e^{\tau\eta\phi}\nabla U\|_{t,x}\right)$$

$$+\left|\,|\tilde{\tau}^{\frac{3}{2}}e^{\tau\eta\phi}U|_{\partial M}\right|_{t,x}+\left|\,|\tilde{\tau}^{\frac{1}{2}}e^{\tau\eta\phi}\nabla U|_{\partial M}\right|_{t,x}+\left|\,|\tilde{\tau}^{\frac{1}{2}}e^{\tau\eta\phi}q|_{\partial M}\right|_{t,x}$$

$$\lesssim \sum_j \left|\,|B_j u|_{\partial M}\right|_{t,x} + \text{source terms}$$

Optimal ?

Optimality: estimation away from the boundary suffices

pressure estimation: $\operatorname{div}\left((\partial_t - \Delta)U + \nabla q\right) = \operatorname{div} F$

$\to (\partial_t - \Delta) h + \Delta q = F \to \Delta(q - h) = \operatorname{div} F - \partial_t h$

Carleman estimate for $q - h$

$$\gamma^{\frac{1}{2}} \| \tilde{\tau}^{-\frac{1}{2}} (q-h) \|_{\tilde{\lambda},1} \lesssim \| F \|_{L^2} + \| \partial_t h \|_{\tilde{\lambda},-1}$$

$$\boxed{\gamma^{\frac{1}{2}} \| \tilde{\tau}^{-\frac{1}{2}} q \|_{\tilde{\lambda},1} \lesssim \| F \|_{L^2} + \| \partial_t h \|_{\tilde{\lambda},-1} + \gamma^{\frac{1}{2}} \| \tilde{\tau}^{-\frac{1}{2}} h \|_{\tilde{\lambda},1}}$$

Velocity estimation

$$V = \begin{pmatrix} U^\phi \\ \Theta^{-1} q^\phi \end{pmatrix} \in \mathbb{R}^{d+1} \qquad \Theta = Op(\lambda)$$

Stokes reads $\quad A\,V = G + \text{error terms}$

$$A = \begin{pmatrix} P^\phi & -i\,\nabla_g^\phi\,\Theta \\ -i\,\Theta\,\mathrm{div}^\phi & 0 \end{pmatrix} \qquad t_a = \begin{pmatrix} P^\phi\, I_d & \lambda\,\zeta \\ \lambda\,{}^t\zeta_g & 0 \end{pmatrix}$$

$$\zeta = \zeta + i\,\tau\, d\phi = \zeta + i\,\tilde{\tau}\, d\psi$$

Eigenvectors

$\begin{pmatrix} \vartheta \\ 0 \end{pmatrix}$ for $\vartheta \perp_g \zeta$ $\qquad p^\phi$ eigenvalue

$\longrightarrow$ $d-1$ vectors

$$\boxed{{}^t\begin{pmatrix} \vartheta \\ 0 \end{pmatrix} V \text{ can be estimated with a loss of } \tfrac{1}{2} \text{ derivative}}$$

$W_q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad W_1 = \begin{pmatrix} \lambda^{-1}\bar{\zeta} \\ 0 \end{pmatrix}$

$\downarrow$ pressure estimate obtained above

$\searrow$ not an eigenvector $\qquad \bar{\zeta} \not\perp_g \zeta$

$${}^tW_1\, a = {}^tW_1\, P + {}^t\zeta_g\,\bar{\zeta}\;\; W_q$$

$${}^tW_1 a = {}^tW_1 P + {}^t\xi\, g\, \bar{\xi}\ W_q$$

${}^tW_1 V$ is estimated from ${}^tW_q V$ with additional loss of $\frac{1}{2}$ derivative

$\Rightarrow$ ${}^tW_1 V$ is estimated with a loss of a full derivative

$\Rightarrow$ loss of a full derivative for the velocity.

Culprit ? All evidences point to $W_1 = \begin{pmatrix} \lambda^{-1}\xi \\ 0 \end{pmatrix}$

El culpable

Optimality

Theorem. Suppose $T > 0$ and there exist $C > 0$ and $\tau_0 > 0$ such that

$$\tau^{\alpha} \| \eta e^{\tau\eta\phi} U \|_{L^2} + \tau^{\beta} \| \eta^{\frac{1}{2}} e^{\tau\eta\phi} q \|_{L^2}$$

$$\leq C \left(\| e^{\tau\eta\phi} ((\partial_t - \Delta) U + \nabla q) \|_{L^2} + \| e^{\tau\eta\phi} \partial_t \operatorname{div} U \|_{\tilde{\lambda}, -1} + \| \tilde{\sigma}^{-1/2} e^{\tau\eta\phi} \operatorname{div} U \|_{\tilde{\lambda}, 1} \right)$$

for $\tau \geq \tau_0$, $U \in \mathcal{C}_c^{\infty}((0,T) \times \Omega; \mathbb{C}^d)$ and $q \in \mathcal{C}_c^{\infty}((0,T) \times \Omega; \mathbb{C})$.

Then, $\alpha \leq 1$ and $\beta \leq 1/2$

How to prove optimality? Elliptic case $P(D) = -\Delta = D^2$

- $\underset{\overset{\|}{0}}{x^0} \in \Omega$, $d\phi(x^0) \neq 0$, $\xi^0 \in \mathbb{R}^d$ such that

$$\begin{cases} {}^t\xi^0 \cdot d\phi(x^0) = 0 \\ \text{and } |\xi^0| = \tau\,|d\phi(x^0)| \end{cases}$$

Then, $a_\varphi = (\xi_j + i\tau\partial_j\phi)^2 = 0$ at (x^0, τ^0, ξ^0)

- Set $w(x) = \zeta^0 \cdot x$ with $\zeta^0 = \xi^0 + i\tau\, d\phi(x^0)$

$$\phi(x) - \operatorname{Im} w(x) = \phi(0) + G(x) + |x|^3 O(1) \qquad G(x) = \frac{1}{2}\sum_{j,k} \partial^2_{jk}\phi(0)\, x_j x_k$$

- $f \in \mathscr{C}^\infty_c(\mathbb{R}^d)$, $f \not\equiv 0$ $\quad u(x) = e^{i\tau(iw(x) - \phi(0))} f_\tau(x)$

$$f_\tau(x) = f(\tau^{1/2} x)$$

$$\bullet \int_{\mathbb{R}^d} e^{2\tau\phi}|u|^2\,dx = \int_{\mathbb{R}^d} e^{2\tau\left(G(x)+|x|^3\mathcal{O}(1)\right)}|f_\tau(x)|^2\,dx$$

$$= \tau^{-d/2}\int_{\mathbb{R}^d} e^{2\left(G(x)+\tau^{-1/2}|x|^3\mathcal{O}(1)\right)}|f(x)|^2\,dx$$

$$\sim \tau^{-d/2}\,\|e^{G}f\|^2_{L^2(\mathbb{R}^d)} \qquad \text{(dominated convergence)}$$

$$\bullet \quad e^{\tau\phi}Pu = e^{\tau(\phi-\phi(0)+i\omega)}\underbrace{e^{\tau(\phi(0)-i\omega)}P(D)e^{\tau(i\omega-\phi(0))}}_{P(D+\tau\xi^0)}f_\tau$$

$$P(D+\tau\xi^0) = \underbrace{P(\tau\xi^0)}_{=0} + P'_\xi(\tau\xi^0)\cdot D_x + \frac{1}{2}P^{(2)}_\xi(\tau\xi^0)(D_x, D_x)$$

$$P(D+\tau\xi^0)f_\tau = \tau^{3/2}\left(P'_\xi(\xi^0)(D_x f)_\tau + \tau^{-1/2}\mathcal{O}(1)\right)$$

$$\| e^{\tau\phi} P(D) u \|_{L^2}^2 \sim K \tau^{3 - d/2}$$

$$\tau^{2\alpha} \| e^{\tau\phi} u \|_{L^2}^2 \lesssim C \| e^{\tau\phi} P(D) u \|_{L^2}^2$$

$$\wr$$

$$\tau^{2\alpha - \frac{d}{2}} \lesssim \tau^{3 - d/2}$$

$$\Longrightarrow \quad \alpha \leq 3/2$$

- Hörmander, 1985 (1963)
- LR, Lebeau, Robbiano 2023

Stokes now

same ζ^0 same $w(x)$

$t \in [0,T] \longleftrightarrow s \in \mathbb{R}$ $\qquad s = \tan\left(\frac{\pi t}{T} - \frac{\pi}{2}\right)$ $\qquad \partial_t = \frac{\pi \langle s \rangle^2}{T} \partial_s$

$\frac{T}{2} \longleftrightarrow s = 0$

$P = \partial_t - \Delta = \frac{a}{T}\partial_s - \Delta$

$U = e^{\tau\eta(i w(x) - \phi(0))} f_\tau(s,x)\, U^0$ with $U^0 = \zeta^{0\#}$

$q = \tau^{1/2} \ell_\tau(s,x)\, e^{\tau\eta(i w(x) - \phi(0))}$

N.B. ${}^tW_1 U$ does not vanish $\longrightarrow$ contribution in the direction with the poor estimate

- $\tau^{2\alpha} \| \eta e^{\tau\eta\phi} U \|^2_{L^2} \gtrsim \tau^{2\alpha - \frac{(d+1)}{2}} |U^0|^2 \, \| e^G f \|^2_{L^2}$
- $\tau^{2\beta} \| \eta^{1/2} e^{\tau\eta\phi} q \|^2_{L^2} \gtrsim \tau^{2\beta + 1 - \frac{(d+1)}{2}} \| e^G \ell \|^2_{L^2}$

- $\| (\tau\eta)^{-\frac{1}{2}} e^{\tau\eta\phi} \operatorname{div} U \|^2_{\tilde{\lambda},1} + \| e^{\tau\eta\phi} \langle s\rangle^2 \partial_s \operatorname{div} U \|^2_{L^2} \lesssim \tau^{2-\frac{(d+1)}{2}}$

$$e^{\tau\eta(\phi(0)-i\omega(x))}(a\partial_s-\Delta)U = -\tau^{3/2}\left(P'_\xi(\xi^0)(D_x f)_\tau + \tau^{-1/2}\mathcal{O}(1)\right)U^0$$

$$e^{\tau\eta(\phi(0)-i\omega(x))}\nabla q = \tau^{3/2}\left(i\, l_\tau U^0 + \tau^{-1/2}\mathcal{O}(1)\right)$$

$U^0 = (\xi^0)^\#$

if $l = P'_\xi(\xi^0)\cdot D_x f$

then $e^{\tau\eta(\phi(0)-i\omega(x))}\left((a\partial_s-\Delta)U + \nabla q\right) = \tau\,\mathcal{O}(1)$

$$\| e^{\tau\eta\phi}\left((a\partial_s-\Delta)U+\nabla q\right)\|^2_{L^2} \lesssim \tau^{2-\frac{(d+1)}{2}}.$$

$$\Rightarrow \quad RHS \lesssim \tau^{2-\frac{(d+1)}{2}}$$

- $\tau^{2\alpha}\|\eta e^{\tau\eta\phi}U\|_{L^2}^2 \gtrsim \tau^{2\alpha-\frac{(d+1)}{2}}|U^0|^2\|e^G f\|_{L^2}^2$
- $\tau^{2\beta}\|\eta^{\frac{1}{2}}e^{\tau\eta\phi}q\|_{L^2}^2 \gtrsim \tau^{2\beta+1-\frac{(d+1)}{2}}\|e^G \ell\|_{L^2}^2$

$$\Rightarrow \quad \boxed{\begin{aligned} \alpha &\le 1 \\ \beta &\le \tfrac{1}{2} \end{aligned} \qquad \text{it's optimal.}}$$

Thank you.

Carleman estimates

Elliptic estimate ideas behind the proof.

$$\tau^{3/2}\|e^{\tau\varphi}u\|_{L^2} + \tau^{1/2}\|e^{\tau\varphi}\nabla u\|_{L^2} \leq C\,\|e^{\tau\varphi}Pu\|_{L^2}$$

$$P_\varphi = e^{\tau\varphi}Pe^{-\tau\varphi}.$$

$$\|P_\varphi v\| \gtrsim \tilde{\tau}^{3/2}\|v\| + \tilde{\tau}^{\frac{1}{2}}\|\nabla v\|$$

Lemma if $|d\psi| \geq c > 0$ and $\varphi = e^{\gamma\psi}$ then subelliptic property

Dependence on γ (Carleman second large parameter)

$$\frac{\gamma^{\mu}}{\tilde{\tau}}(q_2^2+q_1^2) + \{q_2,q_1\} \gtrsim \frac{\gamma}{\tilde{\tau}}(\tilde{\tau}^4+|\xi|^4)$$

$$\|P_\varphi v\| \gtrsim \gamma^{1/2}\left(\|\tilde{\tau}^{3/2}v\| + \|\tilde{\tau}^{\frac{1}{2}}\nabla v\| + \|\tilde{\tau}^{-1/2}D^2 v\|\right)$$

optimal ?