On the linear quadratic problem for evolution equations with finite memory: distinct patterns, challenges, most recent advances

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The talk is based on (ongoing) joint work with

Paolo Acquistapace, Università di Pisa (*Ret.*)

Reference articles:

- [Ac-Bu_2025b] Linear quadratic control of parabolic-like evolutions with memory of the inputs, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.* **36** (2025), no. 1, 167-198.
- [Ac-Bu_2025a] Optimal synthesis control for evolution equations subject to nonlocal inputs, *J. Optim. Theory Appl.* **205**, 37 (2025).
- [Ac-Bu_2024] Riccati-based solution to the optimal control of linear evolution equations with finite memory, *Evol. Equ. Control Theory* **13** (2024), no. 1, 26-66.

Outline

The subject of the talk is in the broad area of optimal control of *deterministic* evolution equations

The core topic: the optimal synthesis in the linear quadratic (LQ) problem for partial differential equations (PDE) with memory

Major focus on a control system with *unbounded* control operator and memory of the inputs

$\rightsquigarrow$ line of argument, obtained results

A glimpse on the latest developments (*ongoing* work)

The core topic: the LQ problem for certain control systems with memory. Introduction

Evolution equations with memory

Application and motivation comes from the study of many interesting physical phenomena, such as

viscoelasticity, diffusion processes in the presence of complex molecular structures (e.g. diffusion in polymers), population dynamics, etc. (2nd half of the XIX century $\rightsquigarrow$)

The resulting model equations (are differential equations which) exhibit *the influence of the past values of one or more variables in play*

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A few bibliographical references:

M. Renardy, W.J. Hrusa, J.A. Nohel, *Mathematical Problems in Viscoelasticity*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 35. Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1987.

J. Prüss, *Evolutionary integral equations and applications*, [2012] reprint of the 1993 edition, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 1993. xxvi+366 pp.

L. Pandolfi, *Systems with persistent memory – controllability, stability, identification*, Interdisciplinary Applied Mathematics, 54. Springer, Cham, [2021], x+356 pp.

The heat equation with finite memory and boundary datum

$\Omega \subset \mathbb{R}^n$ bounded domain, $\partial\Omega =: \Gamma$ *smooth*

$$\begin{cases} w_t(t,x) = \Delta w(t,x) + \int_0^t N(\sigma)\Delta w(t-\sigma,x)\,d\sigma & \text{in } (0,T)\times\Omega =: Q_T \\ w(t,x) = u(t,x) & \text{on } (0,T)\times\Gamma =: \Sigma_T \\ w(0,x) = w_0(x) & \text{in } \Omega \end{cases} \qquad \text{(IBVP)}$$

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The initial-boundary value problem (IBVP) can be recast as the abstract Cauchy problem

$$\begin{cases} w' = Aw + Bu + \int_0^t N(t-\sigma)\big[Aw(\sigma) + Bu(\sigma)\big]\,d\sigma\,, & t \in (0,T) \\ w(0) = w_0 \in H \end{cases}$$

where the state space Y, the (free dynamics) generator $A\colon \mathcal{D}(A) \subset Y \to Y$, and the unbounded control operator B will be specified below

The modeling of boundary control actions, functional-analytic perspective (a digression)

Control acting in the interior of the domain

$\Omega \subset \mathbb{R}^n$ bounded domain, $\partial\Omega =: \Gamma$ *smooth*

$$\begin{cases} y_t(t,x) = \Delta y(t,x) + u(t,x) & \text{in } (0,T) \times \Omega =: Q_T \\ y(t,x) = 0 & \text{on } (0,T) \times \partial\Omega =: \Sigma_T \\ y(0,x) = y_0(x) & \text{in } \Omega \end{cases}$$

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Functional-analytic reformulation:

- $Y = L^2(\Omega)$, $U = L^2(\Omega)$; $y(t) = y(t,\cdot)$, $u(t) = u(t,\cdot)$
- $A\colon \mathcal{D}(A) \subset Y \longrightarrow Y$ is the *realization* of the Laplacian in $L^2(\Omega)$ with (homogeneous) Dirichlet boundary condition, i.e.

$$Af := \Delta f\,, \quad f \in \mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega)$$

- $B = I$ (identity)

Control acting in the interior of the domain (*cont'd*)

Then,

the PDE problem $\rightsquigarrow$ $\begin{cases} y'(t) = Ay(t) + Bu(t)\,, & t \in (0,T) \\ y(0) = y_0 \in Y \end{cases}$

where

- $A\colon \mathcal{D}(A) \subset Y \to Y$ is the infinitesimal generator of a C_0-semigroup $\{e^{tA}\}_{t\ge 0}$ on Y;
- $B \in \mathcal{L}(U,Y)$, with $U = L^2(\Omega)$

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(*mild* solutions) $y(t) = e^{tA}y_0 + \underbrace{\int_0^t e^{(t-\sigma)A}Bu(\sigma)\,d\sigma}_{=:\,Lu(t)}\,, \qquad t \in [0,T]$

Control acting on the boundary

$$\begin{cases} y_t(t,x) = \Delta y(t,x) & \text{in } Q_T \\ y(t,x) = u(t,x) & \text{on } \Sigma_T \\ y(0,x) = y_0(x) & \text{in } \Omega \end{cases}$$

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where

- (still with) $Y = L^2(\Omega)$, and $A\colon \mathcal{D}(A) \subset Y \to Y$ as before,
- now with $U = L^2(\Gamma)$, we have $B \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$ (B is *inherently unbounded*)

Control acting on the boundary (*cont'd*)

Fattorini, Balakrishnan (*70's, early 80's*)

i. The key is the introduction of the operator $D \in \mathcal{L}(L^2(\Gamma), H^{1/2}(\Omega))$ defined as follows:

$$D\colon L^2(\Gamma) \ni \varphi \longmapsto D\varphi =: \psi \iff \boxed{\begin{cases} \Delta\psi = 0 & \text{in } \Omega \\ \psi|_\Gamma = \varphi \end{cases}}$$

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ii. since by construction one has $\Delta D \equiv 0$, the IBVP reads as

$$\begin{cases} y_t(t,x) = \Delta\big(y(t,x) - Du(t,x)\big) & \text{in } Q_T \\ y(t,x) - Du(t,x) = 0 & \text{on } \Sigma_T \\ y(0,x) = y_0(x) & \text{in } \Omega \end{cases}$$

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iii. The above brings about the control system $y' = Ay + Bu$ with $B = -AD$ (note that $A^{-1}B \in \mathcal{L}(U, Y)$)

(Go back to) The full integro-differential control system

$$w' = Aw + Bu + \int_0^t N(t-\sigma)\big[Aw(\sigma) + Bu(\sigma)\big]\, d\sigma\,, \quad t \in (0,T) \qquad \text{(full-eq)}$$

Distinct challenges:

- the differential operator A occurs in the convolution integral
- the control operator B is *unbounded*
- the memory terms pertain to *both* the dynamics and the control

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Remark

Aiming at studying the optimal control problem with quadratic functionals for the integro-differential control system (full-eq) *and more specifically to attain the optimal synthesis by solving appropriate operator equations, we pursued a plan where the technical challenges are addressed in stages.*

A project which has developed over time

i) In a first step, we assumed $B \in \mathcal{L}(U, Y)$, omitted the memory of the control as well as the differential operator inside the convolution integral (MacCamy's trick provides a justification for this), namely,

$$w' = Aw + Bu + \int_0^t N(t-\sigma)w(\sigma)\, d\sigma\,, \quad t \in (0,T)\,;$$

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ii) next, we tackled the LQ problem for a control system where *the sole memory of the inputs* is kept (still with $B \in \mathcal{L}(U, Y)$), i.e.

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see [Ac-Bu_2025a]

iii) the obtained results have been subsequently extended to the case when B is unbounded, provided e^{tA} is an *analytic semigroup* – see [Ac-Bu_2025b]

Our most recent developments, ahead of the latest (ongoing) step

A control system with the (sole) memory of the inputs

$\mathcal{H}$, U separable complex Hilbert spaces; $0 < T < +\infty$

A linear control system whose dynamics depends (also) on the *past value of the inputs*

$$\begin{cases} w'(t) = Aw(t) + Bu(t) + \int_0^t k(t-\sigma)Bu(\sigma)\,d\sigma\,, & t \in (0,T) \\ w(0) = w_0 \in \mathcal{H} \end{cases} \qquad \text{(system)}$$

under the following assumptions:

Assumptions

A1. *The linear operator* $A\colon \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ *is the infinitesimal generator of a* C_0*-semigroup* $\{e^{tA}\}_{t\ge 0}$ *on* $\mathcal{H}$*, which is also analytic;*

A2. $B \in \mathcal{L}(U, [\mathcal{D}(A^*)]')$*, and there exists* $\gamma \in (0,1)$ *such that* $(\lambda_0 - A)^{-\gamma}B \in \mathcal{L}(U,\mathcal{H})$ *for some* $\lambda_0 > 0$*;*

A3. $k \in L^2(0,T;\mathcal{L}(\mathcal{H}))$*, and* $e^{tA}k = ke^{tA}$.

Quadratic optimal control

Associated quadratic functional on a *finite* time horizon:

$$J(u) = \int_0^T \left(\|Cw(t)\|_H^2 + \|u(t)\|_U^2 \right) dt \,, \qquad \text{(cost)}$$

with

- $C \in \mathcal{L}(\mathcal{H})$ (observation/weighting operator)

The optimal control problem

Given $w_0 \in \mathcal{H}$, seek a control function $\hat{u}(\cdot)$ that minimizes the functional (cost) within the class $\mathcal{U} = L^2(0, T; U)$, where $w(\cdot) = w(\cdot; w_0, u)$ is the solution to (system) corresponding to the control function $u(\cdot)$ and with initial datum w_0.

Main feature and aim:

- focus on the *finite* time horizon problem, for an evolution with *finite* memory,
- we seek a *closed-loop* representation of the optimal control, whose synthesis is attained by way of solving appropriate Riccati equations (or *analogues* of them)

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Works addressing more general /different optimal control problems (semilinear equations, non necessarily quadratic functionals, *infinite* memory, and yet with distributed control):

- [Cannarsa-Frankowska-Marchini, 2013], [Casas-Yong, 2023]

The authors establish first- or second-order optimality conditions, respectively

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References which are actually more pertinent – even when dealing with finite dimensional systems –, as they share the more specific goal:

- [Pandolfi, 2018], [Pandolfi, 2025]

The literature is richer in the realm of stochastic equations:

- [Wang T., 2018], [Bonaccorsi-Confortola, 2020], [Abi Jaber-Miller-Pham, 2021], [Han-Lin-Yong, 2023], [Wang H.-Yong-Zhou, 2023], [Hamaguchi-Wang T., 2024]

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The wider background: the other great questions of control theory for integro-differential PDE, such as *reachability, controllability, unique continuation, observability and inverse problems via Carleman estimates, stability and uniform decay rates*

- a huge literature (some references are given at the end)

The dynamic programming approach

Richard E. Bellman (1957)

- the optimal control problem is embedded in a family of similar optimization problems, depending on suitable parameters, here the initial time $s \in [0, T)$ and a corresponding initial state $x \in \mathcal{H}$

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- then, the value function $V(s,x) = \inf_{u(\cdot)} J_s(u)$ is introduced, where $J_s(u)$ denotes the integral over (s, T) (and the evolution at time s equals x);

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- whether the operator $P(\cdot)$ actually satisfies the differential Riccati equation (to which the Bellman's equation reduces)

$$\begin{aligned}\frac{d}{dt}(P(t)x,y)_H + (P(t)x,Ay)_H + (Ax,P(t)y)_H + (Cx,Cy)_H \\ -(B^*P(t)x,B^*P(t)y)_U = 0\,, \quad x,y \in \mathcal{D}(A)\end{aligned} \tag{DRE}$$

  (along with $P(T) = 0$) is the key issue

Line of argument (and what is meant for full synthesis)

- We introduce a *suitable state space* (also shared by Pandolfi);
- we adapt an established path for the study of the LQ problem for relevant classes of *memoryless* PDE (Lasiecka-Triggiani, '80s $\rightsquigarrow$)

Recall (memoryless case):

existence of a unique minimizer (*open-loop* optimal control)

$\rightsquigarrow$ optimality condition, which in particular brings about

$\rightsquigarrow$ an operator $P(t)$ which enters a *feedback* formula $\boxed{\hat{u}(t) = -B^* P(t)\hat{w}(t)}$

$\rightsquigarrow$ $P(t)$ *does* solve the DRE (*existence*)

$\rightsquigarrow$ *uniqueness* for the DRE $\rightsquigarrow$ *closed-loop* optimal control

A history spanning (more than) four decades

Remark

In the memoryless case, when B is unbounded, the actual meaning of $B^*P(t)$ – or lack thereof – is the central issue that must be tackled. It is here that the analysis of parabolic or hyperbolic dynamics split apart.

parabolic-like PDE:
Balakrishnan (1977), Lasiecka-Triggiani (1983), Pritchard-Salamon (1984), Flandoli (1984), Da Prato-Ichikawa (1985);

hyperbolic-like PDE:
Da Prato-L-T (1986), Flandoli (1987), Salamon (1987), Flandoli-L-T (1988), Barbu-L-T (2000)

composite systems of PDE
Avalos-Lasiecka (1996), Lasiecka (2000-2002), L-T (2002, 2004); Lasiecka-Tuffaha (2008, 2009);
Acquistapace-B.-Lasiecka (2005, 2013), Acquistapace-B. (2022-2023)

The parametric problem. Enlarged state space, mild solutions

With the initial time s allowed to vary in $[0, T)$, we set

$$Y_s := \begin{cases} \mathcal{H} & \text{if } s = 0 \\ \mathcal{H} \times L^2(0, s; U) & \text{if } s \in (0, T) \end{cases}, \qquad X_0 = \begin{cases} w_0 & \text{if } s = 0 \\ \begin{pmatrix} w_0 \\ \eta(\cdot) \end{pmatrix} & \text{if } s \in (0, T) \end{cases}$$

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The representation formula for *mild* solutions

$$\begin{aligned} w(t) &= e^{A(t-s)} w_0 + L_s u(t) + H_s u(t) + \mathcal{K}_s \eta(t) \\ &= E(t, s) X_0 + \big[(L_s + H_s) u\big](t)\,, \end{aligned}$$

with

The parametric problem. Enlarged state space, mild solutions (*cont'd*)

$$L_s u(t) \equiv (L_s u)(t) := \int_s^t e^{A(t-q)} Bu(q)\, dq$$

$$H_s u(t) \equiv (H_s u)(t) := \int_s^t e^{A(t-q)} \int_s^q k(q-p) Bu(p)\, dp\, dq$$

$$\mathcal{K}_s \eta(t) \equiv (\mathcal{K}_s \eta)(t) := \int_0^s \lambda(t,p,s)\eta(p)\, dp\,,$$

having set

$$\lambda(t,p,s) := \int_s^t e^{A(t-q)} k(q-p) B\, dq$$

Our findings

Theorem (Acquistapace-B., Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. 2025)

With reference to the optimal control problem (system)-(cost), *under the standing assumptions, the following statements are valid for any* $s \in [0, T)$.

A1. *For each* $X_0 \in Y_s$ *there exists a unique optimal pair* $(\hat{u}(\cdot, s, X_0), \hat{w}(\cdot, s, X_0))$ *which satisfies*

$$\hat{u}(\cdot, s, X_0) \in C([s, T], U), \quad \hat{w}(\cdot, s, X_0) \in C([s, T], \mathcal{H}).$$

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$$\hat{u}(\cdot, s, X_0) \in C([s, T], U), \quad \hat{w}(\cdot, s, X_0) \in C([s, T], \mathcal{H}).$$

A2. *There exist three linear bounded operators – denoted by* $P_0(s)$*,* $P_1(s, p)$*,* $P_2(s, p, q)$ *– such that the optimal cost is given by*

$$J_s(\hat{u}) = \big(P_0(s)w_0, w_0\big)_{\mathcal{H}} + 2Re \int_0^s \big(P_1(s, p)\eta(p), w_0\big)_{\mathcal{H}} \, dp + \int_0^s \int_0^s \big(P_2(s, p, q)\eta(p), \eta(q)\big)_U \, dp \, dq \equiv \big(P(s)X_0, X_0\big)_{Y_s}.$$

$P_0(s)$ *and* $P_2(s, p, q)$ *are self-adjoint and non-negative operators in the respective functional spaces* $\mathcal{H}$ *and* $L^2(0, s; U)$*; in addition,* $P_2(s, p, q) = P_2(s, q, p)$*.*

Theorem *(cont'd)*

A3. *The optimal control admits the following representation in closed-loop form:*

$$\boxed{\begin{aligned}\hat{u}(t,s,X_0) = &-\big[B^*P_0(t)+P_1(t,t)^*\big]\,\hat{w}(t,s,X_0)\\ &-\int_0^t \big[B^*P_1(t,p)+P_2(t,p,t)\big]\theta(p)\,dp\,,\end{aligned}}$$

with

$$\theta(\cdot) = \begin{cases}\eta(\cdot) & \text{in } [0,s)\\ \hat{u}(\cdot,s,X_0) & \text{in } [s,t)\end{cases}$$

Theorem *(cont'd)*

A4. *The operators* $P_0(t)$, $P_1(t,p)$, $P_2(t,p,q)$ *– as from* **A3.** *– satisfy the following coupled system of equations, for every* $t\in[0,T)$, $s,q\in[0,t]$, *and for any* $x,y\in\mathcal{D}(A)$, $v,u\in U$:

$$\begin{cases}
\frac{d}{dt}\big(P_0(t)x,y\big)_{\mathcal{H}}+\big(P_0(t)x,Ay\big)_{\mathcal{H}}+\big(Ax,P_0(t)y\big)_{\mathcal{H}}+\big(C^*Cx,y\big)_{\mathcal{H}} \\
\qquad -\big([B^*P_0(t)+P_1(t,t)^*]x,[B^*P_0(t)+P_1(t,t)^*]y\big)_U=0 \\
\frac{\partial}{\partial t}\big(P_1(t,p)v,y\big)_{\mathcal{H}}+\big(P_1(t,p)v,Ay\big)_{\mathcal{H}}+\big(k(t-p)Bv,P_0(t)y\big)_{\mathcal{H}} \\
\qquad -\big([B^*P_1(t,p)+P_2(t,p,t)]v,[B^*P_0(t)+P_1(t,t)^*]y\big)_U=0 \\
\frac{\partial}{\partial t}\big(P_2(t,p,q)u,v\big)_U+\big(P_1(t,p)u,k(t-q)Bv\big)_{\mathcal{H}}+\big(k(t-p)Bu,P_1(t,q)v\big)_{\mathcal{H}} \\
\qquad -\big([B^*P_1(t,p)+P_2(t,p,t)]u,[B^*P_1(t,q)+P_2(t,q,t)]v\big)_U=0
\end{cases}$$

with final conditions

$$P_0(T)=0\,,\;\; P_1(T,p)=0\,,\;\; P_2(T,p,q)=0\,. \qquad (*)$$

Theorem *(cont'd)*

A5. **(Uniqueness)** *There exists a unique triplet* $(P_0(t), P_1(t,p), P_2(t,p,q))$ *that solves the coupled system above and fulfils the final conditions* (*)*, within the class of linear bounded operators (in the respective spaces), the former and the latter being self-adjoint and non-negative.*

Theorem *(cont'd)*

A5. **(Uniqueness)** *There exists a unique triplet* $(P_0(t), P_1(t,p), P_2(t,p,q))$ *that solves the coupled system above and fulfils the final conditions* (*), *within the class of linear bounded operators (in the respective spaces), the former and the latter being self-adjoint and non-negative.*

Remark

These findings appear to be the first ones of their kind; furthermore, they extend the classical theory of the LQ problem and Riccati equations for parabolic PDE with boundary control

Remarks about the proofs

Principal technical challenges:

- the novel solution formula naturally accounts for the more involved computations *at any step* of the investigation;
- (differently from the memoryless case) it is not evident that the operators P_i, $i \in \{0,1,2\}$ – the building-blocks of the optimal cost operator $P(t)$ – actually occur in a first feedback formula that follows from the optimality condition;
- proving uniqueness for the Riccati equation is rendered *more complicated* by the fact that the state space $Y_t = \mathcal{H} \times L^2(0,t;\mathcal{H})$ depends on t

Back to the original (full) control system

An approach akin to the one of Dafermos

In order to address the LQ problem for the actual control system (full-eq), we mimick the celebrated "history approach" introduced by Dafermos in 1970 (to deal with PDE with *infinite* memory) and introduce an auxiliary variable, that is

$$z(t,r) := w(t-r) \qquad r \in [0,t];$$

this satisfies

$$z_r(t,r) = -w'(t-r)\,, \quad z(0,\cdot) = 0\,.$$

The original model equation becomes

$$w'(t) = Aw(t) + \int_0^t k(\sigma)Az(t,\sigma)\,d\sigma + Bu(t) + \int_0^t k(t-\sigma)Bu(\sigma)\,d\sigma, \quad t \in [0,T]\,. \tag{1}$$

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Next, we introduce the operators $W(t)$ (for $t \in [0,T]$) and D defined as follows:

$$W(t)\varphi := \int_0^t k(t-r)A\varphi(r)\,dr, \qquad \varphi \in L^2(0,t;\mathcal{D}(A))$$

$$D := -\frac{d}{dr}\,, \quad \mathcal{D}(D) = \{\varphi \in H^1(0,t;H) \colon \varphi(0) = 0\}\,.$$

An approach akin to the one of Dafermos (*cont'd*)

With the augmented variable $y(t) := (w(t), z(t,\cdot))$ and the product space $X = \mathcal{H} \times L^2(0,T;\mathcal{H})$, we attain the novel control system

$$\boxed{y' = \mathcal{A}(t)y + \mathcal{B}u(t) + \int_0^t k(t-\sigma)\mathcal{B}u(\sigma)\,d\sigma\,, \qquad t \in (0,T]\,,} \qquad \text{(non-autonomous)}$$

where

$$\begin{cases} \mathcal{A}(t)\begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} A(x + A^{-1}W(t)f) \\ Df \end{pmatrix}, \quad \begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{D}(\mathcal{A}(t))\,, \\ \mathcal{D}(\mathcal{A}(t)) = \big\{(x,f)^T \in X\colon\ x + A^{-1}W(t)f \in \mathcal{D}(A),\ f \in H^1(0,t;\mathcal{H}),\ f(0) = 0\big\}\,; \end{cases}$$

$$\mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}.$$

Remarks

It is important to emphasize that

- *on one side the (free) dynamics operator* $A(t)$ *which arises is time-dependent,*
- *on the other side the control operator* $\mathcal{B}$ *has the simple structure* $\begin{pmatrix} B \\ 0 \end{pmatrix}$*;*
- *the sole memory of the control function is present.*

Novel tasks and challenges

- A usable representation formula for the solutions would follow if $A(t)$ generates an evolution operator $U(t,s)$ in the function space $X = \mathcal{H} \times L^2(0,T;\mathcal{H})$; this in turn would hold true, provided the (very technical) results by H. Tanabe (1979) apply;
- even so, regularity estimates (in time and space) for the operator $U(t,s)\mathcal{B}$ would be necessitated.

Contributions to the study of equations with memory

general monographs: [Gripenberg-Londen-Staffans, 1990], [Prüss, 1993], [Pandolfi, 2021]

former works in the ∞-dimensional case: V. Barbu (1975); G. Gripenberg, S.-O. Londen (70's); G. Da Prato-M. Iannelli (1980), G. Da Prato-M. Iannelli-E. Sinestrari (1985), D. Sforza (1986-), G. Di Blasio (1994)

quadratic optimal control (finite dimensional case): ..., A. Pritchard-Y. You (1996), ..., L. Pandolfi (2018, 2025)

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controllability, reachability, observability, inverse problems:
P. Gomboa-V. Komornik-O. Veira (2016), S. Ivanov-L. Pandolfi (2009), P. Loreti-D. Sforza (2009-), A. Doubova-E. Fernandez-Cara (2012), S. Guerrero-O. Yu. Imanuvilov (2013), L-S-Yamamoto (2017), . . . , E. Fernandez-Cara-J.L. Machado-D.A. Souza (2020)

well-posedness, (interior and) boundary regularity: . . . , P. Cannarsa-D. Sforza (2002-2004), . . . , P. Loreti-D. Sforza (2016), B.-Pandolfi (2019), B.-Eller (2021)

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Thank you for your attention!

On MacCamy's trick

Consider for simplicity the (abstract) initial value problem

$$\begin{cases} w_t = Aw + \int_0^t N(t-s)Aw(s)\,ds \\ w(0) = w_0\,. \end{cases} \qquad \text{(model-1)}$$

1. The idea underlying MacCamy's trick is to regard the integro-differential equation in (model-1) as a Volterra integral equation in the unknown Aw, namely

$$Aw + \int_0^t N(t-s)Aw(s)\,ds = w_t\,,$$

with w_t *in the role of a given function*.

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with w_t *in the role of a given function*.

By using the resolvent $R(t)$ (of the convolution kernel $N(t)$), which satisfies

$$R(t) + \int_0^t N(t-s)R(s)\,ds = N(t)\,,$$

we first find

$$Aw = w_t - \int_0^t R(t-s)w_t(s)\,ds\,.$$

On MacCamy's trick (*cont'd*)

2. Then,

$$\begin{aligned} Aw(t) &= w_t - \int_0^t R(t-s)w_t(s)\,ds \\ &= w_t - \Big[R(t-s)w(s)\big|_{s=0}^{s=t} + \int_0^t R'(t-s)w(s)\,ds\Big] \\ &= w_t - R(0)w(t) + R(t)w_0 - \int_0^t R'(t-s)w(s)\,ds\,, \end{aligned}$$

where it is assumed $N \in H^1(0,T)$ ($R \in H^1(0,T)$, as a consequence) in order to justify integration by parts.

On MacCamy's trick (*cont'd*)

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where it is assumed $N \in H^1(0,T)$ ($R \in H^1(0,T)$, as a consequence) in order to justify integration by parts.

Nice outcome: the convolution term in the obtained integro-differential problem, that is

$$w_t = Aw(t) + R(0)w(t) + \int_0^t R'(t-s)w(s)\,ds - R(t)w_0\,, \quad w(0) = w_0\,,$$

involves only *lower order terms*

On MacCamy's trick (*cont'd*)

3. The obtained integro-differential equation can be further simplified, in order to get rid of the term $R(0)w(t)$, via the change of variable $w(t) = e^{R(0)t}v(t)$, thereby attaining

$$\begin{cases} v_t = Av(t) + \int_0^t L(t-s)v(s)\,ds - e^{-R(0)t}R(t)w_0 \\ v(0) = w_0\,, \end{cases} \qquad \text{(model-2)}$$

with convolution kernel $L(t) := e^{-R(0)t}R'(t)$.