Internal null controllability of a fourth-order parabolic system

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Presentation and motivation

Can we obtain null-controllability for two coupled 4th order parabolic equations (Kuramoto-Sivashinsky type) on $Q = (0, L) \times (0, T)$,

$$\begin{cases} \partial_t u_1 + \partial_x^4 u_1 = \sum_{j=1}^{2} (a_{1j}\partial_x^3 u_j + b_{1j}\partial_x^2 u_j + c_{1j}\partial_x u_j + d_{1j}u_j) + 1_\omega h, & \text{in } Q, \\ \partial_t u_2 + \partial_x^4 u_2 = \sum_{j=1}^{2} (a_{2j}\partial_x^3 u_j + b_{2j}\partial_x^2 u_j + c_{2j}\partial_x u_j + d_{2j}u_j), & \text{in } Q, \\ (u_1, u_2)(0, t) = (u_1, u_2)(L, t) = 0, & \text{in } (0, T), \\ \partial_x (u_1, u_2)(0, t) = \partial_x (u_1, u_2)(L, t) = 0, & \text{in } (0, T), \\ (u_1, u_2)(x, 0) = (u_1, u_2)^0(x), & \text{in } (0, L), \end{cases}$$

- $(u_1, u_2)^0 \in L^2(0, L)^2$ is the initial condition and $h \in L^2(Q)$ is the control.
- coupling terms $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ are assumed to be in $C_c^\infty(Q)$.

Same system written in a matrix form

$$\begin{cases} \partial_t U + \partial_x^4 U = A\partial_x^3 U + B\partial_x^2 U + C\partial_x U + DU + 1_\omega Gh, & \text{in } Q, \\ U(0,t) = U(L,t) = 0, & \text{in } (0,T), \\ \partial_x U(0,t) = \partial_x U(L,t) = 0, & \text{in } (0,T), \\ U(x,0) = U^0(x), & \text{in } (0,L), \end{cases}$$

with $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ and $G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where (A, B, C, D) are supposed to be 2×2 matrices with coefficients in $C_c^\infty(Q)$.

Previous results

- Control of a single KS equation in 1D
  - Cerpa-Mercado 2011, Cerpa-Guzman-Mercado 2017 with boundary controls, proofs based on moments theory and Carleman estimates
  - Gao 2016 with internal control, and f a globally lipschitz function, proofs based on Carleman estimates

Gao 2016

If $y^0 \in H_0^2(0,L)$, there exists $h \in L^2(0,T)$ such that the solution of

$$\begin{cases} \partial_t y + \partial_x^4 y + f(y, \partial_x y, \partial_x^2 y) = 1_\omega h, & \text{in } Q, \\ y(0,t) = y(L,t) = 0, & \text{in } (0,T), \\ \partial_x y(0,t) = \partial_x y(L,t) = 0, & \text{in } (0,T), \\ y(x,0) = y^0(x), & \text{in } (0,L), \end{cases}$$

satisfies $y(.,T) = 0$.

  - Guzmann 2016 with internal control, other boundary conditions.

- Control of a single KS equation
  - Kassab 2020 in 2D,
  - Guerrero-Kassab 2019, Zongo-Robbiano 2025 in nD, with different boundary conditions
- Control of KS-KdV system

$$\begin{cases} \partial_t y + \gamma \partial_x^4 y + \partial_x^3 y + a\partial_x^2 y + y\partial_x y = \partial_x z + 1_\omega h, & \text{in } Q, \\ \partial_t z - \Gamma \partial_x^2 z + c\partial_x z = \partial_x y + 1_\omega h, & \text{in } Q, \\ y(0,t) = y(L,t) = 0, & \text{in } (0,T), \\ \partial_x y(0,t) = \partial_x y(L,t) = 0, & \text{in } (0,T), \\ z(0,t) = z(L,t) = 0, & \text{in } (0,T), \\ y(x,0) = y^0(x)\, z(x,0) = z^0(x), & \text{in } (0,L), \end{cases}$$

  - Cerpa-Mercado-Pazoto 2015, One control on KS equation
  - Carreno-Cerpa 2016, One control on heat equation

Use Carleman estimates and the equation to eliminate one observation

Our problem

Null controllability of

$$\begin{cases} \partial_t U + \partial_x^4 U = A\partial_x^3 U + B\partial_x^2 U + C\partial_x U + DU + 1_\omega Gh, & \text{in } Q, \\ U(0,t) = U(L,t) = 0, & \text{in } (0,T), \\ \partial_x U(0,t) = \partial_x U(L,t) = 0, & \text{in } (0,T), \\ U(x,0) = U^0(x), & \text{in } (0,L), \end{cases}$$

with $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, $D = (d_{ij})$ and $G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where (A, B, C, D) are supposed to be 2×2 matrices with coefficients in $C_c^\infty(Q)$.

- Only 1 control
- highly coupled system (up to order 3)
- coupling coefficients in time and space.

Core idea

Use fictitious control method with some algebraic resolution based on the articles:

- Coron-Lissy 2014 null-controllability of the 3d Navier Stokes equation
- Duprez-Lissy 2016 and 2018 null controllability of parabolic system with coupling of order 0 and 1.

Ideas

- ▷ First prove that the system is controllable with 2 controls (Fictitious control)
- ▷ Remove the new control with algebraic manipulations, needs a sufficiently regular control

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- ▷ First prove that the system is controllable with 2 controls (Fictitious control)
- ▷ Remove the new control with algebraic manipulations, needs a sufficiently regular control

Algebraic resolution

We suppose first that we already have the null-controllability result with 2 controls

to be proved later

Suppose that $U^0 \in L^2(0,L)^2$, there exists a control $H \in L^2(Q)^2$ with support $\subsetneq \omega \times (0,T) := Q_\omega$ such that

$$\begin{cases} \partial_t \tilde{U} + \partial_x^4 \tilde{U} = A\partial_x^3 \tilde{U} + B\partial_x^2 \tilde{U} + C\partial_x \tilde{U} + D\tilde{U} + H, & \text{in } Q, \\ \tilde{U}(0,t) = \tilde{U}(L,t) = 0, & \text{in } (0,T), \\ \partial_x \tilde{U}(0,t) = \partial_x \tilde{U}(L,t) = 0, & \text{in } (0,T), \\ \tilde{U}(x,0) = U^0(x), & \text{in } (0,L), \end{cases}$$

satisfies $\tilde{U}(.,T) = 0$.

Algebraic resolution

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satisfies $\tilde{U}(.,T) = 0$.

Idea of algebraic resolution

find a trajectory $(\hat{U}, \hat{g})$ with support $\subsetneq \omega \times (0,T)$ such that

$$\begin{cases} \partial_t \hat{U} + \partial_x^4 \hat{U} = A\partial_x^3 \hat{U} + B\partial_x^2 \hat{U} + C\partial_x \hat{U} + D\hat{U} + H + G\hat{g}, & \text{in } Q_\omega, \\ \hat{U}(0,t) = \hat{U}(L,t) = 0, & \text{in } (0,T), \\ \partial_x \hat{U}(0,t) = \partial_x \hat{U}(L,t) = 0, & \text{in } (0,T), \\ \hat{U}(x,0) = 0, & \text{in } (0,L), \\ \hat{U}(x,T) = 0, & \text{in } (0,L). \end{cases}$$

Then $(U,h) = (\tilde{U} - \hat{U}, -\hat{g})$ is a solution of the problem.

Algebraic resolution- How to construct $(\hat{U}, \hat{g})$?

Let $H = (h_1, h_2)$ with $\operatorname{supp}(H) \subsetneq Q_\omega$.
Denoting $\hat{U} = (\hat{u}_1, \hat{u}_2)$, we rewrite the problem as solving $\mathcal{L}(\hat{u}_1, \hat{u}_2, \hat{g}) = (h_1, h_2)$ where

$$\mathcal{L}(\hat{u}_1, \hat{u}_2, \hat{g}) := \begin{pmatrix} \mathcal{L}_1(\hat{u}_1, \hat{u}_2) - \hat{g} \\ \mathcal{L}_2(\hat{u}_1, \hat{u}_2) \end{pmatrix}$$

$$:= \begin{pmatrix} \partial_t \hat{u}_1 + \partial_x^4 \hat{u}_1 - \sum_{j=1}^{2} (a_{1j}\partial_x^3 \hat{u}_j + b_{1j}\partial_x^2 \hat{u}_j + c_{1j}\partial_x \hat{u}_j + d_{1j}\hat{u}_j) - \hat{g} \\ \partial_t \hat{u}_2 + \partial_x^4 \hat{u}_2 - \sum_{j=1}^{2} (a_{2j}\partial_x^3 \hat{u}_j + b_{2j}\partial_x^2 \hat{u}_j + c_{2j}\partial_x \hat{u}_j + d_{2j}\hat{u}_j) \end{pmatrix}$$

Can we find a partial differential operator $\mathcal{M}$ such that

$$\mathcal{L} \circ \mathcal{M} = Id \text{ in } Q_\omega?$$

We first solve $\mathcal{L}_2(\hat{u}_1, \hat{u}_2) = h_2$ and then take $\hat{g} = \mathcal{L}_1(\hat{u}_1, \hat{u}_2) - h_1$. Solving $\mathcal{L}_2 \circ \mathcal{M}_2 = Id$ in Q_ω is equivalent to solving

$$\mathcal{M}_2^* \circ \mathcal{L}_2^* = Id \text{ in } Q_\omega.$$

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Algebraic resolution

How to solve

$$\mathcal{M}_2^* \circ \mathcal{L}_2^* = Id \text{ in } Q_\omega?$$

$$\mathcal{L}_2^*\phi := \begin{pmatrix} \mathcal{A}_1\phi \\ \mathcal{A}_2\phi \end{pmatrix} = \begin{pmatrix} \partial_x^3(a_{21}\phi) - \partial_x^2(b_{21}\phi) + \partial_x(c_{21}\phi) - d_{21}\phi \\ -\partial_t\phi + \partial_x^4\phi + \partial_x^3(a_{22}\phi) - \partial_x^2(b_{22}\phi) + \partial_x(c_{22}\phi) - d_{22}\phi \end{pmatrix}.$$

$\Longrightarrow$ algebraic manipulations with $\mathcal{A}_1\phi$ and $\mathcal{A}_2\phi$ in order to get ϕ.
$\Longrightarrow$ differentiate $\mathcal{A}_1$ k times in space and (once in time and l times in space), and the operator $\mathcal{A}_2$ n times in space in order to construct a square matrix M: $k = 6,\ l = 2,\ n = 5$

Algebraic resolution

$$\mathcal{L}_2^*\phi := \begin{pmatrix}\mathcal{A}_1\phi \\ \mathcal{A}_2\phi\end{pmatrix} = \begin{pmatrix} \partial_x^3(a_{21}\phi) - \partial_x^2(b_{21}\phi) + \partial_x(c_{21}\phi) - d_{21}\phi \\ -\partial_t\phi + \partial_x^4\phi + \partial_x^3(a_{22}\phi) - \partial_x^2(b_{22}\phi) + \partial_x(c_{22}\phi) - d_{22}\phi \end{pmatrix}.$$

$$\mathcal{Q}(\phi) = \begin{pmatrix} \mathcal{A}_1\phi \\ \partial_x\mathcal{A}_1\phi \\ \partial_x^2\mathcal{A}_1\phi \\ \partial_x^3\mathcal{A}_1\phi \\ \partial_x^4\mathcal{A}_1\phi \\ \partial_x^5\mathcal{A}_1\phi \\ \partial_x^6\mathcal{A}_1\phi \\ \partial_t\mathcal{A}_1\phi \\ \partial_t\partial_x\mathcal{A}_1\phi \\ \partial_t\partial_x^2\mathcal{A}_1\phi \\ \mathcal{A}_2\phi \\ \partial_x\mathcal{A}_2\phi \\ \partial_x^2\mathcal{A}_2\phi \\ \partial_x^3\mathcal{A}_2\phi \\ \partial_x^4\mathcal{A}_2\phi \\ \partial_x^5\mathcal{A}_2\phi \end{pmatrix} = \begin{pmatrix} I & 0 \\ \partial_x & 0 \\ \partial_x^2 & 0 \\ \partial_x^3 & 0 \\ \partial_x^4 & 0 \\ \partial_x^5 & 0 \\ \partial_x^6 & 0 \\ \partial_t & 0 \\ \partial_t\partial_x & 0 \\ \partial_t\partial_x^2 & 0 \\ 0 & I \\ 0 & \partial_x \\ 0 & \partial_x^2 \\ 0 & \partial_x^3 \\ 0 & \partial_x^4 \\ 0 & \partial_x^5 \end{pmatrix} \circ \mathcal{L}_2^*\phi = \mathcal{S}\circ\mathcal{L}_2^*\phi := M \begin{pmatrix} \phi \\ \phi_x \\ \phi_{2x} \\ \phi_{3x} \\ \phi_{4x} \\ \phi_{5x} \\ \phi_{6x} \\ \phi_{7x} \\ \phi_{8x} \\ \phi_{9x} \\ \phi_t \\ \phi_{tx} \\ \phi_{txx} \\ \phi_{t3x} \\ \phi_{t4x} \\ \phi_{t5x} \end{pmatrix}.$$

We clearly see that $\mathcal{S}$ is an operator of degree 6 in space, 1 in time, and 1-2 in time-space, and M is a 16×16 square matrix depending on the coefficients $(a_{2,j}, b_{2,j}, c_{2,j}, d_{2,j})_{j=1,2}$ and their derivatives in time and space.

If

$$|\det M(x,t)| > C, \quad \text{for every } (x,t) \in \mathcal{O} \times (T_1, T_2) \subsetneq Q_\omega \tag{1}$$

P_1 := projection on the first component

$$\underbrace{P_1 M^{-1} \mathcal{S}}_{\mathcal{M}_2^*} \circ \mathcal{L}_2^* \phi = \phi.$$

$(\hat{u}_1, \hat{u}_2) = \mathcal{M}_2 h_2, \Longrightarrow$ derive h_2, 1 in time, six times in space, and 1-2 times in time-space

$\hat{g} = \mathcal{L}_1(\hat{u}_1, \hat{u}_2) - h_1 \Longrightarrow$ derive again h_2, finaly at a maximum of 2 times in time, 10 times in space, and 1-6 and 2-2 times in time-space.

regularity needed

$$h_2 \in H^2(L^2) \cap L^2(H^{10}) \cap H^1(H^6) \cap H^2(H^2) = L^2(H^{10}) \cap H^2(H^2).$$

Carleman inequality

How to get a Carleman inequality with 10 derivatives on the LHS?

1. get a Carleman estimate for one KS equation with non-homogeneous boundary conditions
2. Apply it to $\partial_x^8 \phi$

Previous Carleman estimates for KS equations :

- Carreno-Cerpa 2016: Carleman for KS with non-homogeneous Dirichlet conditions but only ϕ on the LHS $\implies$ no hope to get $\partial_x^2 \phi$ on the LHS.
- Gao 2016: Carleman for KS with homogeneous Neumann conditions but with $\partial_x^2 \phi$ on the LHS .

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Carleman inequality for KS with NH Neumann BC

$$\begin{cases} -\varphi_t + \varphi_{4x} = B_0 + \partial_x B_1 + \partial_{xx} B_2, & \text{in } Q, \\ \varphi_{2x}(0,t) = b_1(t),\ \varphi_{2x}(L,t) = b_2(t), & \text{in } (0,T), \\ \varphi_{3x}(0,t) = b_3(t),\ \varphi_{3x}(L,t) = b_4(t), & \text{in } (0,T), \\ \varphi(x,T) = \varphi^T(x), & \text{in } (0,L). \end{cases}$$

Use the work of Fernandez-Cara, Gonzales-Burgos, Guerrero, Puel 2006.

- $\eta \in C^4([0,L])$ be a function satisfying,

$$\begin{cases} \eta(x) > 0,\ \forall x \in (0,L),\ \eta(0) = \eta(L) = 0, \\ |\eta'(x)| \geq \delta > 0,\ \forall x \in [0,L] \setminus \omega_0, \end{cases}$$

  for some $\omega_0 \Subset \omega$ where this means that $\bar{\omega}_0 \subset \omega$. Thus, we have

$$\eta'(0) \geq \delta \text{ and } -\eta'(L) \geq \delta.$$

- usual exponential weight functions with $\lambda > 1$, $k > m > 0$.

$$\alpha(x,t) := \frac{e^{k\frac{m+1}{m}\lambda\|\eta\|_\infty} - e^{\lambda(k\|\eta\|_\infty + \eta(x))}}{t^m(T-t)^m}, \qquad \xi(x,t) := \frac{e^{\lambda(k\|\eta\|_\infty + \eta(x))}}{t^m(T-t)^m},$$

$$\alpha^*(t) := \max_{x\in[0,L]} \alpha(x,t) = \alpha(0,t) = \alpha(L,t),$$

$$\xi^*(t) := \min_{x\in[0,L]} \xi(x,t) = \xi(0,t) = \xi(L,t),$$

Theorem

Let $B_0 \in L^2(Q)$, $B_1 \in L^2(0,T;H^{1/2}(0,L))$, $B_2 \in L^2(0,T;H^{3/2}(0,L))$, $b_1, b_2, b_3, b_4 \in L^2(0,T)$, $m \geq 1/2$ *and* $\omega \subset (0,L)$. $\exists \lambda_0 > 0$ *and* $C > 0$ *s.t. any solution* φ *of*

$$\begin{cases} -\varphi_t + \varphi_{4x} = B_0 + \partial_x B_1 + \partial_{xx} B_2, & \text{in } Q, \\ \varphi_{2x}(0,t) = b_1(t),\ \varphi_{2x}(L,t) = b_2(t), & \text{in } (0,T), \\ \varphi_{3x}(0,t) = b_3(t),\ \varphi_{3x}(L,t) = b_4(t), & \text{in } (0,T), \\ \varphi(x,T) = \varphi^T(x), & \text{in } (0,L), \end{cases}$$

satisfies, for every $\lambda \geq \lambda_0$ *and* $s \geq C(T^{2m} + T^{2m-1})$, *that*

$$\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 + s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 + s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2\,dxdt$$

$$\leq C\Bigg(\iint_Q e^{-2s\alpha}(|B_0|^2 + s^2\lambda^2\xi^2|B_1|^2 + s^4\lambda^4\xi^4|B_2|^2)dxdt$$

$$+\iint_{Q_\omega} s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2\,dxdt$$

$$+\int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|b_1(t)|^2 + |b_2(t)|^2 + |B_2(L,t)|^2 + |B_2(0,t)|^2)dt$$

$$+\int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|b_3(t)|^2 + |b_4(t)|^2 + |B_1(L,t)|^2 + |B_1(0,t)|^2$$

$$+ |\partial_x B_2(L,t)|^2 + |\partial_x B_2(0,t)|^2)dt\Bigg).$$

Carleman for the adjoint system

Let $\Psi^T \in L^2(0,L)^2$, $\bar\eta \in L^2(Q)^2$, $(\bar b_1, \bar b_2, \bar b_3, \bar b_4) \in L^2(Q)^2$, $(\tilde\Pi, \tilde\Lambda, \tilde\Gamma, \tilde\Theta) \in C_c^\infty(Q)$, let $m > 1/2$ and $\omega \subset (0,L)$. There exists $\lambda_0 > 0$ and $C > 0$ such that any solution Ψ of

$$\begin{cases} -\partial_t \Psi + \partial_x^4 \Psi = \tilde\Pi \partial_x^3 \Psi + \tilde\Lambda \partial_x^2 \Psi + \tilde\Gamma \partial_x \Psi + \tilde\Theta \Psi + \bar\eta, & \text{in } Q, \\ \Psi_{2x}(0,t) = \bar b_1(t), \ \Psi_{2x}(L,t) = \bar b_2(t), & \text{in } (0,T), \\ \Psi_{3x}(0,t) = \bar b_3(t), \ \Psi_{3x}(L,t) = \bar b_4(t), & \text{in } (0,T), \\ \Psi(x,T) = \Psi^T(x), & \text{in } (0,L), \end{cases}$$

satisfies for every $\lambda \geq \lambda_0$ and $s \geq C(T^{2m} + T^{2m-1})$,

$$\iint_Q s^3 \lambda^4 \xi^3 e^{-2s\alpha} |\Psi_{2x}|^2 + s^7 \lambda^8 \xi^7 e^{-2s\alpha} |\Psi|^2 dxdt$$
$$\leq C \left(\iint_Q e^{-2s\alpha} |\bar\eta|^2 dxdt + s^7 \lambda^8 \iint_{Q_\omega} e^{-2s\alpha} \xi^7 |\Psi|^2 dxdt \right.$$
$$\left. + s^3 \lambda^3 \int_0^T e^{-2s\alpha^*} (\xi^*)^3 (|\bar b_1|^2 + |\bar b_2|^2) dt + s\lambda \int_0^T e^{-2s\alpha^*} \xi^* (|\bar b_3|^2 + |\bar b_4|^2) dt \right).$$

Well-Posedness results

For the direct problem,

$$\begin{cases} \partial_t U + \partial_x^4 U = A\partial_x^3 U + B\partial_x^2 U + C\partial_x U + DU + F, & \text{in } Q, \\ U(0,t) = U(L,t) = 0, & \text{in } (0,T), \\ \partial_x U(0,t) = \partial_x U(L,t) = 0, & \text{in } (0,T), \\ U(x,0) = U^0(x), & \text{in } (0,L), \end{cases}$$

Theorem

Let (A, B, C, D) in $C_c^\infty(Q)$.
If $U^0 \in L^2(0,L)^2$ and $F \in L^1(0,T; L^2(0,L)^2)$,
then there exists a unique solution
$U \in C([0,T]; L^2(0,L)^2) \cap L^2(0,T; H_0^2(0,L)^2)$ *and* $\exists C > 0$ *such that*

$$||U||_{C([0,T];L^2(0,L)^2)\cap L^2(0,T;H^2(0,L)^2)} \leq C \left\{ ||U^0||_{L^2(0,L)^2} + ||F||_{L^1(0,T;L^2(0,L)^2)} \right\}.$$

For the adjoint problem,

$$\begin{cases} -\partial_t \Phi + \partial_x^4 \Phi = \Pi \partial_x^3 \Phi + \Lambda \partial_x^2 \Phi + \Gamma \partial_x \Phi + \Theta \Phi + \eta, & \text{in } Q, \\ \Phi(0,t) = \Phi(L,t) = 0, & \text{in } (0,T), \\ \partial_x \Phi(0,t) = \partial_x \Phi(L,t) = 0, & \text{in } (0,T), \\ \Phi(x,T) = \Phi^T(x), & \text{in } (0,L). \end{cases} \quad (2)$$

Theorem

Rewrite (2) as $-\Phi_t := L(t)\Phi + \eta$*, where* $L(t) = L(t, x, \partial_x)$ *and let* $d \in \mathbb{N}$*. If* $\Phi^T \in H^{4d+2}(0,L)^2$ *and* $\eta \in L^2(0,T; H^{4d}(0,L)^2) \cap H^d(0,T; L^2(0,L)^2)$ *satisfy the compatibility conditions:*

$$\begin{cases} \bar{g}_0 := \Phi^T \in H_0^2(0,L)^2, \\ \bar{g}_1 := -L(T)\bar{g}_0 - \eta(T,.) \in H_0^2(0,L)^2, \\ \bar{g}_d := -\left(\sum_{k=0}^{d-1} \binom{d-1}{k} \partial_t^k L(T) \bar{g}_{d-1-k}\right) - \partial_t^{d-1}\eta(T,.) \in H_0^2(0,L)^2, \end{cases}$$

then $\Phi \in [C(0,T; H^{4d+2}(0,L)) \cap L^2(0,T; H^{4d+4}(0,L)) \cap H^{d+1}(0,T; L^2(0,L))]^2$*, and* $\exists C > 0$ *such that*

$$\|\Phi\|_{L^2(0,T;H^{4d+4}(0,L)) \cap H^{d+1}(0,T;L^2(0,L))} \leq C \left(\|\eta\|_{L^2(0,T;H^{4d}(0,L)^2) \cap H^d(0,T;L^2(0,L)^2)} + \|\Phi_T\|_{H^{4d+2}(0,L)^2} \right).$$

Carleman with 10 derivatives on the LHS

Let Φ^T in $C_c^\infty(0,L)^2$, with the WP property, $\forall d \in \mathbb{N}$

$$\Phi \in [C(0,T;H^{4d+2}(0,L)) \cap L^2(0,T;H^{4d+4}(0,L)) \cap H^{d+1}(0,T;L^2(0,L))]^2$$

We derive the adjoint system 8 times in space and with $\Psi = \Phi_{8x}$,

$$\begin{cases} -\partial_t \Psi + \partial_x^4 \Psi = \tilde{\Pi}\partial_x^3\Psi + \tilde{\Lambda}\partial_x^2\Psi + \tilde{\Gamma}\partial_x\Psi + \tilde{\Delta}\Psi + \bar{\eta}, & \text{in } Q, \\ \Psi_{2x}(0,t) = \Phi_{10x}(0,t),\ \Psi_{2x}(L,t) = \Phi_{10x}(L,t), & \text{in } (0,T), \\ \Psi_{3x}(0,t) = \Phi_{11x}(0,t), \Psi_{3x}(L,t) = \Phi_{11x}(L,t), & \text{in } (0,T), \\ \Psi(x,T) = \Psi^T(x) := \partial_{8x}\Phi^T(x), & \text{in } (0,L), \end{cases}$$

where $\bar{\eta} = \bar{\eta}(\Phi, \partial_x\Phi, \ldots, \partial_{7x}\Phi) \in L^2(Q)$, $(\tilde{\Pi}, \tilde{\Lambda}, \tilde{\Gamma}, \tilde{\Delta}) \in C_c^\infty(Q)$. Furthermore, we get $\Phi_{10x}(0,t)\ \Phi_{10x}(L,t)\ \Phi_{11x}(0,t)\ \Phi_{11x}(L,t) \in L^2(0,T)$.

$\Longrightarrow$ we use our Carleman estimate for KS with non-homogeneous Neumann boundary conditions

$$\begin{cases} -\partial_t \Psi + \partial_x^4 \Psi = \tilde{\Pi}\partial_x^3\Psi + \tilde{\Lambda}\partial_x^2\Psi + \tilde{\Gamma}\partial_x\Psi + \tilde{\Delta}\Psi + \bar{\eta}, & \text{in } Q, \\ \Psi_{2x}(0,t) = \Phi_{10x}(0,t),\ \Psi_{2x}(L,t) = \Phi_{10x}(L,t), & \text{in } (0,T), \\ \Psi_{3x}(0,t) = \Phi_{11x}(0,t),\ \Psi_{3x}(L,t) = \Phi_{11x}(L,t), & \text{in } (0,T), \\ \Psi(x,T) = \Psi^T(x) := \partial_{8x}\Phi^T(x), & \text{in } (0,L), \end{cases}$$

Thus, we can apply the Carleman estimate with $\omega_8 \Subset \omega$ to get

$$\begin{aligned} &\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\Psi_{2x}|^2 + s^7\lambda^8\xi^7 e^{-2s\alpha}|\Psi|^2 dxdt \\ &\quad \le C\Bigg(\iint_Q e^{-2s\alpha}|\bar{\eta}|^2 dxdt + s^7\lambda^8 \iint_{Q_{\omega_8}} e^{-2s\alpha}\xi^7|\Psi|^2 dxdt \\ &\qquad + s^3\lambda^3\int_0^T e^{-2s\alpha^*}(\xi^*)^3(|\Phi_{10x}(0,t)|^2 + |\Phi_{10x}(L,t)|^2)dt \\ &\qquad\qquad + s\lambda\int_0^T e^{-2s\alpha^*}\xi^*(|\Phi_{11x}(0,t)|^2 + |\Phi_{11x}(L,t)|^2)dt\Bigg). \end{aligned}$$

Final Carleman estimate

- Step 1. estimate the boundary terms on the RHS.
- Step 2. compare $\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\Psi_{2x}|^2 + s^7\lambda^8\xi^7 e^{-2s\alpha}|\Psi|^2 dxdt$ with the desired LHS with $\Psi = \partial_x^8\Phi$
- Step 3. re-estimate the right-hand side.

Carleman result

There exists $C := C(L,\omega)$, $\lambda_0 > 0$ such that for all $\Phi^T \in L^2(0,L)^2$, $m > 1/2$ the corresponding solution Φ of adjoint system satisfies

$$\iint_Q e^{-2s\alpha}\sum_{r=0}^{10}(s\xi)^{23-2r}\lambda^{24-2r}|\partial_x^r\Phi|^2 dxdt \leq C\iint_{Q_\omega} e^{-2s\alpha}s^{23}\lambda^{24}\xi^{23}|\Phi|^2 dxdt,$$

for every $s \geq C(T^{2m} + T^{2m-1})$ and $\lambda \geq \lambda_0$.

Control result with One fictitious control

Null controllability with 2 very regular controls

The system

$$\begin{cases} \partial_t U + \partial_x^4 U = A\partial_x^3 U + B\partial_x^2 U + C\partial_x U + DU + \theta H, & \text{in } Q, \\ U(0,t) = U(L,t) = 0, & \text{in } (0,T), \\ \partial_x U(0,t) = \partial_x U(L,t) = 0, & \text{in } (0,T), \\ U(x,0) = U^0(x), & \text{in } (0,L), \end{cases}$$

is null controllable at any time T, i.e., for any $U^0 \in L^2(0,L)^2$, there exists a control $H \in L^2(Q)^2$ such that the solution satisfies $U(.,T) = 0$ in $(0,L)$.
$\forall K \in (0,1)$, $e^{Ks_0\alpha^*} H \in L^2(0,T;H^{10}(0,L)^2) \cap H^2(0,T;H^2(0,L)^2)$,

$$\|e^{Ks_0\alpha^*} H\|_{L^2(0,T;H^{10}(0,L)^2)\cap H^2(0,T;H^2(0,L)^2)} \leq C\|U^0\|_{L^2(0,L)^2}.$$

Final result

we use

- the null-controllability with very regular controls
- the algebraic resolution

Let $T > 0$, $L > 0$ and ω be a non-empty open subset of $(0, L)$. Let assume that the matrix is invertible. For any $U^0 \in L^2(Q)^2$, there exists a control $h \in L^2(Q)$ such that the solution satisfies $U(\cdot, T) = 0$ in $(0, L)$.

Conclusion:

- null controllability of a highly coupled 4th order parabolic system
- New Carleman estimates for a 4th order equation with Neumann non-homogeneous conditions
- Coupling terms acting on ω+invertible matrix

Perspectives:

- Case of boundary controls?
- What about the 2D or 3D case?