Reconstruction of degeneracy region and power for parabolic equations and systems

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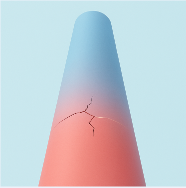
Degenerate parabolic equations: Motivations & Goal

Motivations

Climatology, financial mathematics, population dynamics, ...

Heat diffusion in a body with a **conductivity failure**:

$$\begin{cases} \partial_t w - \nabla\left(K(x)\nabla w\right) = f(x,w) \\ w(0,x) = w_0(x) \\ +B.C. \end{cases}$$



- $w(x,t)$: temperature at point x and time t;
- $K(x)$: thermal conductivity can **degenerate** (ability to resist heat transfer).

Goal

Inverse problem: determine $K(x)$ from some suitable additional observations.

The degeneracy identification problem

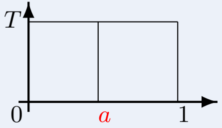
Starting point: Cannarsa, Doubova, Yamamoto, Inverse Problems (2024)

Assume: 1-D scalar model, linear decay: $K(x) = |x - a|$, $a \in (0,1)$

Inverse Problem: interior degeneracy at a point $a \in (0,1)$.

$$\begin{cases} \partial_t w - \partial_x(|x-a|\partial_x w) = 0 & (x,t) \in (0,1) \times (0,T) \\ w(0,t) = w(1,t) = 0 & t \in (0,T) \\ w(x,0) = w_0(x) & x \in (0,1) \end{cases}$$

Find $a \in (0,1)$ (degeneracy point) from suitable measurements η of w.



Our aim

- Identification problem for degenerate systems.
- Strong degeneracy: $K(x) = |x-a|^{\theta}$, $\theta \in [1,2)$, $a \in (0,1)$.
- Reconstruction of the degeneracy power.

Our Identification Problem: unknown interior degeneracy point $a \in (0,1)$

Complex parabolic equation with $\theta \in [1,2)$:

$$\begin{cases} \partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0 & (x,t) \in (0,1) \times (0,T) \\ w(0,t) = w(1,t) = 0 & t \in (0,T) \\ w(x,0) = w_0(x) & x \in (0,1) \end{cases}$$

with given $c = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, $w_0 = u_0 + iv_0$, u_0, v_0 real-valued functions.
$+$ additional observations at the boundary: $\eta(t) = \partial_x w(1,t)$.

$w(x,t) = u(x,t) + iv(x,t) \Longrightarrow$ **Coupled real system**:

$$\begin{cases} \partial_t u - \partial_x(|x-a|^\theta \partial_x u) - \alpha u + \beta v = 0, & (x,t) \in (0,1) \times (0,T), \\ \partial_t v - \partial_x(|x-a|^\theta \partial_x v) - \alpha v - \beta u = 0, & (x,t) \in (0,1) \times (0,T), \\ +\text{B. C. and I. C.} \end{cases}$$

First step: introduce energy spaces which depend on the degeneracy.

Functional setting

$X := L^2(0,1;\mathbb{C})$, the operator $\mathbb{A} : D(\mathbb{A}) \subset X \to X$ is defined by

$$D(\mathbb{A}) := H^2_\theta(0,1;\mathbb{C}) \quad \text{and} \quad \mathbb{A}w := Au + iAv \quad A := \partial_x(|x-a|^\theta \partial_x) \ ,$$

$$\forall w = u + iv \in D(\mathbb{A}) \quad \text{and u,v} \quad \mathbb{R}-\text{valued functions,}$$

where $H^2_\theta(0,1;\mathbb{C}) := \left\{ w \in H^1_\theta(0,1;\mathbb{C}) \ \mid (|x-a|^\theta w'(x))' \in X \right\}$ and

$$H^1_\theta(0,1;\mathbb{C}) := \Big\{ w \in X \ \mid \ w \text{ locally abs. cont. in } (a,1] \text{ and in } [0,a),$$
$$|x-a|^{\theta/2} w'(x) \in X \text{ and } w(0) = 0 = w(1) \Big\} \ .$$

Operator $\mathbb{A}$

- $\mathbb{A} : D(\mathbb{A}) \subset X \to X$ is **dissipative**, **self-adjoint**, with dense domain.
- $\mathbb{A}$ is the infinitesimal generator of an **analytic semigroup** of contractions $e^{t\mathbb{A}}$ on X and $t \mapsto w(\cdot,t)$ is an analytic map for all $t > 0$.

The problem can be recast in the abstract form

$$\begin{cases} w'(t) = (\mathbb{A} + c\mathbb{I})w(t) & t \geq 0, \\ w(0) = w_0. \end{cases}$$

Direct Problem: well-posedness

The function $w \in \mathcal{C}^0([0,T];X) \cap L^2(0,T;H^1_\theta(0,1;\mathbb{C}))$, given by

$$w(\cdot,t) = e^{t(\mathbb{A}+c\mathbb{I})}w_0 = e^{(\alpha+i\beta)t}\left(e^{tA}u_0 + ie^{tA}v_0\right) \ ,$$

is the **solution** of the problem in the sense of semigroup theory.

A function

$$w \in \mathcal{C}^0([0,T];H^1_\theta(0,1;\mathbb{C})) \cap H^1(0,T;X) \cap L^2(0,T;D(\mathbb{A}))$$

is a **strict solution** if satisfies $\partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0$ a.e. in $(0,1)\times(0,T)$, and the initial and boundary conditions for all $t \in [0,T]$ and all $x \in [0,1]$.

If $w_0 \in H^1_\theta(0,1;\mathbb{C})$, then the solution is the unique strict solution.

Inverse Problem: main questions

1. **Uniqueness**: w^1 and w^2 solutions corresponding to a_1 and a_2, respectively and $\eta^1 \equiv \eta^2$. Then, do we have $a_1 = a_2$?
2. **Stability**: can we estimate $|a_1 - a_2|$ in terms of $|\eta^1 - \eta^2|$?
3. **Reconstruction**: how to compute (numerically) a from η?

Reconstruction of strong degeneracy region

$$\begin{cases} \partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0 & (x,t) \in (0,1)\times(0,T) \\ w(0,t) = w(1,t) = 0 & t \in (0,T) \\ w(x,0) = w_0(x) & x \in (0,1) \end{cases}$$

Strong degeneracy: $|x-a|^\theta \partial_x w|_{x=a} = 0 \Longrightarrow$ no transmission of information from one side of the degeneracy to the other.

The strongly degenerate problem can be decoupled into two sub-problems of boundary degeneracy in $(0,a)$ and $(a,1)$.

$$\begin{cases} \partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0 & (x,t) \in (0,a)\times(0,T) \\ w(0,t) = 0, \quad (a-x)^\theta \partial_x w(x,t)\big|_{x=a} = 0 & t \in (0,T) \\ w(x,0) = w_0(x), & x \in (0,a) \end{cases}$$

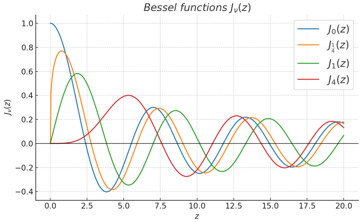
$$\begin{cases} \partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0 & (x,t) \in (a,1)\times(0,T) \\ w(1,t) = 0, \quad (x-a)^\theta \partial_x w(x,t)\big|_{x=a} = 0 & t \in (0,T) \\ w(x,0) = w_0(x) & x \in (a,1) \end{cases}$$

Spectral Analysis for the diffusion operator

$$\begin{cases} -((x-a)^\theta \phi'(x))' = \lambda\phi, & x \in (a,1) \\ \phi(1) = 0, & (x-a)^\theta \phi'(x)\big|_{x=a} = 0 \end{cases}$$

can be solved by using Bessel functions J_{ν_θ} of the first kind and of order $\nu_\theta = \frac{\theta-1}{2-\theta}$, obtaining

$$\lambda_1 < \lambda_2 < \dots \quad , \quad \text{with}$$



$$\lambda_n = k_\theta^2 \frac{j_{\nu_\theta,n}^2}{(1-a)^{2k_\theta}}, \quad \phi_n(x) = \frac{\sqrt{2k_\theta}}{|J'_{\nu_\theta}(j_{\nu_\theta,n})|} \left(\frac{x-a}{1-a}\right)^{\frac{1-\theta}{2}} J_{\nu_\theta}\left(j_{\nu_\theta,n}\left(\frac{x-a}{1-a}\right)^{k_\theta}\right)$$

Explicit form of the normal derivative of $w^a(x,t) = u^a(x,t) + iv^a(x,t)$

$$\begin{pmatrix} \partial_x u^a(1,t) \\ \partial_x v^a(1,t) \end{pmatrix} = e^{\alpha t} \sum_{n=1}^{\infty} \frac{2k_\theta e^{-\lambda_n(a)t}}{J'_{\nu_\theta}(j_{\nu_\theta,n})\,(j_{\nu_\theta,n})^{\frac{1}{2k_\theta}}} R(\beta t) \begin{pmatrix} U_n^0(a) \\ V_n^0(a) \end{pmatrix},$$

$$U_n^0(a) := \int_0^{j_{\nu_\theta,n}} u_0\left(a + (1-a)\left(\frac{s}{j_{\nu_\theta,n}}\right)^{\frac{1}{k_\theta}}\right) s^{\frac{1}{2k_\theta}} J_{\nu_\theta}(s)\, ds \ ,$$

$$V_n^0(a) := \int_0^{j_{\nu_\theta,n}} v_0\left(a + (1-a)\left(\frac{s}{j_{\nu_\theta,n}}\right)^{\frac{1}{k_\theta}}\right) s^{\frac{1}{2k_\theta}} J_{\nu_\theta}(s)\, ds \ .$$

Reformulation of the identification problem

The problem (boundary degeneracy) for $w^a(x,t) = u^a(x,t) + iv^a(x,t)$:

$$\begin{cases} \partial_t w - \partial_x((x-a)^\theta \partial_x w) - cw = 0 & (x,t) \in (a,1)\times(0,T) \\ w(1,t) = 0, \quad (x-a)^\theta \partial_x w(x,t)\big|_{x=a} = 0 & t \in (0,T) \\ w(x,0) = w_0(x) & x \in (a,1) \end{cases}$$

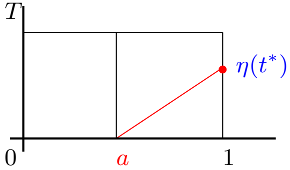
Measurement: $\eta(t^*) = \partial_x w^a(1,t^*)$ for some $t^* \in (0,T)$.

Goal: obtain a Lipschitz stability estimate.

Reformulation of IP

Are there initial data u_0, v_0 such that for some $t \in (0,T)$ and $\forall\, a \in [\tau,\gamma] \subset (0,1)$

$$|\partial_a(\partial_x w^a(1,t))| > 0 \ ?$$



Such a property would guarantee the injectivity of the map $a \mapsto \partial_x w^a(1,t)$ and the Lipschitz stability estimate:

$$|\partial_x w^{a_2}(1,t) - \partial_x w^{a_1}(1,t)| \geq C|a_2 - a_1| \ .$$

Lipschitz stability with one-point measurement

Theorem

Let $\theta \in [1,2)$ and assume $u_0, v_0 \in Lip([0,1])$, $|u_0| > 0$ or $|v_0| > 0$, $x \in (a,1)$.
Let $a_1, a_2 \in [\tau, \gamma] \subset (0,1)$, and w^{a_1}, w^{a_2} the corresponding solutions to

$$\begin{cases} \partial_t w - \partial_x((x-a)^\theta \partial_x w) - (\alpha + i\beta) w = 0 & (x,t) \in (a,1) \times (0,T) \\ w(1,t) = 0, \quad (x-a)^\theta \partial_x w(x,t)\big|_{x=a} = 0 & t \in (0,T) \\ w(x,0) = w_0(x) & x \in (a,1) \end{cases}$$

Then, $\exists\, t_0(u_0, v_0, \theta) > 0$ and a constant $C > 0$ such that the following holds:

$$|a_2 - a_1| \le C |\partial_x w^{a_2}(1,t) - \partial_x w^{a_1}(1,t)| \ ,$$

- for all $a_1, a_2 \in [\tau, \gamma]$ and for all $t \in [t_0, t_1]$, if $\lambda_1(\gamma) > \alpha$;
- for all $a_1, a_2 \in [\tau, \gamma]$ and for all $t \ge t_0$, if $\lambda_1(\gamma) \le \alpha$,

where $\lambda_1(\gamma) = k_\theta^2 \dfrac{j^2_{\nu_\theta,1}}{(1-\gamma)^{2k_\theta}}$.

Idea of the proof: Check if $a \mapsto |\partial_a \partial_x w^a(1,t)| > 0$ for t large enough.

1. Explicit expressions for w^a and $\partial_x w^a(1,t)$ in terms of Bessel functions.
2. Final estimates, neglecting suitable terms for large t.

Examples of initial conditions determining stability

Example: $u_0 = 0,\ v_0 = 1,\ \theta = 1.3,\ \alpha = 1,\ \beta = 1/2$

Lipschitz stability estimate: $\forall$ t large enough, $\forall\ a \in [\tau, \gamma] \subset (0, 1)$

$$|a_2 - a_1| \leq C|\partial_x w^{a_2}(1,t) - \partial_x w^{a_1}(1,t)|$$

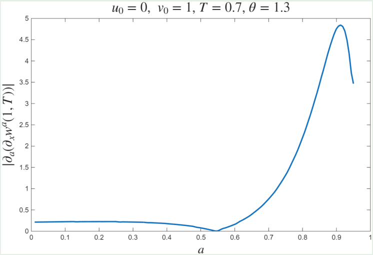


Figure: lack of stability, $T = 0.7$

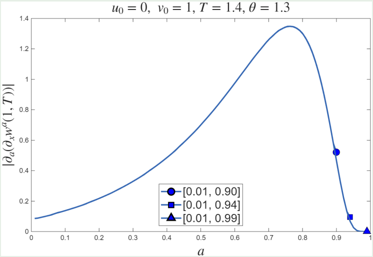


Figure: stability for t large, $T = 1.4$

Two more general inverse problems

Problem 1 Unknowns: degeneracy point a and initial data u_0, v_0

$$\begin{cases} \partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0 & (x,t) \in (a,1)\times(0,T) \\ w(1,t) = 0, \quad (x-a)^\theta \partial_x w(x,t)\big|_{x=a} = 0 & t \in (0,T) \\ w(x,0) = w_0(x) & x \in (a,1) \end{cases}$$

Measurements: $\eta(t) = \partial_x w(1,t)$ for all $t \in (t_1,t_2)$.

Problem 2 Unknowns: degeneracy point a, coefficient c and initial data u_0, v_0

$(a,1)$:

$$\begin{cases} \partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0 \\ w(1,t) = 0, \\ (x-a)^\theta \partial_x w(x,t)\big|_{x=a} = 0 \\ w(x,0) = w_0(x) \end{cases}$$

$(0,a)$:

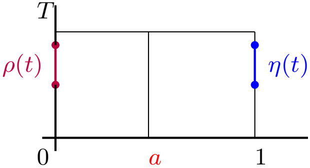
$$\begin{cases} \partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0 \\ w(0,t) = 0, \\ (x-a)^\theta \partial_x w(x,t)\big|_{x=a} = 0 \\ w(x,0) = w_0(x) \end{cases}$$

Measurements: $\eta(t) = \partial_x w(1,t)$ and $\rho(t) = \partial_x w(0,t)$ for all $t \in (t_1,t_2)$.

Inverse Problems

1. Find a and w_0 from $\eta(t)$
2. Find a, c and w_0 from $\eta(t)$ and $\rho(t)$

distributed measurements over a time interval



Uniqueness results for “distributed” measurements

Problem 1

Unknowns: degeneracy point a and initial data $w_0 = u_0 + iv_0$

$$\begin{cases} \partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0 & (x,t) \in (a,1) \times (0,T) \\ w(1,t) = 0, \quad (x-a)^\theta \partial_x w(x,t)\big|_{x=a} = 0 & t \in (0,T) \\ w(x,0) = w_0(x) & x \in (a,1) \end{cases}$$

Measurements: $\eta(t) = \partial_x w(1,t)$ for all $t \in (t_1, t_2)$.

Theorem

Let $\theta \in [1,2)$, $0 < a_1, a_2 < 1$ and $0 < t_1 < t_2$. Let w^{a_1} and w^{a_2} be two solutions corresponding to $w_0 = u_0 + iv_0$ and $\widetilde{w}_0 = \widetilde{u}_0 + i\widetilde{v}_0$, respectively. Assume $|w_0| > 0$ in $(a,1)$, $|\widetilde{w}_0| > 0$ in $(a,1)$. Then, if for all $t_1 < t < t_2$

$$\partial_x w^{a_1}(1,t) = \partial_x w_x^{a_2}(1,t) \quad \Longrightarrow \quad a_1 = a_2 \quad \text{and} \quad w_0 = \widetilde{w}_0 \quad \text{in } (a,1)$$

Idea of the proof: explicit expression of $\partial_x w^a(1,t)$ + analyticity for all $t > 0$

+ monotonicity of the first eigenvalue $\lambda_1 = k_\theta^2 \dfrac{j_{\nu_\theta,1}^2}{(1-a)^{2k_\theta}}$ w.r.t. a.

Uniqueness results for "distributed" measurements

Problem 2

Unknowns: degeneracy point a, coefficient c and initial data $w_0 = u_0 + iv_0$

$(a, 1)$:

$$\begin{cases} \partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0 \\ w(1,t) = 0, \\ (x-a)^\theta \partial_x w(x,t)\big|_{x=a} = 0 \\ w(x,0) = w_0(x) \end{cases}$$

$(0, a)$:

$$\begin{cases} \partial_t w - \partial_x(|x-a|^\theta \partial_x w) - cw = 0 \\ w(0,t) = 0, \\ (x-a)^\theta \partial_x w(x,t)\big|_{x=a} = 0 \\ w(x,0) = w_0(x) \end{cases}$$

Measurements: $\eta(t) = \partial_x w(1,t)$ and $\rho(t) = \partial_x w(0,t)$ for all $t \in (t_1, t_2)$.

Theorem

Let $\theta \in [1,2)$, $0 < a_1, a_2 < 1$ and $0 < t_1 < t_2$. Let w^{a_1} and w^{a_2} be two solutions corresponding to the initial data $w_0 = u_0 + iv_0$ and $\widetilde{w}_0 = \widetilde{u}_0 + i\widetilde{v}_0$ and the coefficients c and $\widetilde{c}$, respectively.
Assume $|w_0| > 0$ in $(0,1)$, $|\widetilde{w}_0| > 0$ in $(0,1)$. Then, if for all $t_1 < t < t_2$

$$\begin{cases} \partial_x w^{a_1}(1,t) = \partial_x w^{a_2}(1,t) \\ \partial_x w^{a_1}(0,t) = \partial_x w^{a_2}(0,t) \end{cases} \quad \Longrightarrow \quad \begin{cases} a_1 = a_2, \quad c = \widetilde{c} \\ \text{and} \quad w_0 = \widetilde{w}_0 \quad \text{in } (0,1) \end{cases}$$

Coupled systems:

$$\begin{cases} \partial_t u - \partial_x(|x-a|^\theta \partial_x u) - \alpha u + \beta v = 0 & (x,t) \in (0,1)\times(0,T) \\ \partial_t v - \partial_x(|x-a|^\theta \partial_x v) - \alpha v - \beta u = 0 & (x,t) \in (0,1)\times(0,T) \\ \begin{pmatrix} u(0,t) \\ v(0,t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & t \in (0,T) \\ \begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} & x \in (0,1) \end{cases}$$

Test performed

1. **Test 1**: Find a and also the initial data from distributed two-side measurements $\eta(t) = \partial_x u(1,t)$ and $\zeta(t) = \partial_x v(1,t)$, $\rho(t) = \partial_x u(0,t)$ and $\kappa(t) = \partial_x v(0,t)$, for $t \in (t_1, t_2)$.
2. **Test 2**: Find a and also the initial data from distributed one-side measurements $\eta(t) = \partial_x u(1,t)$ and $\zeta(t) = \partial_x v(1,t)$ for $t \in (t_1, t_2)$.

Test 1: distributed two-side measurements

Given: two distributed measurements in (t_1, t_2) at $x = 0, 1$, initial datum v_0.

Inverse Problem: find $a \in \mathcal{U}_{ad}^a = \{a : a \in (\delta, 1-\delta)\}$ ($\delta > 0$ small) and piecewise-constant initial data

$$u_0 = \begin{cases} u_{01} & \text{if } 0 < x < a, \\ u_{02} & \text{if } a < x < 1, \end{cases}$$

such that u^a and v^a satisfy:

$$\partial_x u^a(1,t) = \eta(t)\ , \qquad \partial_x v^a(1,t) = \zeta(t)\ , \qquad \text{for } t \in (t_1, t_2)\ , \quad 0 \le t_1 \le t_2 \le T\ ,$$
$$\partial_x u^a(0,t) = \rho(t)\ , \qquad \partial_x v^a(0,t) = \kappa(t)\ , \qquad \text{for } t \in (t_1, t_2)\ , \quad 0 \le t_1 \le t_2 \le T\ .$$

Reformulation (optimization problem): find $a \in \mathcal{U}_{ad}^a$ and u_0 that

$$\text{Minimize} \quad \mathcal{H}(a, u_{01}, u_{02}),$$

where $\mathcal{H} : (a, u_{01}, u_{02}) \in \mathcal{U}_{ad}^a \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is defined as follows:

$$\mathcal{H}(a, u_{01}, u_{02}) =$$
$$= \frac{1}{2}\int_{t_1}^{t_2} |\eta(t) - \partial_x u^{a,u_{01},u_{02}}(1,t)|^2\,dt \quad + \frac{1}{2}\int_{t_1}^{t_2} |\zeta(t) - \partial_x v^{a,u_{01},u_{02}}(1,t)|^2\,dt$$
$$+ \frac{1}{2}\int_{t_1}^{t_2} |\rho(t) - \partial_x u^{a,u_{01},u_{02}}(0,t)|^2\,dt + \frac{1}{2}\int_{t_1}^{t_2} |\kappa(t) - \partial_x v^{a,u_{01},u_{02}}(0,t)|^2\,dt.$$

MATLAB Optimization ToolBox: `fmincon`

Test 1: $v_0 = 1$, $\theta = 1.5$, $\alpha = 1$, $\beta = 1$, $T = 4$, $t_1 = 0$, $t_2 = T$.
Initial guess: $u_{01in} = 0.5$, $u_{02in} = 1.5$, $a_i = 0.1$.

Desired: $a_d = 0.5$, $u_{01d} = 1$, $u_{02d} = 2$.

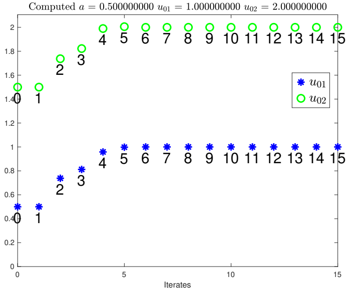


Figure: Iterations in the computation of u_{01} and u_{02}.

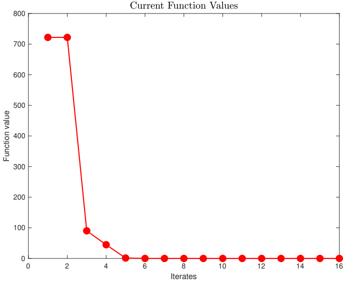


Figure: The evolution of the cost. Final cost $\approx 1.e-18$

We can solve the IP in $(0,1)$ with a two-side distributed measurement!

Test 2 - A: distributed one-side measurements

Given: a distributed measurement in (t_1, t_2) at point $x = 1$, initial datum v_0.

Inverse Problem: find $a \in \mathcal{U}_{ad}^a = \{a : a \in (\delta, 1-\delta)\}$ ($\delta > 0$ small) and piecewise-constant initial data

$$u_0 = \begin{cases} u_{01} & \text{if } 0 < x < a, \\ u_{02} & \text{if } a < x < 1, \end{cases}$$

such that u^a and v^a satisfy:

$$\partial_x u^a(1,t) = \eta(t)\ , \qquad \partial_x v^a(1,t) = \zeta(t)\,, \quad \text{for } t \in (t_1, t_2)\,, \quad 0 \le t_1 \le t_2 \le T\,.$$

Reformulation (optimization problem): find $a \in \mathcal{U}_{ad}^a$ and u_0 that

$$\text{Minimize} \quad \mathcal{M}(a, u_{01}, u_{02}),$$

where $\mathcal{M} : (a, u_{01}, u_{02}) \in \mathcal{U}_{ad}^a \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is defined as follows:

$$\mathcal{M}(a, u_{01}, u_{02}) = \frac{1}{2}\int_{t_1}^{t_2} \left|\eta(t) - \partial_x u^{a,u_{01},u_{02}}(1,t)\right|^2 dt + \frac{1}{2}\int_{t_1}^{t_2} \left|\zeta(t) - \partial_x v^{a,u_{01},u_{02}}(1,t)\right|^2 dt.$$

Numerical results

Test 2 - A: $v_0 = 0$, $\theta = 1.5$, $\alpha = 1$, $\beta = 1$, $T = 2$, $t_1 = 0$, $t_2 = T$.
Initial guess: $u_{01in} = 0.5$, $u_{02in} = 1.8$ $a_i = 0.1$.

Desired: $a_d = 0.5$, $u_{01d} = 1$, $u_{02d} = 2$.

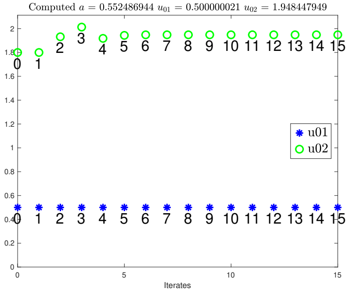


Figure: Iterations in the computation of u_{01} and u_{02}.

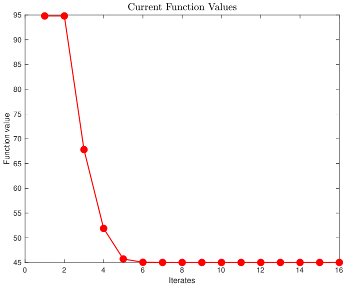


Figure: The evolution of the cost.

We cannot solve the IP in $(0, 1)$ with only a one-side distributed measurement!

Numerical results

Test 2 - B: $v_0 = 0$, $u_{01} = 1$, $\alpha = 1$, $\beta = 1$, $\theta = 1.5$, $T = 2$, $t_1 = 0$, $t_2 = T$.
Initial guess: $u_{02in} = 1.8$ $a_i = 0.1$.

Desired: $a_d = 0.5$, $u_{02d} = 2$.

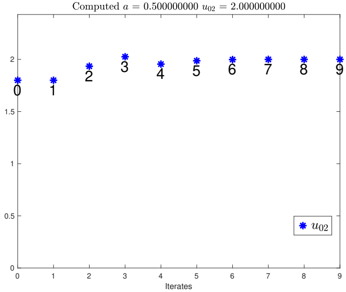


Figure: Iterations in the computation of u_{02}.

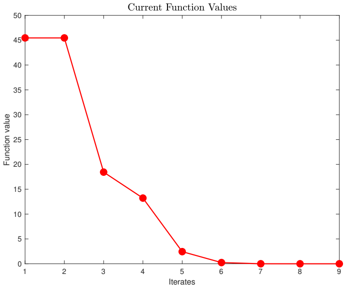


Figure: The evolution of the cost. Final cost $\approx 1.e-18$

We can solve the IP in $(a, 1)$ with a one-side distributed measurement in $x = 1$!

Reconstruction of the degeneracy power

Problem

Unknowns: degeneracy power θ and initial data $w_0 = u_0 + iv_0$

$$\begin{cases} \partial_t w - \partial_x(|x-a|^{\theta}\partial_x w) - cw = 0 & (x,t) \in (a,1) \times (0,T) \\ w(1,t) = 0, \quad (x-a)^{\theta}\partial_x w(x,t)\big|_{x=a} = 0 & t \in (0,T) \\ w(x,0) = w_0(x) & x \in (a,1) \end{cases}$$

where a is given.

Measurements: $\eta(t) = \partial_x w(1,t)$ for all $t \in (t_1, t_2)$.

Goal: uniqueness result with a one-side distributed measurements.

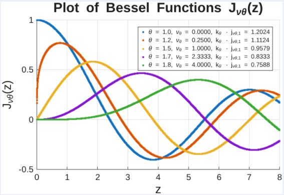
Theorem

Let $\theta_1, \theta_2 \in [1,2)$ and $0 < t_1 < t_2$. Let w^{θ_1} and w^{θ_2} be two solutions corresponding to $w_0 = u_0 + iv_0$ and $\widetilde{w}_0 = \widetilde{u}_0 + i\widetilde{v}_0$, respectively. Assume $|w_0| > 0$ in $(a,1)$, $|\widetilde{w}_0| > 0$ in $(a,1)$. Then, if for all $t_1 < t < t_2$

$$\partial_x w^{\theta_1}(1,t) = \partial_x w_x^{\theta_2}(1,t) \quad \Longrightarrow \quad \theta_1 = \theta_2 \quad \text{and} \quad w_0 = \widetilde{w}_0 \quad \text{in } (a,1)$$

Idea of the proof

1. Explicit expression of $\partial_x w^\theta(1,t)$ + Analyticity for all $t > 0$.
2. Monotonicity of the first eigenvalue $\lambda_1 = k_\theta^2 \dfrac{j_{\nu_\theta,1}^2}{(1-a)^{2k_\theta}}$ w.r.t. θ.



Abstract framework

- $\theta < 2$: compact injection of H^1_θ into L^2.
- Variational formulation of the first eigenvalue:

$$\lambda_1(\theta) := \min_{\phi_1 \in H^1_\theta \setminus \{0\}} \frac{\int_a^1 (x-a)^\theta \, |\phi_1'|^2 \, dx}{\int_a^1 |\phi_1|^2 \, dx},$$

where ϕ_1 is an associated eigenfunction.
- We can prove that if $\theta_1 < \theta_2 \implies \lambda_1(\theta_1) > \lambda_2(\theta_2)$.

Conclusions

Our result

- Several identification problems for degenerate coupled systems.
- Strong degeneracy: $K(x) = |x-a|^{\theta}$, $\theta \in [1,2)$, $a \in (0,1)$.
- Unknowns: degeneracy point and power, zero-order coefficient and initial data.

Reference: P. Cannarsa, V. Danesi, A. Doubova, *Reconstruction of degeneracy region and power for parabolic equations and systems.*
https://arxiv.org/abs/2509.13962v2

Open questions & work in progress

- Other techniques?
- Degeneracy region and power in higher dimension?
- Weak degeneracy with $\theta \in [0,1)$?

Thanks for your attention

Idea of the proof

Check if $|\partial_a (\partial_x w^a(1,t))| > 0$ for t large enough:

1. Explicit expressions for w^a and $\partial_x w^a(1,t)$ in terms of Bessel functions.
2. Estimates, neglecting suitable terms for large t.

$$\begin{pmatrix} \partial_x u^a(1,t) \\ \partial_x v^a(1,t) \end{pmatrix} = e^{\alpha t} \sum_{n=1}^{\infty} \frac{2k_\theta e^{-\lambda_n(a)t}}{J'_{\nu_\theta}(j_{\nu_\theta,n}) \left(j_{\nu_\theta,n}\right)^{\frac{1}{2k_\theta}}} R(\beta t) \begin{pmatrix} U_n^0(a) \\ V_n^0(a) \end{pmatrix},$$

$$\left| \partial_a \begin{pmatrix} \partial_x u^a(1,t) \\ \partial_x v^a(1,t) \end{pmatrix} \right| = \left| e^{\alpha t} \sum_{n=1}^{\infty} \frac{2k_\theta e^{-\lambda_n t}}{d_{\nu_\theta,n}} R(\beta t) \begin{pmatrix} F_n^1(a) \\ F_n^2(a) \end{pmatrix} \right|$$

$$\geq e^{\alpha t} e^{-\lambda_1 t} 2k_\theta \left(\frac{1}{|d_{\nu_\theta,1}|} \left| \begin{pmatrix} F_1^1(a) \\ F_1^2(a) \end{pmatrix} \right| - \sum_{n=2}^{\infty} \frac{e^{-(\lambda_n - \lambda_1)t}}{|d_{\nu_\theta,n}|} \left| \begin{pmatrix} F_n^1(a) \\ F_n^2(a) \end{pmatrix} \right| \right)$$

where $d_{\nu_\theta,n} := J'_{\nu_\theta}(j_{\nu_\theta,n}) \left(j_{\nu_\theta,n}\right)^{\frac{1}{2k_\theta}}$ and

$$F_n^1(a) := \left[(U_n^0)'(a) - 2k_\theta^3 \frac{j_{\nu_\theta,n}^2 t}{(1-a)^{2k_\theta+1}} U_n^0(a) \right],$$

$$F_n^2(a) := \left[(V_n^0)'(a) - 2k_\theta^3 \frac{j_{\nu_\theta,n}^2 t}{(1-a)^{2k_\theta+1}} V_n^0(a) \right].$$

Idea of the proof

$$|\partial_a(\partial_x w^a(1,t))| \geq e^{\alpha t} e^{-\lambda_1 t} 2k_\theta \left(\frac{1}{|d_{\nu_\theta,1}|} \left| \begin{pmatrix} F_1^1(a) \\ F_1^2(a) \end{pmatrix} \right| - \sum_{n=2}^{\infty} \frac{e^{-(\lambda_n-\lambda_1)t} \left(|F_n^1(a)| + |F_n^2(a)|\right)}{|d_{\nu_\theta,n}|} \right)$$

- First term: From the hypothesis $|u_0| > 0$ or $|v_0| > 0$ in $x \in (a,1)$, we get $\left| \begin{pmatrix} U_1^0(a) \\ V_1^0(a) \end{pmatrix} \right| \geq \delta, \forall\, a \in [\tau,\gamma] \Longrightarrow \left| \begin{pmatrix} F_1^1(a) \\ F_1^2(a) \end{pmatrix} \right| \geq L \;\; \forall\, a \in [\tau,\gamma], \quad \forall\, t \geq \bar{t}.$
- Other terms: $\displaystyle\sum_{n=2}^{\infty} \frac{e^{-(\lambda_n-\lambda_1)t} \left(|F_n^1(a)| + |F_n^2(a)|\right)}{|d_{\nu_\theta,n}|} \leq K_1(t,a) + K_2(t,a)$
  and $\displaystyle\lim_{t\to+\infty} K_1(t,a) = 0$, $\displaystyle\lim_{t\to+\infty} K_2(t,a) = 0$.

$\Longrightarrow$ for all $a_1, a_2 \in [\tau,\gamma]$, we get

$$|\partial_x w^{a_2}(1,t) - \partial_x w^{a_1}(1,t)| \geq e^{\alpha t} e^{-\lambda_1(\gamma)t} 2k_\theta \frac{L}{|d_{\nu_\theta,1}|} |a_2 - a_1|, \quad \forall\, t \geq t_0(u_0, v_0, \delta, L, \theta)\,.$$

By fixing t, we get the stability estimate with C depending on $\lambda_1(\gamma)$ and α.

Test 3: distributed one-side measurements

Given: a distributed measurement in (t_1, t_2) at point $x = 1$, the degeneracy point a, the initial datum v_0 and the initial datum u_0 in $(0, a)$.

Inverse Problem: find $\theta \in \mathcal{V}_{ad}^{\theta} = \{\theta : \theta \in [1, 2-\delta]\}$ ($\delta > 0$ small) and piecewise-constant initial data

$$u_0 = \begin{cases} u_{01} & \text{if } 0 < x < a, \\ u_{02} & \text{if } a < x < 1, \end{cases}$$

such that u^a and v^a satisfy:

$$\partial_x u^a(1,t) = \eta(t) \, , \qquad \partial_x v^a(1,t) = \zeta(t) \, , \quad \text{for } t \in (t_1, t_2) \, , \quad 0 \leq t_1 \leq t_2 \leq T \, .$$

Reformulation (optimization problem): find $\theta \in \mathcal{V}_{ad}^{\theta}$ and u_{02} that

$$\text{Minimize} \quad \mathcal{N}(\theta, u_{02}),$$

where $\mathcal{N} : (\theta, u_{02}) \in \mathcal{V}_{ad}^{\theta} \times \mathbb{R} \mapsto \mathbb{R}$ is defined as follows:

$$\mathcal{N}(\theta, u_{02}) = \frac{1}{2} \int_{t_1}^{t_2} |\eta(t) - \partial_x u^{\theta, u_{02}}(1,t)|^2 \, dt + \frac{1}{2} \int_{t_1}^{t_2} |\zeta(t) - \partial_x v^{\theta, u_{02}}(1,t)|^2 \, dt.$$

Numerical results

Test 3: $a = 0.5$, $v_0 = 0$, $u_{01} = 1$, $\alpha = 1$, $\beta = 1$, $T = 4$, $t_1 = 0$, $t_2 = T$.
Initial guess: $u_{02in} = 1.5$ $\theta_{ini} = 1.1$.

Desired: $\theta_d = 1.5$, $u_{02d} = 2$.

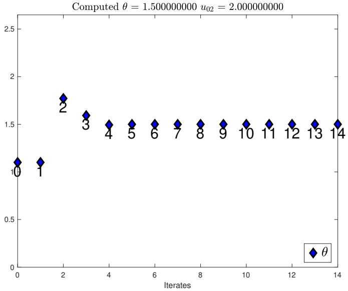


Figure: Iterations in the computation of θ.

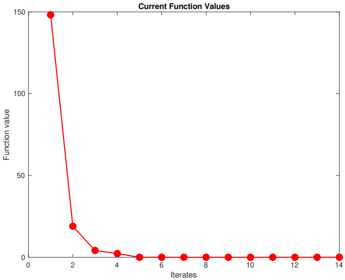


Figure: The evolution of the cost. Final cost $\approx 1.e - 26$

We can solve the IP in $(a, 1)$ with a one-side distributed measurement in $x = 1$!

Test : one-point measurements

Given: some $t^* \in (0,T)$ and a one-point measurement.

Inverse Problem: find $a \in \mathcal{U}^a_{ad} = \{a : a \in (\delta, 1-\delta)\}$ ($\delta > 0$ small) such that u^a and v^a satisfy:

$$\partial_x u^a(1,t^*) = \eta(t^*) \ , \qquad \partial_x v^a(1,t^*) = \zeta(t^*) \ .$$

Reformulation (optimization problem): find $a \in \mathcal{U}^a_{ad}$

$$J(a) \leq J(a') \quad \forall\, a' \in \mathcal{U}^a_{ad}$$

where $J : a \in \mathcal{U}^a_{ad} \mapsto \mathbb{R}$ is

$$J(a) = \frac{1}{2}|\eta(t^*) - \partial_x u^a(1,t^*)|^2 + \frac{1}{2}|\zeta(t^*) - \partial_x v^a(1,t^*)|^2$$

for some $t^* \in (0,T)$.

Numerical results

Test : $\color{blue}{u_0 = 1,\ v_0 = 1}$, $\theta = 1.5$, $T = 4$, $t^* = 1.99$. Initial guess: $a_i = 0.1$.
Desired: $\color{red}{a_d = 0.5}$, Computed: $\color{red}{a_c}$

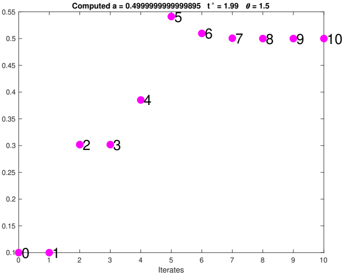


Figure: Iterations in the computation of a

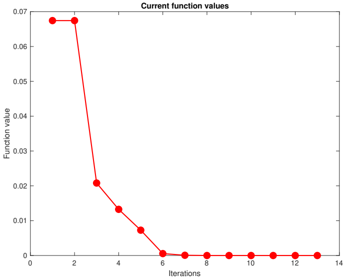


Figure: The evolution of the cost

Noise	Cost	Iterations	a_c
1%	1.e-15	16	0.4967806209438190
0.1%	1.e-16	10	0.5010448098047946
0.01%	1.e-19	10	0.5000110001604564
0%	1.e-27	10	0.4999999999999895

Test 2 - B: distributed one-side measurements

Given: a distributed measurement in the interval (t_1, t_2) at point $x = 1$, initial datum v_0, initial datum u_0 in $(0, a)$.

Inverse Problem: find $a \in \mathcal{U}^a_{ad} = \{a : a \in (\delta, 1 - \delta)\}$ ($\delta > 0$ small) and initial data u_{02} in $(a, 1)$

$$u_0 = \begin{cases} u_{01} & \text{if } 0 < x < a, \\ u_{02} & \text{if } a < x < 1, \end{cases}$$

such that u^a and v^a satisfy:

$$\partial_x u^a(1,t) = \eta(t) \,, \qquad \partial_x v^a(1,t) = \zeta(t) \,, \quad \text{for } t \in (t_1, t_2) \,, \quad 0 \le t_1 \le t_2 \le T \,.$$

Reformulation (optimization problem): find $a \in \mathcal{U}^a_{ad}$ and u_{02} that

$$\text{Minimize} \quad \mathcal{K}(a, u_{02}),$$

where $\mathcal{K} : (a, u_{02}) \in \mathcal{U}^a_{ad} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is defined as follows:

$$\mathcal{K}(a, u_{02}) = \frac{1}{2} \int_{t_1}^{t_2} \left| \eta(t) - \partial_x u^{a,u_{02}}(1,t) \right|^2 dt + \frac{1}{2} \int_{t_1}^{t_2} \left| \zeta(t) - \partial_x v^{a,u_{02}}(1,t) \right|^2 dt.$$

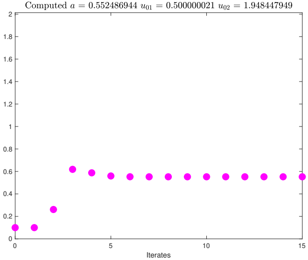


Figure: Test 2-A: Iterations in the computation of a.

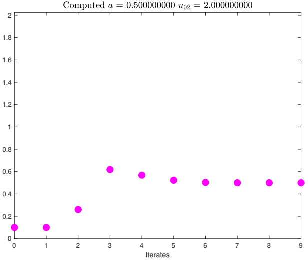


Figure: Test 2-B: Iterations in the computation of a.

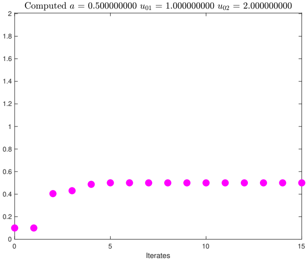


Figure: Test 1: Iterations in the computation of a.