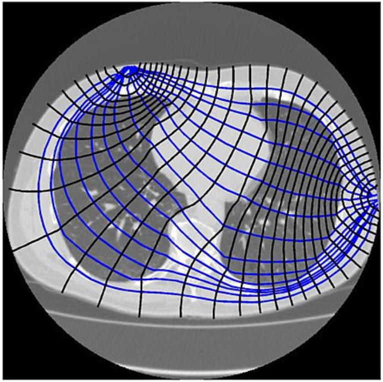
The Born approximation in the reconstruction of the Calderon inverse problem

C. Castro[+], F. Macià, C. Meroño and D. Sánchez-Mendoza
[+]Univ. Politécnica de Madrid

December 1, 2025

Electrical Impedance Tomography (wikipedia)



Stream lines (blue) and equipotential lines (black)

The Calderon problem

Consider the following system in the unit ball

$$(P1)\begin{cases} -div\,(\gamma(x)\nabla u(x)) = 0, & x \in \Omega, \\ u = f, \quad x \in \partial\Omega. \end{cases}$$

Given $\gamma \in L^\infty(\Omega)$ we define the Dirichlet to Neumann map (DtN) as

$$\begin{array}{rcl} \Lambda_\gamma : H^{1/2}(\partial\Omega) & \to & H^{-1/2}(\partial\Omega) \\ f & \to & \gamma \frac{\partial u}{\partial n}\Big|_{\partial\Omega} \end{array}$$

The reconstruction problem: Can we find γ from Λ_γ?

In other words, can we invert the following map?

$$\gamma \to \Phi(\gamma) = \Lambda_\gamma$$

Equivalent potential problem

$$(P2)\begin{cases} -\Delta u(x) + q(x)\, u(x) = 0, & x \in \Omega, \\ u = f, & x \in \partial\Omega. \end{cases}$$

Given $q \in L^\infty(\Omega)$ we define the Dirichlet to Neumann map (DtN) as

$$\begin{array}{rcl} \Lambda_q : H^{1/2}(\partial\Omega) & \to & H^{-1/2}(\partial\Omega) \\ f & \to & \frac{\partial u}{\partial n} \end{array}$$

The reconstruction problem: Can we find q from Λ_q?
Remark. For smooth γ, (P1) is equivalent to (P2) with

$$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$$

Is the problem well-posed?

Uniqueness: Does $\Lambda_\gamma = \Phi(\gamma)$ determines γ?

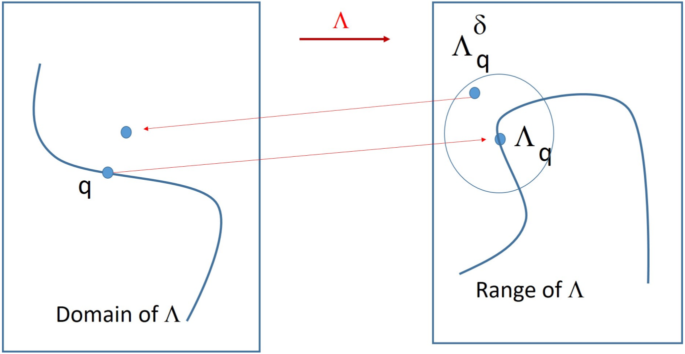
- J. Sylvester and G. Uhlmann (1987). Dimensión $d \geq 3$ and smooth γ.
- A. Nachman (1996). Dimension $d = 2$ and $\gamma \in C^2$.
- M. Lassas, L. Päivärinta and G. Uhlmann (2003). Dimension $d \geq 3$ and γ Lipschitz.
- K. Astala and L. Päivärinta (2006). Dimension $d = 2$ and $\gamma \in L^\infty$.

Stability:

- G. Alessandrini (1988) proved logarithmic stability for $d \geq 3$.
- Mandache (2001) proved the optimality of this log-stability result for $d \geq 2$.

Ill-posed problem

Close DtN may be associated with very different q, but also one can be outside the domain of Λ [Siltanen and Mueller 2012]



Reconstruction methods

- **NOSER** (Cheney, Isaacson and Newell, 1990). A suitable choice of a basis gives the finite dimensional system

$$\gamma \sim \gamma_N = (\gamma_i)_{1\leq i\leq N}, \quad \Lambda_\gamma \sim \Lambda_\gamma^M = (\Lambda_\gamma)_{1\leq i,j\leq M} = \Phi^M(\gamma)$$

Now we can approximate γ^* by minimizing the least square functional

$$J(\gamma_N) = \frac{1}{2}\|\Phi^M(\gamma_N) - \Lambda_{\gamma^*}^M\|^2 + R$$

where γ^* is the unknown conductivity and R is a regularization term, typically Tikhonov or total variation. Optimality condition

$$dJ(\gamma_N^*) = \left(\Phi^M(\gamma_N^*) - \Lambda_{\gamma^*}^M\right) d\Phi^M(\gamma_N^*) = F^N(\gamma_N^*) = 0$$

is solved by a one-step Newton method from the constant conductivity $\gamma_N^0 = 1$:

$$\gamma_N^1 = \gamma_N^0 - [dF^N(\gamma_N^0)]^{-1} F^N(\gamma_N^0)$$

- **Direct.** Use the CGO solutions in the unicity results (Knudsen, Lassas, Mueller and Siltanen, 2009, and Bikowski, Knudsen and Mueller, 2012).

$$(\Lambda_\gamma, \Lambda_1) \to \psi(x,\zeta)|_{\partial\Omega} \to t(\xi,\zeta) \to q(x) \to \gamma$$

where

$$\psi(x,\zeta) = e^{ix\cdot\zeta} - \int_{\partial\Omega} G_\zeta(x-y)(\Lambda_\gamma - \Lambda_1)\psi(y,\zeta)\, dS(y),$$

$$t(\xi,\zeta) = \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)}(\Lambda_\gamma - \Lambda_1)\psi(x,\zeta)\, dS(x).$$

$$\hat{q}(\xi) = \lim_{\zeta\to\infty} t(\xi,\zeta).$$

Linearization around $\gamma = 1$: Born approximation.

$$(\Lambda_\gamma, \Lambda_1) \to t^{exp}(\xi,\zeta) \to q(x) \to \gamma$$

where

$$t^{\exp}(\xi,\zeta) = \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)}(\Lambda_\gamma - \Lambda_1)e^{ix\cdot\zeta}\, dS(x).$$

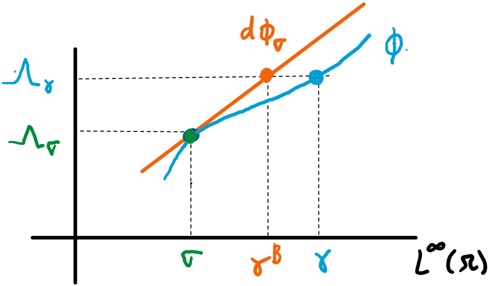
$$\hat{q}^{exp}(\xi) = \lim_{\zeta\to\infty} t^{\exp}(\xi,\zeta).$$

Our approach: Direct linearization

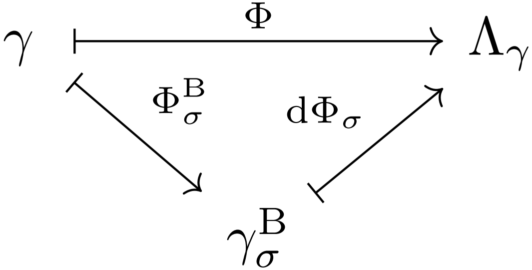
$$\begin{aligned}\Phi : L^\infty(\Omega) &\to \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega)) \\ \gamma &\to \Lambda_\gamma\end{aligned}$$

Take a reference background conductivity σ and look for γ_σ^B s.t.

$$\mathrm{d}\boldsymbol{\Phi}_\sigma(\gamma_\sigma^{\mathrm{B}} - \sigma) = \Lambda_\gamma - \boldsymbol{\Lambda}_\sigma.$$



This can be used to define a reconstruction strategy in two steps:



Main questions:

1. $\gamma_\sigma^{\mathrm{B}}$ exists?
2. Can we compute $\mathrm{d}\Phi_\sigma^{-1}$?
3. Can we compute $(\Phi_\sigma^B)^{-1}$?

Existence of γ_σ^B

Formally

$$\gamma_\sigma^{\mathrm{B}} = \sigma - \mathrm{d}\Phi_\sigma^{-1}(\Lambda_\sigma - \Lambda_\gamma) = \mathrm{d}\Phi_\sigma^{-1}(\Lambda_\gamma),$$

since it can be proved that $\mathrm{d}\Phi_\sigma(\sigma) = \Lambda_\sigma$. Note that

$$\gamma_\sigma^{\mathrm{B}} = \sigma - [\mathrm{d}\Psi_\sigma]^{-1}\Psi(\sigma), \quad \Psi(s) = \Phi(s) - \Phi(\gamma).$$

Theorem (CC, Macià, Meroño, Sánchez-Mendoza, 25')

Suppose that Ω is the unit ball in $\mathbb{R}^d$, that $\gamma \in W_+^{2,\infty}(\Omega)$ is a **radial function**, *and that $\kappa \in (-\infty, \lambda_{\nu_d,1}^2)$. Then there exists a* **radial function** $\gamma_{\sigma_{\kappa,d}}^{\mathrm{B}} \in W^{1,1}(\Omega)$ *such that:*

$$\mathrm{d}\Phi_{\sigma_{\kappa,d}}(\gamma_{\sigma_{\kappa,d}}^{\mathrm{B}}) = \Lambda_\gamma.$$

Here

$$\sigma_{\kappa,d}(x) = \left(c_d \frac{J_{\nu_d}(\sqrt{\kappa}|x|)}{(\sqrt{\kappa}|x|)^{\nu_d}}\right)^2, \qquad \nu_d = \frac{d-2}{2}, \quad c_d = \Gamma(\nu_d+1)2^{\nu_d}.$$

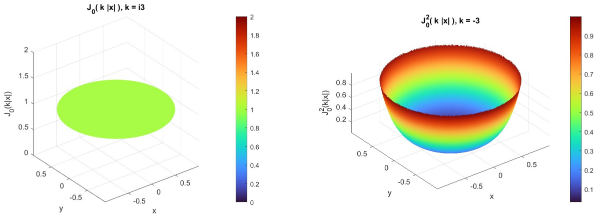


Figure: Background conductivity.

Idea: Reduce to the potential case where a similar result is known for background potential $q = cte$ ([T. Daudé, F. Macià, C. Meroño, and F. Nicoleau, 2025], [F. Macià, C. Meroño, D. Sánchez-Mendoza, 2025])

Invert $\mathrm{d}\Phi_\sigma$, $(d = 2)$

We want to find $\gamma_\kappa^{\mathrm{B}}$ such that

$$\mathrm{d}\Phi_{\sigma_\kappa}(\gamma_\kappa^{\mathrm{B}}) = \Lambda_\gamma,$$

Let $(e_m)_{m\in\mathbb{N}}$ be an orthonormal basis in $L^2(\partial\Omega)$. Define the matrix,

$$\mathfrak{m}^\kappa_{\ell,m}[F] := \langle e_\ell, \mathrm{d}\Phi_{\sigma_\kappa}(F) e_m\rangle_{H^{1/2}\times H^{-1/2}}, \quad F \in L^\infty(\mathbb{D}).$$

Then

$$\mathfrak{m}^\kappa_{\ell,m}[\gamma^{\mathrm{B}}] = \langle e_\ell, \Lambda_\gamma e_m\rangle_{H^{1/2}\times H^{-1/2}}.$$

Theorem (CC, Macià, Meroño, Sánchez-Mendoza, 25')

For every $\gamma \in L^\infty(\mathbb{D})$, $\kappa \in (-\infty, \lambda_{0,1}^2)$, *and* $\ell, m \in \mathbb{Z}$ *one has*

$$\mathfrak{m}_{\ell,m}^{\kappa}[\gamma] = \sigma_\kappa(1) \int_{\mathbb{D}} \frac{\gamma(x)}{\sigma_\kappa(x)} \Phi(\kappa, \ell, m, |x|) \mathrm{d}x,$$

where $\Phi(\kappa, \ell, m, |x|)$ *is an explicit function that depends on the Bessel functions.*

For $\kappa = 0$ *we have the simpler formula*

$$\mathfrak{m}_{\ell,m}^{0}[\gamma] = \int_{\mathbb{D}} \gamma(z) \frac{\ell m}{\pi} \overline{z}^{\ell-1} z^{m-1} \, m(\mathrm{d}z).$$

Inversion algorithm

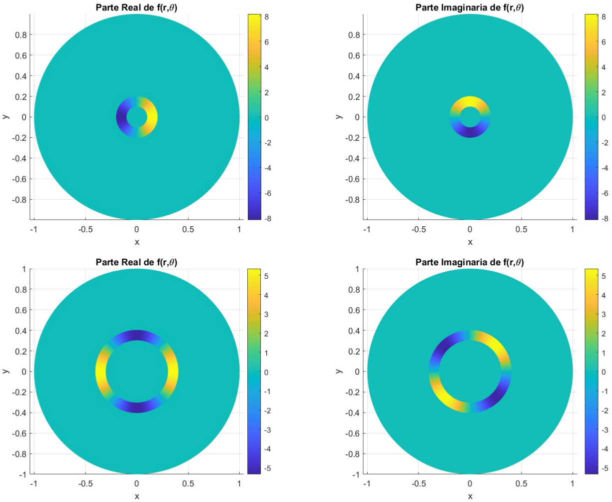
Find $\gamma^{\mathrm{B}} \in L^\infty$ such that

$$\sigma_\kappa(1)\int_{\mathbb{D}} \frac{\gamma^{\mathrm{B}}(x)}{\sigma_\kappa(x)}\Phi(\kappa,\ell,m,|x|)\mathrm{d}x = \langle e_\ell, \Lambda_\gamma e_m\rangle_{H^{1/2}\times H^{-1/2}}\,.$$

Galerkin approximation: Introduce the finite dimensional space $V^{I,L} = span\,(f_{i,j}(r,\theta))_{i=1,j=-L}^{I,L}$ where

$$f_{i,j}(r,\theta)\frac{I}{\sqrt{i-1/2}}\chi_{(\frac{i-1}{I},\frac{i}{I}]}(r)e_j(\theta), \quad i=1,\dots I,\ j=-L,\dots,L.$$

$$\sum_{i=1}^{I}\sum_{j=-L}^{L} x_{i,j}\mathfrak{m}^{\kappa}_{\ell,m}[f_{i,j}] = \langle e_\ell, \Lambda_\gamma e_m\rangle_{L^2(\mathbb{S}^1)}\,, \qquad \ell, m = 1,\dots,L.$$

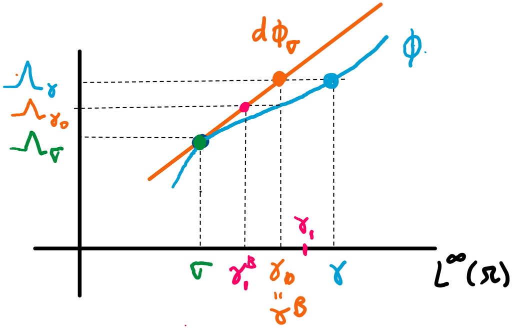

Parte Real de f(r,θ)
Parte Imaginaria de f(r,θ)
Parte Real de f(r,θ)
Parte Imaginaria de f(r,θ)
x
y

$$\begin{cases} \gamma_0 = \gamma_1^{\mathrm{B}}, \\ \gamma_n = \gamma_{n-1} + \gamma_1^{\mathrm{B}} - (\gamma_{n-1})_1^{\mathrm{B}}. \end{cases}$$

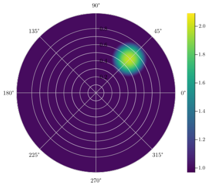
For each iteration $n-1 \to n$:

$$\gamma_{n-1} \to \Lambda_{\gamma_{n-1}} \to \left\langle e_\ell, \Lambda_{\gamma_{n-1}} e_m \right\rangle_{L^2(\mathbb{S}^1)} \to (\gamma_{n-1})_1^{\mathrm{B}} \to \gamma_n$$

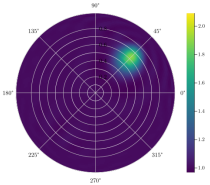
(The direct problem $\gamma_{n-1} \to \Lambda_{\gamma_{n-1}}$ is solved with a spectral method)



Background conductivity $\sigma = 1$

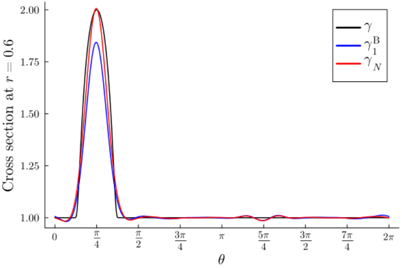


γ

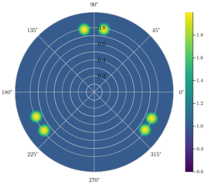


$\gamma_1^{\mathrm{B}} = \gamma_0$

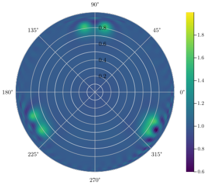
$n \setminus p$	1	2	∞
0	0.03895	0.06904	0.3338
1	0.03432	0.05210	0.2698
2	0.03670	0.05064	0.2510



Background conductivity $\sigma = 1$: precision



γ



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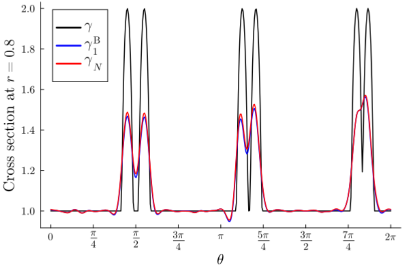

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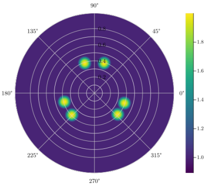
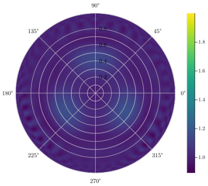
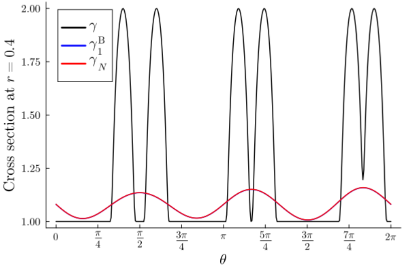


γ



$\gamma_1^{\mathrm{B}} = \gamma_0$

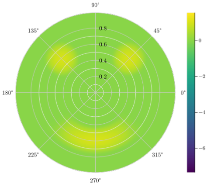


Convergence in terms of L ($I = 50$)

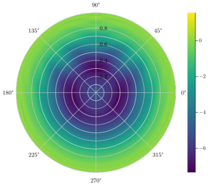
$L \setminus p$	1	2	∞
24	0.26730	0.24074	0.91152
29	0.19799	0.20441	0.89030
34	0.12702	0.15496	0.69693
39	0.10260	0.13376	0.67875
44	0.09744	0.13329	0.62283
49	0.08933	0.13038	0.61826

$\left\|\gamma - \gamma_1^{\mathrm{B}}\right\|_{L^p(\mathbb{D})}$ for the conductivity using different values of L.

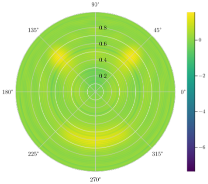
This experiment shows that whenever γ is a small perturbation of (a multiple of) σ_κ, then $\gamma^{\mathrm{B}}_{\sigma_\kappa}$ is a much better approximation to γ than $\gamma^{\mathrm{B}}_{\sigma_0} = \gamma^{\mathrm{B}}_{1}$.



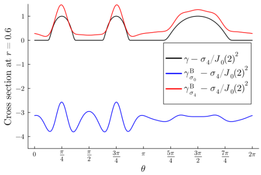
$$\gamma - \frac{\sigma_4}{J_0(2)^2}$$



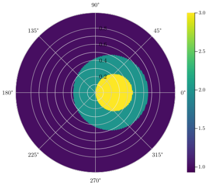
$$\gamma^{\mathrm{B}}_{\sigma_0} - \frac{\sigma_4}{J_0(2)^2}$$



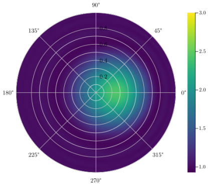
$$\gamma^{\mathrm{B}}_{\sigma_4} - \frac{\sigma_4}{J_0(2)^2}$$



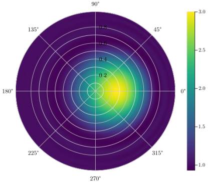
Angular cross section



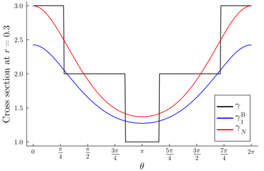
γ



$\gamma_1^{\mathrm{B}} = \gamma_0$

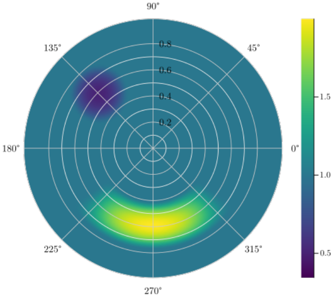


γ_N

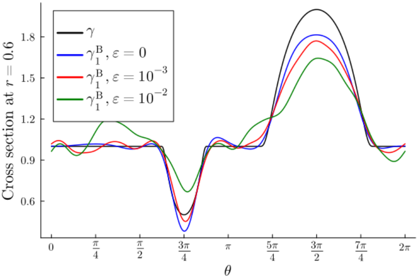


Angular cross section

Noisy data



γ



Angular cross section