Control and Inverse Problems

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Outline

1. Control of the 1-d Schrödinger equation with moving Diracs and moving domains (with A. Duca)
2. Spectral approximation of control and inverse problems (With S. Boumimez)
3. The Born approximation in the inverse Calderon problem. (with J.A. Barceló, F. Macià and C. Meroño)

Control of the 1-d Schrödinger equation with moving Diracs and moving domains

We consider the system with a potential $a(t,x)$

$$\begin{cases} i\partial_t\psi = -\partial^2_{xx}\psi + a(t,x)\psi, & x \in \Omega = (0,1),\ t \geq 0, \\ \psi(t,0) = \psi(t,1) = 0, & t \geq 0, \\ \psi(0) = \psi^0 \in L^2(\Omega,\mathbb{C}). \end{cases}$$

The L^2-norm is conserved, i.e.

$$\|\psi(t)\|_{L^2} = \|\psi^0\|_{L^2}, \quad t > 0.$$

We are interested in the following controllability result: Given ψ^f with $\|\psi^f\|_{L^2} = \|\psi^0\|_{L^2}$, find $T > 0$ and $a(t,x)$ such that

$$\psi(T,x) = \psi^f(x).$$

Our result (dimension $d = 1$ and $\Omega = (0, 1)$)

Theorem (CC-A. Duca, MCRF 23')

Assume that

$$\psi^0 = \sum_{j=1}^{N_0} c_j \varphi_j(x), \quad \psi^f = \sum_{j=1}^{N_f} d_j \varphi_j(x).$$

For any $\varepsilon > 0$ there exist $T > 0$ (large) and $a(t, x)$ such that the solution of the above system can be written as

$$\psi(T) = \sum_{k=1}^{\infty} c_j(T) \varphi_j,$$

where

$$\sum_{k=1}^{K} ||c_k(T)| - |d_k||^2 + \sum_{k=K+1}^{\infty} |c_k(T)|^2 < \varepsilon$$

Example

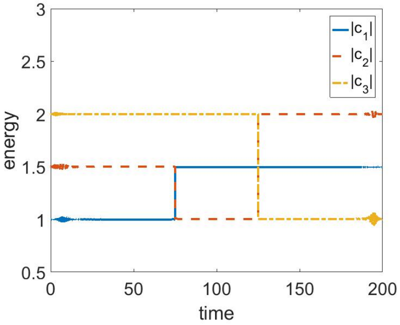
Assume that we want to permute the 'energy' of the first three modes:

$$\psi^0 = \varphi_1 + \tfrac{3}{2}\varphi_2 + 2\varphi_3$$
$$\psi^f = \tfrac{3}{2}\varphi_1 + 2\varphi_2 + \varphi_3$$

The control will produce a solution for which

$$\psi(x,T) \sim c_1(T)\varphi_1 + c_2(T)\varphi_2 + c_3(T)\varphi_3,$$

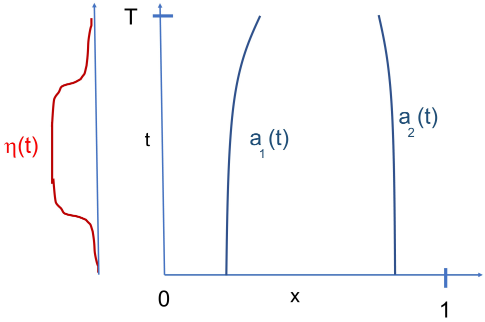
where $|c_1(T)| \sim 3/2$, $|c_2(T)| \sim 2$ and $|c_3(T)| \sim 1$.



The control $a(t, x)$

The control is explicit and given by

$$a(t,x) = \eta(t) \sum_{k=1}^{K} \delta_{a_k(t)}(x)$$



The controls are $\{a_k(t)\}_{k=1}^{K}$

Related works

Most of the works consider electric fields $a(t, x) = v(t)\mu(x)$ where $v(t)$ is the intensity of the field (control) and $\mu(x)$ the dipolar moment (smooth)

- **Global approximate controllability:** Mirrahimi and Beauchard' 09, Boscain and Adami' 05, Boscain, Chittaro, Gauthier, Mason, Rossi and Sigalotti' 12, Boussaid, Caponigro and Chambrion' 22, ...
- **Local exact controllability:** Ball, Marsden and Slemrod' 85 (Negative result), Beauchard and Laurent 11', Puel' 16...
- Nonlinear models, systems, networks, etc...

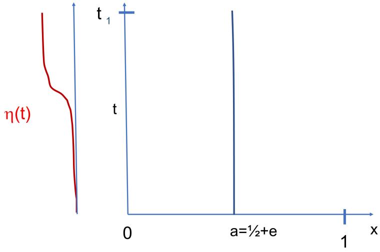
The peculiarity of our result is in the explicit form of the control. It produces an adibatic regime almost any time.

Example

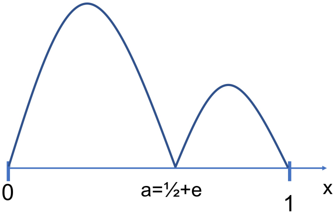
Assume that we want to permute the energy of the first two modes. Take

$$\psi^0 = 2\varphi_1 + \varphi_2, \quad \psi^f = 1\varphi_1 + 2\varphi_2$$

First subinterval $(0, t_1)$ Take $a(t, x) = \eta(t)\delta_{a_1}(x)$ with $a_1 = 1/2 + \varepsilon$ and $\eta(t)$ growing slowly from zero to a large value η_M (adiabatic regime)

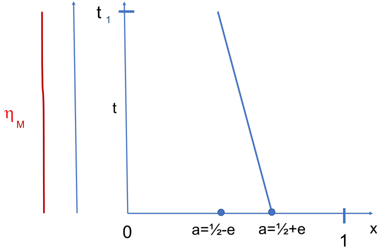


$\eta(t)$ and $a(t)$

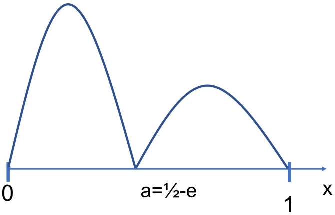


$|\psi(t_1, x)|^2$

Second subinterval (t_1, t_2) Take $a(t,x) = \eta_M \delta_{a_1(t)}(x)$ with $a_1(t)$ that moves fast from $a_1 + \varepsilon$ to $a_1 - \varepsilon$ (continuity result)

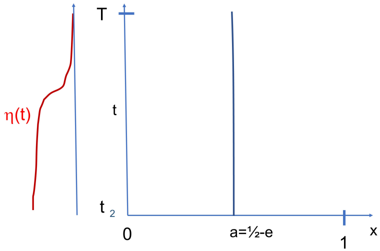


$\eta(t)$ and $a(t)$

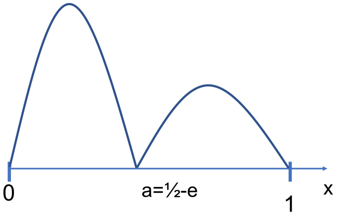


$|\psi(t_1, x)|^2$

Third subinterval (t_2, T) Take $a(t,x) = \eta(t)\delta_{a_1}(x)$ with $a_1 = 1/2 - \varepsilon$ and $\eta(t)$ descending slowly from the large value η_M to zero (adiabatic regime)



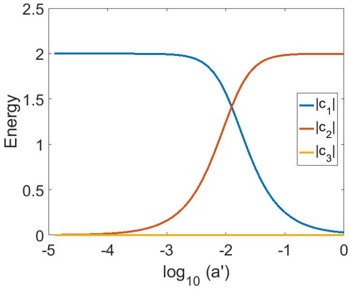
$\eta(t)$ and $a(t)$



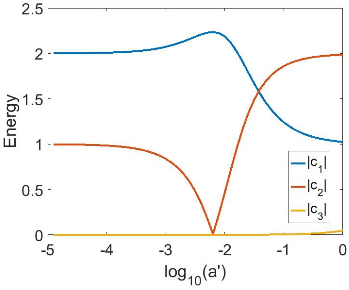
$|\psi(t_1,x)|^2$

Then $\psi(x,T) = c_1(T)\varphi_1 + c_2(T)\varphi_2 + O(\varepsilon)$ with $|c_1(T)| = 1$ and $|c_2(T)| = 2$

Remark: The permutation of energy in the modes is performed in step 2. This is obtained with a rapid nonadiabatic movement of the Dirac support $a(t)$. Sufficiently slow movements will produce an adiabatic regime with no permutation of energy. Intermediate situations are also possible



$$\psi_0(x) = 2\phi_1(x,0)$$



$$\psi_0(x) = 2\phi_1(x,0) + \phi_2(x,0)$$

Numerical approximation: Spectral method

Take $X = L^2(0,1)$, $\mathbf{a} = (a_1, ..., a_K)$ and consider the associated eigenpairs of $A^{\eta,\mathbf{a}}$

$$(\lambda_k(t), \phi_k(x,t)), \quad k \geq 1.$$

Consider also the eigenpairs of the Dirichlet Laplacian

$$(\mu_k(t), w_k(x,t)), \quad k \geq 1.$$

Define

$$X^N = span\{w_k\}_{k=1}^N, \qquad P^N : X \to X_N.$$

Discrete problem: Find $\psi_N(t) \in X_N$ such that,

$$\begin{cases} i\partial_t \psi_N = P^N A^{\eta,\mathbf{a}}(t)\psi_N, & t > 0 \\ \psi_N(0) = P^N \psi^0. \end{cases}$$

Theorem (CC-A. Duca, MCRF 23')

Assume that a and η satisfy the hypotheses to guarantee the existence of a solution $\psi \in C([0,T]; H_0^1)$ with initial data $\psi^0 \in H_0^1$. Let ψ_N be the solution of the corresponding finite dimensional approximation. Then, for $t \in [0,T]$,

$$\|\psi(t) - \psi_N(t)\|_{L^2} \leq \left(1 + 2T\frac{\eta_M}{\pi}\right) \frac{\sqrt{\eta_M}}{\sqrt{3}\sqrt{N}} \|\psi(t)\|_{L^\infty((0,T);H_0^1)},$$

where $\eta_M = \max_{t\in[0,T]} \eta(t)$.

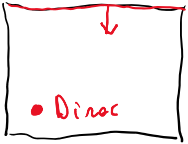
Remark The estimate depends on η and T that are large. Therefore it requires N large.

Comments

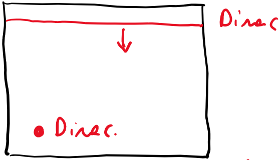
1. For simplicity we have focused on permutations of energy states. However, the technique can be adapted to any redistribution of the energy in a finite number of Fourier coefficients.
2. Probably the paths of Diracs (controls) can be optimized.
3. The idea can be adapted to higher dimensions, at least in simple situations.

Higher dimensions

Two strategies



- Shrink the domain
- Include a point Dira

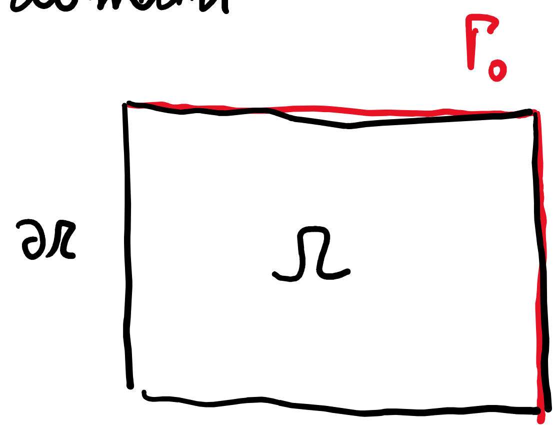


- Move a Dirac supported on a line
- Include a point Dirac

2. Spectral approximation of control and inverse pb's

Wave eq. in a rectangular domain

$$\begin{cases} u_{tt} - \Delta u = 0, & t \in (0,T), \ x \in \Omega \\ u = 0 & t \in (0,T), \ x \in \partial\Omega \\ u(x,0) = u^0(x), & x \in \Omega \\ u_t(x,0) = u^1(x), & x \in \Omega \end{cases}$$



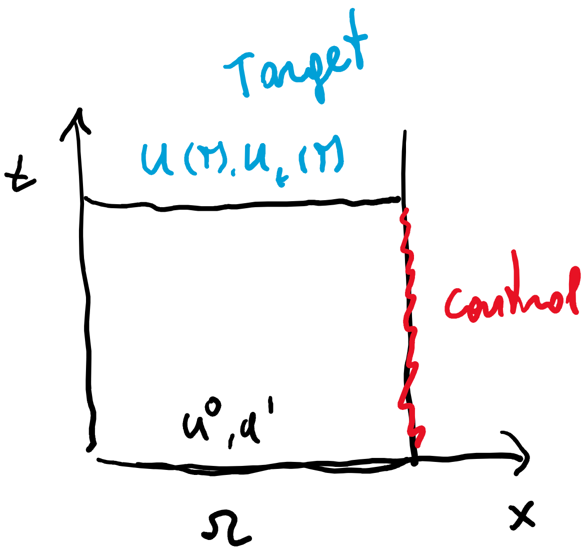
Observability inequality: For $T>0$ sufficiently large there exists $C>0$ such that,

$$E(0) \leq C \int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 ds\, dt, \quad \forall (u^0, u^1)$$

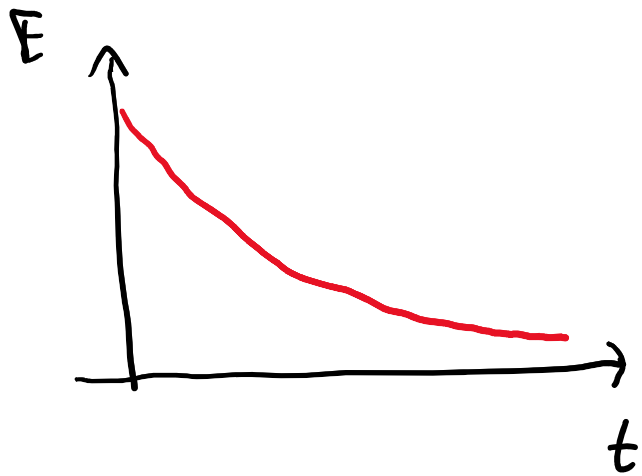
$$E(t) = \frac{1}{2} \left[\int_\Omega |\nabla u|^2 dx + \int_\Omega |u_t|^2 dx \right]$$

The observability inequality is useful in

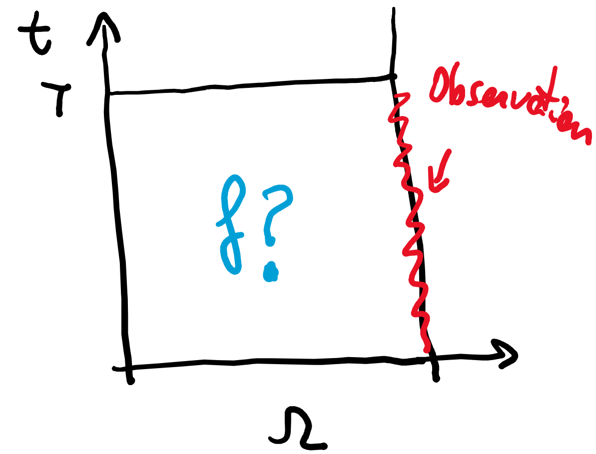
Boundary control

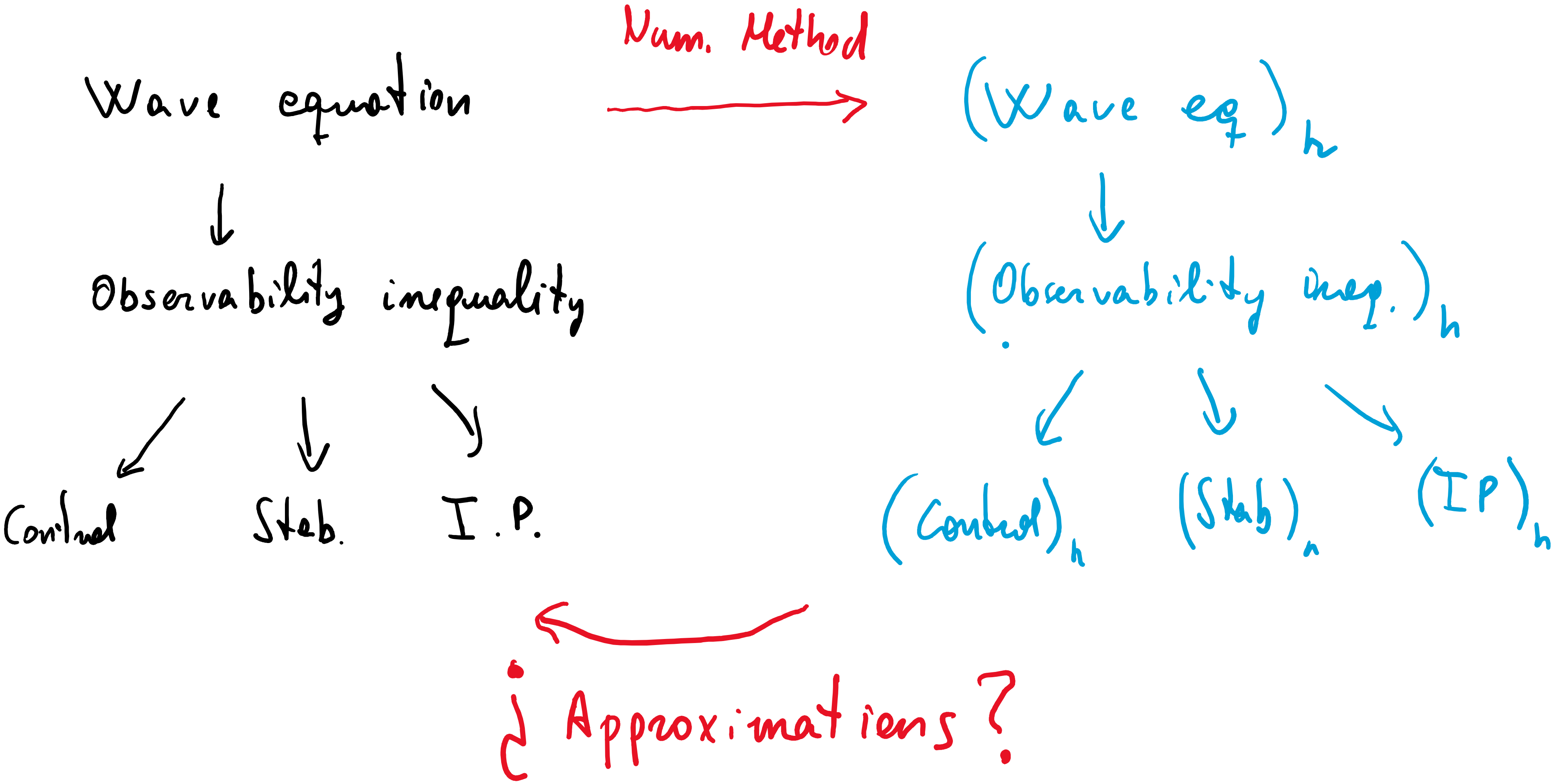


Boundary stabilization



Inverse pb from boundary data

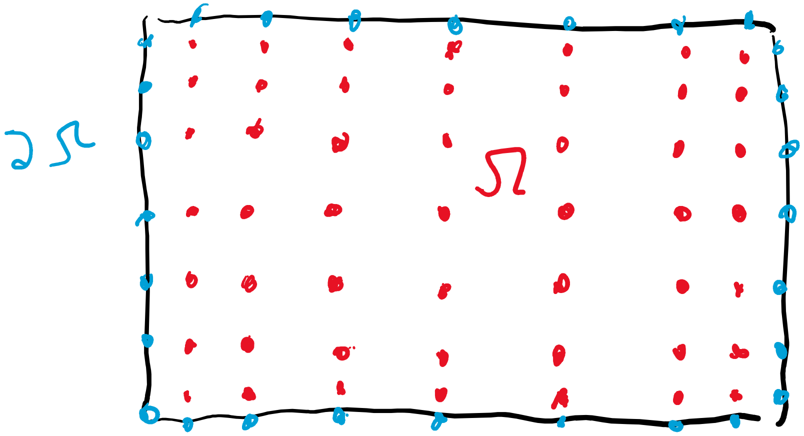



Num. Method
Wave equation
(Wave eq)h
Observability inequality
(Observability ineq.)h
Control
Stab.
I.P.
(Control)h
(Stab)h
(IP)h
¿Approximations?

Main problem: when considering discretizations of the wave equation the corresponding observability inequality is NOT uniform with respect to h (the discretization parameter)

$$E_h(0) \leq C_h \int_0^T \int_{\Gamma_0} \left| \frac{\partial u^h}{\partial n} \right| ds\, dt \quad , \quad C_h \xrightarrow[h \to 0]{} \infty$$

Spectral collocation



$u^N \in \mathbb{P}^N$ (polin. de grado $\leq N$)

$P_i \in \Omega$

$P_j \in \partial\Omega$

$$\begin{cases} u_{tt} - \Delta u = 0, & t \in (0,T), \; x \in \Omega \\ u = 0, & t \in (0,T), \; x \in \partial\Omega \\ u(x,0) = u^0(x), & x \in \Omega \\ u_t(x,0) = u^1(x), & x \in \Omega \end{cases}$$

$$\begin{cases} [u^N_{tt} - \Delta u^N](P_i, t) = 0, & t \in (0,T) \\ u^N(P_j, t) = 0, & t \in (0,T) \\ u^N(P_i, 0) = u^0(P_i) \\ u^N_t(P_i, 0) = u^1(P_i) \end{cases}$$

Advantages

- High order
- Easy to implement

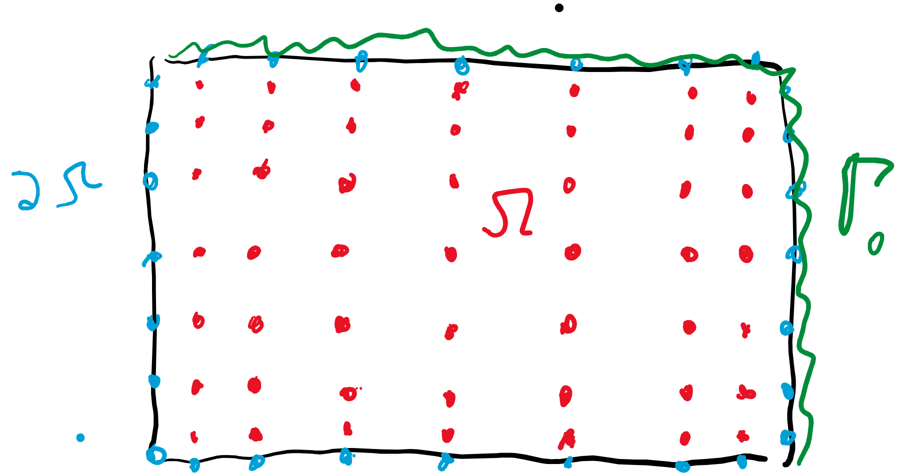
Drawbacks

- Rectangular domains
- Preconditioning techniques required

Uniform observability [Boumimez - C, COAM 23']

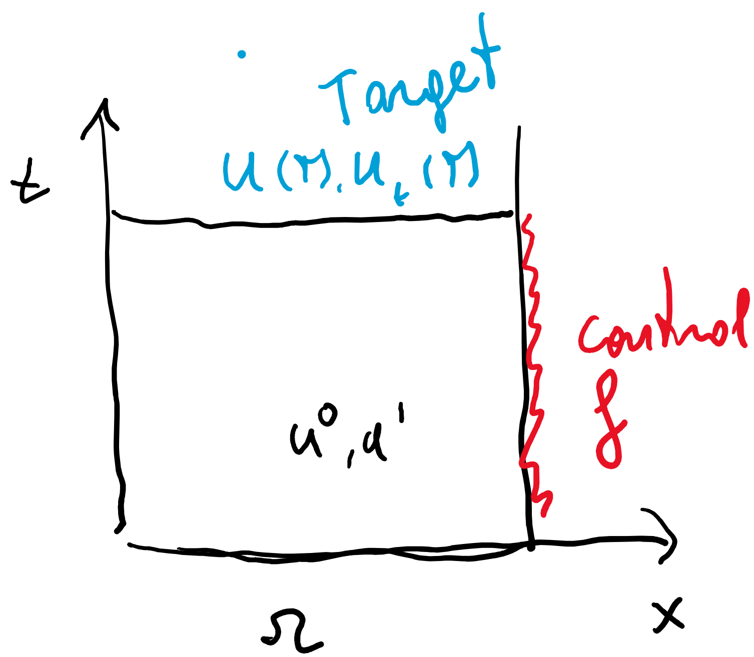
$$E^N(0) \leq C \left[\int_0^T \int_{\Gamma_0} \left| \frac{\partial u^N}{\partial n} \right|^2 dx\,dt + \int_0^T \int_{\partial\Omega} \left| \frac{\partial^2 u^N}{\partial n^2} \right|^2 dx\,dt \right]$$

$\forall \, (u^{0,N}, u^{1,N})$



Consequences (Boundary control)

Uniform boundary controllability



- $f^N \longrightarrow f$ ✓

- Spectral convergence?

$$\| f^N - f \|_{L^2} \leq C N^{-\alpha} \| u^0, u^1 \|_{H^\alpha \times H^{\alpha-1}}$$

Only numerical evidences!!

Open questions:

Boundary control

- Spectral convergence of controls for the wave eq.
- " " " elasticity system

Inverse source problem

- Spectral convergence of source terms from boundary observations (wave eq. & elasticity system)

Consider the following system in the 3-d unit ball

$$(P1)\begin{cases} -div\,(\gamma(x)\nabla u(x)) = 0, & x \in B, \\ u = f, \quad x \in \partial\Omega. \end{cases}$$

Given $\gamma \in L^\infty(B)$ we define the Dirichlet to Neumann map (DtN) as

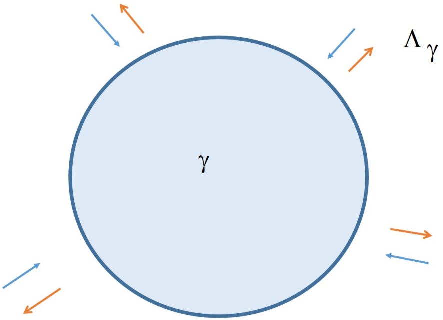
$$\begin{array}{rcl} \Lambda_\gamma : H^{1/2}(\partial B) & \to & H^{-1/2}(\partial B) \\ f & \to & \gamma \frac{\partial u}{\partial n}\Big|_{\partial\Omega} \end{array}$$

The reconstruction problem: Can we find γ from Λ_γ?

Main idea: use a linearization of the map

$$\Lambda_\gamma \to \gamma$$

Mathematical formulation of the classical problem where we want to derive the conductivity from boundary measurements



Multiple applications: EIT, nonintrusive defect detection in materials, geophysics, etc.

Equivalent potential problem

$$(P2)\begin{cases} -\Delta u(x) + q(x)\, u(x) = 0, & x \in B, \\ u = f, & x \in \partial\Omega. \end{cases}$$

Given $q \in L^\infty(B)$ we define the Dirichlet to Neumann map (DtN) as

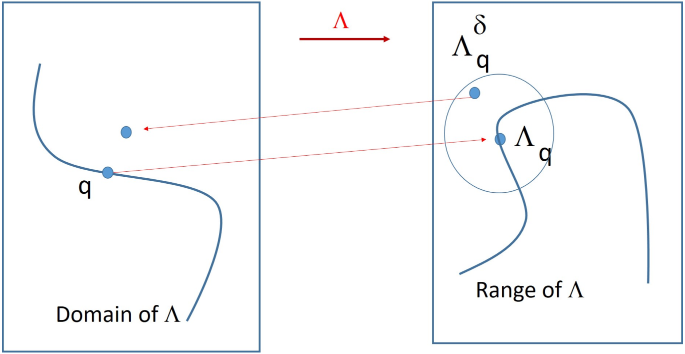
$$\begin{array}{rcl} \Lambda_q : H^{1/2}(\partial B) & \to & H^{-1/2}(\partial B) \\ f & \to & \frac{\partial u}{\partial n} \end{array}$$

The reconstruction problem: Can we find q from Λ_q?
Remark. For smooth γ, (P1) is equivalent to (P2) with

$$q = \frac{\Delta \gamma^{1/2}}{\gamma^{1/2}}$$

Ill-posed problem

Close DtN may be associated with very different q, but also one can be outside the domain of Λ [Siltanen and Mueller 2012]



The linearization: 3D problem

In the direct reconstruction numerical algorithm given in [Bikowski, Knudsen, Mueller, 2011], the following Born approximation was also proposed:

$$(\widehat{\gamma_{exp} - 1})(\xi) = -\frac{2}{|\xi|^2} \lim_{|\zeta|\to\infty} \int_{\partial B_R} e^{-ix\cdot(\xi+\zeta)} (\Lambda_\gamma - \Lambda_1) e^{ix\cdot\zeta} dx, \quad \xi \in \mathbb{R}^3,$$

where $\zeta \in \mathcal{V}_\xi = \{\eta \in \mathbb{C}^d \backslash \{0\} \ : \ \eta\cdot\eta = 0, \ (\xi+\eta)^2 = 0\}$.

Remarks:

- It is deduced for smooth γ but can be used for less regular ones
- It is formal in the sense that the limit may not exist, and even if it exists the right hand side may be not the Fourier transform of an L^∞ function
- It requires an high frequency limit which is difficult to compute numerically.
- For $\xi \sim 30$ the right hand side has some inestabilities and we only can compute a (very) low pass filter.

A simplified problem: Radial conductivity $\gamma(x) = \gamma(|x|)$

We assume that γ is a radial function in the unit ball B, i.e. $\gamma(x) = \gamma(|x|)$ and that $\gamma(x) = 1$ in a neighbourhood of the boundary. In this case, $\gamma : [0, 1] \to \mathbb{R}$ is represented by a one-variable function. The following holds:

- The Spherical harmonics are eigenfunctions of Λ_γ. The sequence of eigenvalues $\{\lambda_k[\gamma]\}_{k\geq 0}$ characterize the DtN map Λ_γ
- The eigenvalues of Λ_γ satisfy

$$|\lambda_k[\gamma] - k| \leq Ce^{-k}$$

- It is possible to compute $\lambda_k[\gamma]$ when γ is piecewise constant.

Radial conductivities

Formal definition [Barceló, C, Macià, Meroño, 22']: Let $\gamma \in L^\infty(B)$ a radial function,

$$\widehat{\gamma^{exp} - 1}(\xi) = -\pi^{3/2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!\Gamma(k+3/2)} \left(\frac{|\xi|}{2}\right)^{2k-2} (\lambda_k[\gamma] - k) \, .$$

Remark

- The series contain terms which are the product of very large and very small numbers. Usual float64 number representation provides an accurate sum only for $\xi \sim 30$, which is the limit found before. Convergence for larger ξ require larger accuracy

Experiment 1: Piecewise constant conductivity

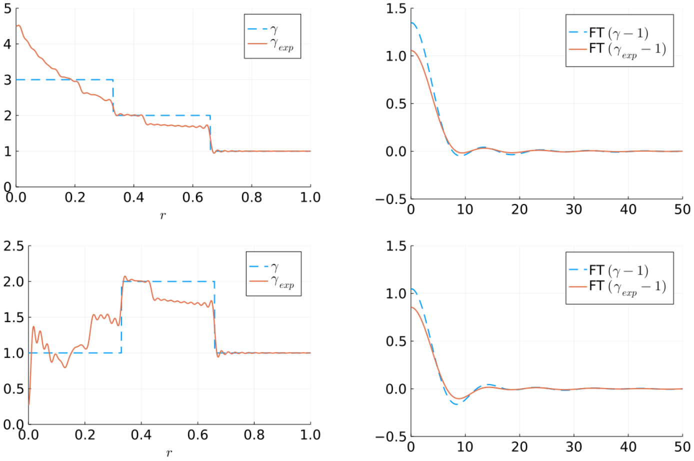


Figure: Experiment 1: A step conductivity (left) and its Fourier transform (right) versus the Fourier transform of its Born approximation (right).

Experiment 2: Unique continuation from the boundary

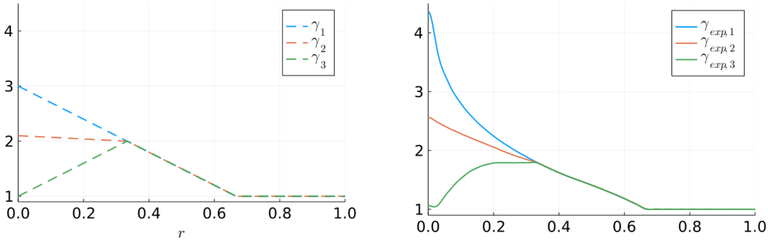


Figure: Experiment 5: Three different conductivities that coincide in the interval $(1/3, 1)$ (left) and their Born approximations (right). We observe that they also coincide in this same interval $(1/3, 1)$.

The Born approximation is more accurate close to the boundary. We have computed the error $e(r) = \frac{1}{N_s}\sum_{i=1}^{N_s} |\gamma(r) - \gamma_{exp}(r)|$ for a random sampling of $N_s = 100$ conductivities.

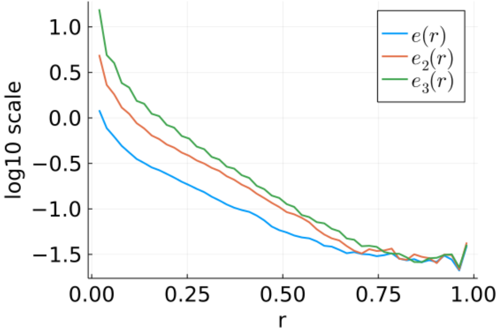


Figure: Experiment 6: average error distribution of the Born approximation, $e_\alpha(x)$, computed with 100 samples, where Fourier coefficients are chosen randomly in different intervals

2. Improving the Born approximation: Fixed point iteration

Based on the good behavior of the Born approximation we propose the following iterative algorithm

$$\begin{cases} \gamma^0 &= \gamma_{exp} \\ \gamma^{n+1} &= \gamma_{exp} + \gamma^n - [\gamma^n]_{\mathrm{exp}} \end{cases}$$

Remarks.

- We initialize with the Born approximation
- At each step, the DtN map of γ^n must be computed. This requires an approximation by piecewice constant conductivities.

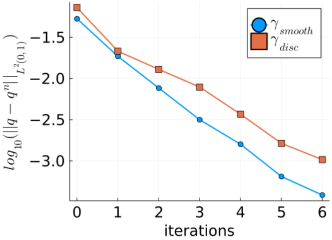
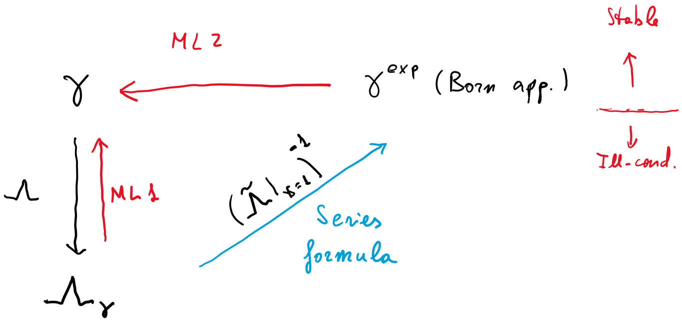


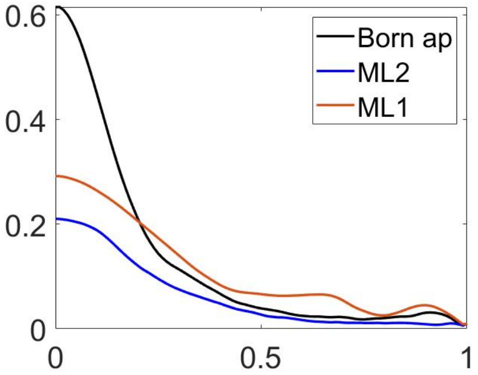
Figure: L^2-error (left) and L^∞-error (right) in log_{10}-scale for the iterative algorithm when considering both the smooth and Lipschitz conductivities



$\tilde{\Lambda}|_{\gamma=1} \to$ linearization of Λ at $\gamma=1$

Validation

Distributed error with 1000 validation potentials



The Born approximation in the non-radial case: d=2

We propose the following formula for the "Born approximation"

$$\widehat{\gamma - 1}(t\cos\theta, t\sin\theta) = \frac{-2}{t^2}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} i^{\ell+k}\frac{\lambda_{k,\ell}[\gamma] - k\delta_{k,\ell}}{k!\ell!}\left(\frac{t}{2}\right)^{k+\ell} e^{i(k-\ell)\theta},$$

where

$$\lambda_{k,\ell}[\gamma] - k\delta_{k,\ell} = \langle e^{-i\ell\theta}, (\Lambda_\gamma - \Lambda_0)e^{ik\theta}\rangle.$$

Fourier variable in polar coordinates $(\xi_1, \xi_2) = (t\cos\theta, t\sin\theta)$

Conclussions

- We have deduced a new formula to compute the Born approximation for radial conductivities. This formula recovers the Fourier transform of γ^{exp} as a series.
- The numerical implementation of the formula requires large precision.
- The Born approximation recovers fairly well the conductivity if γ is not far from 1.
- A suitable fixed point iteration provides a convergent sequence of approximations to the conductivity.
- The Born approximation can be extended to non-radial conductivities.

Future work

- Numerical approximation of the Born approximation for the non-radial case in dimension $d = 2$. This requires high order methods to compute the DtN map (Spectral collocation).
- Reconstruction with noisy data.
- Reconstruction with partial data.
- Use of ML techniques to improve the reconstruction.