Control of the Schrödinger equation by domain deformation

C. Castro (joint work with A. Duca)

Univ. Politécnica de Madrid, Spain.

December 3, 2024

We consider the Schrödinger equation

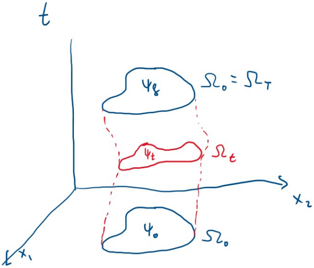
$$\begin{cases} i\partial_t\psi + \Delta\psi = 0 & x \in \Omega_t,\ t \geq 0, \\ \psi = 0, & x \in \partial\Omega_t,\ t \geq 0, \\ \psi(0) = \psi^0 \in L^2(\Omega_0, \mathbb{C}). & \end{cases}$$

where $\Omega_t \subset \mathbb{R}^2$ is a continuous deformation of a bounded domain Ω_0 depending on the time $t \geq 0$
The L^2-norm is conserved, i.e.

$$\|\psi(t)\|_{L^2} = \|\psi^0\|_{L^2}, \quad t > 0.$$

We are interested in the following controllability result: Given ψ^f with $\|\psi^f\|_{L^2} = \|\psi^0\|_{L^2}$, find $T > 0$ and Ω_t, with $\Omega_0 = \Omega_T$ such that

$$\psi(T, x) = \psi^f(x).$$



Main motivation

In a recent paper by A. Duca, R. Joly and D. Turaev, (23)' it is shown a theoretical method to control the Schrödinger equation by domain deformation.

Question: Approximate numerically this process.

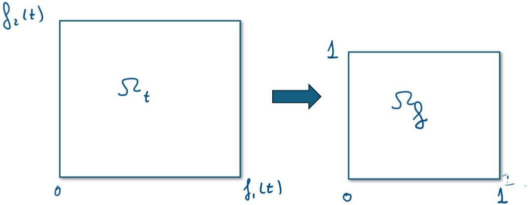
Difficulties:

1. Domain deformations are based on a 'not very explicit' diffeomorphism that preserves the adiabatic behavior.
2. Control time is not explicit either and can be extremely large.
3. Simulations in domains depending on the time are challenging.

Idea: We exploit a special simple formula for domain deformations of the form

$$\Omega(t) = (0, f_1(t)) \times (0, f_2(t)),$$

to simulate the process (A. Duca and R. Joly 21').



After the change of variables

$$x_1 = f_1(t)y_1, \quad x_2 = f_2(t)y_2, \quad (y_1, y_2) \in (0,1) \times (0,1) = \Omega_f$$

$$\begin{cases} i\partial_t w + \frac{w_{y_1 y_1}}{f_1^2(t)} + \frac{w_{y_2 y_2}}{f_2^2(t)} - V(t,y)w = 0 & w \in \Omega_f,\ t \geq 0, \\ y = 0, & w \in \partial\Omega_f,\ t \geq 0, \\ w(0) = w^0 \in L^2(\Omega_f, \mathbb{C}). \end{cases}$$

with

$$V(t,y) = \frac{f_1''(t)f_1(t)y_1^2 + f_2''(t)f_2(t)y_2^2}{4}, \quad w = e^{-i\psi(t,x)}u$$

$$\psi(t,x) = \frac{1}{4}\left(\frac{f_1'(t)}{f_1(t)}x_1^2 + \frac{f_2'(t)}{f_2(t)}x_2^2\right)$$

Our result adapt the idea by A. Duca, R. Joly and D. Turaev, (23)' to a simpler setting where we can approximate numerically the control. Some simplifications:

- Domain deformations are restricted to rectangular domains
- A localized potential in the neighbourhood of a point must be included in the process, i.e. a potential of the form

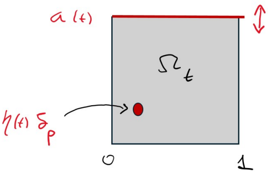
$$a(t,x) = \eta(t) exp(-\gamma|x - x_0|^2), \quad \gamma >> 1,$$

  where $x_0 \in \Omega_t$ for $t \in (0, T)$ and $\eta(t)$ is complactly supported in $(0, T)$.
- An approximate control is obtained.

$$\begin{cases} i\partial_t\psi + \Delta\psi + \eta(t)exp(-\gamma|x-x_0|^2)\psi = 0 & x \in \Omega_t,\ t \geq 0, \\ \psi = 0, & x \in \partial\Omega_t,\ t \geq 0, \\ \psi(0) = \psi^0 \in L^2(\Omega_0, \mathbb{C}). \end{cases}$$

Here $\Omega_t = (0,1) \times (0, a(t))$.
The controls are:

$$\eta(t), \quad a(t)$$



δ_P represents either a Dirac delta supported on P or a

Theorem

Let $\{\varphi_j(x)\}_{j\geq 1}$ *the sequence of eigenfunctions of the Laplace operator in* Ω_0. *Assume that*

$$\psi^0 = \sum_{j=1}^{\infty} c_j \varphi_j(x), \quad \psi^f = \sum_{j=1}^{\infty} d_j \varphi_j(x).$$

For any $\varepsilon > 0$ *there exist* $T > 0$ *(large),* $a(t)$, *with* $a(0) = a(T)$ *and* $\eta(t) \in C_o(0, T)$ *such that the solution of the above system can be written as*

$$\psi(T) = \sum_{k=1}^{\infty} c_j(T)\varphi_j,$$

where

$$\sum_{k=1}^{\infty} ||c_k(T)| - |d_k||^2 < \varepsilon$$

Example

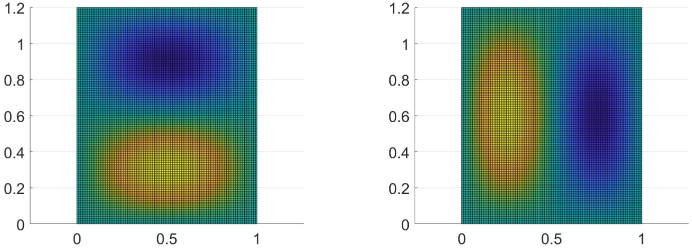
Assume that we want to permute the 'energy' of the second and third modes:

$$\psi^0 = \varphi_2$$
$$\psi^f = \varphi_3$$

The control will produce a solution for which

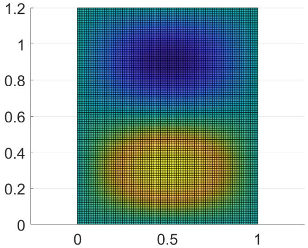
$$\psi(x, T) \sim c_3(T)\varphi_3,$$

where $|c_3(T)| \sim 1$.

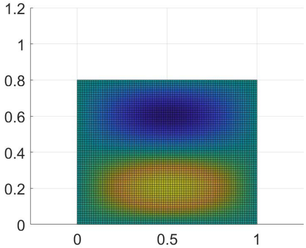


The control combine two ideas:

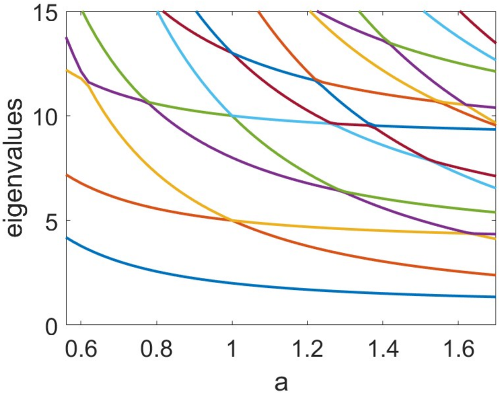
- **Adiabaticity (Born and Fock 1928'):** For smooth and sufficiently slow time-varying coefficients the Schrödinger equation preserves the energy of the modes, as soon as the modes are **simple**. This is also true for sufficiently slow deformation of the domain.
- **Continuity** in time of the solutions. When a deformation produces a multiple eigenvalue, the energy is exchanged between the two modes.



second mode

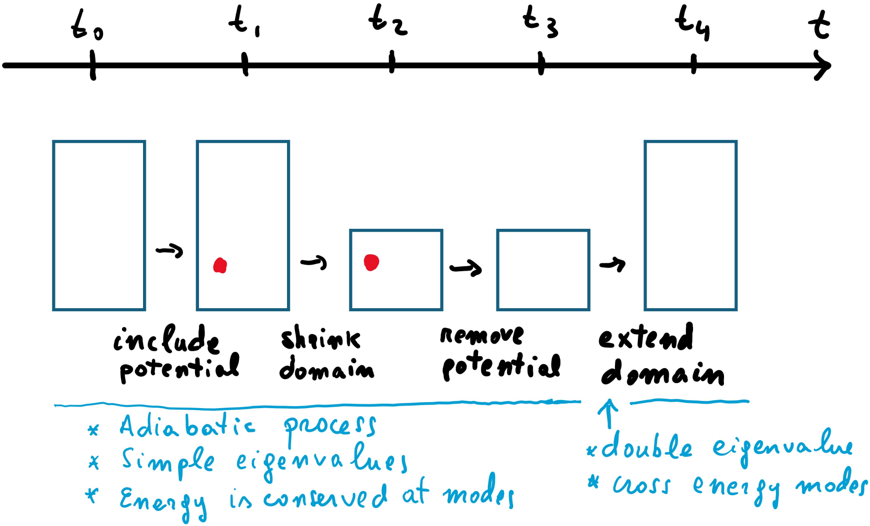


third mode

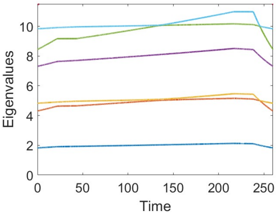


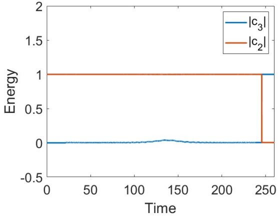
Eigenvalues for different deformations: $a \in [0.6; 1.6]$

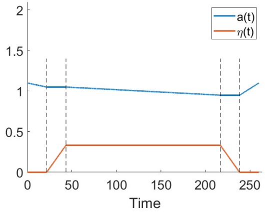
The control to exchange the energy of the second and third mode



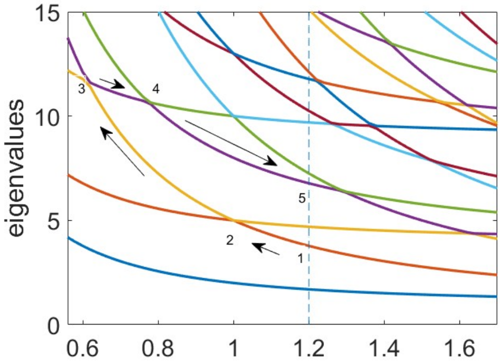
The control to exchange the energy of the second and third mode





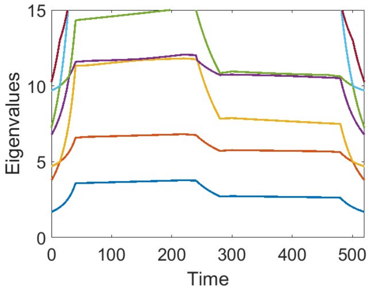


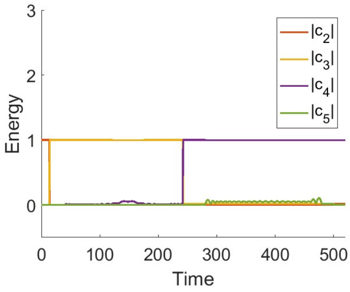
transfer the energy of the second and fourth modes



Eigenvalues for different deformations: $a \in [0.6; 1.6]$

Transfer the energy of the second and fourth mode





Numerical approximation: Spectral method

Take $X = L^2(\Omega_f)$ and consider the associated eigenpairs of $A^{\eta,\mathbf{a}}$

$$(\lambda_k(t), \phi_k(x,t)), \quad k \geq 1.$$

Consider also the eigenpairs of the Dirichlet Laplacian

$$(\mu_k, w_k(x)), \quad k \geq 1.$$

Define

$$X^N = span\{w_k\}_{k=1}^N, \qquad P^N : X \to X_N.$$

Discrete problem: Find $\psi_N(t) \in X_N$ such that,

$$\begin{cases} i\partial_t \psi_N = P^N A^{\eta,\mathbf{a}}(t)\psi_N, & t > 0 \\ \psi_N(0) = P^N \psi^0. \end{cases}$$

Theorem

Assume that a and η satisfy the hypotheses to guarantee the existence of a solution $\psi \in C([0,T]; H_0^1)$ with initial data $\psi^0 \in H_0^1$. Let ψ_N be the solution of the corresponding finite dimensional approximation. Then, for $t \in [0,T]$,

$$\|\psi(t) - \psi_N(t)\|_{L^2} \leq \left(1 + 2T\frac{\eta_M}{\pi}\right) \frac{\sqrt{\eta_M}}{\sqrt{3}\sqrt{N}} \|\psi(t)\|_{L^\infty((0,T);H_0^1)},$$

where $\eta_M = \max_{t\in[0,T]} \eta(t)$.

Remark The estimate depends on η and T that are large. Therefore it requires N large.

Comments

1. For simplicity we have focused on permutations of energy states. However, the technique can be adapted to any redistribution of the energy in a finite number of Fourier coefficients.
2. The idea can be adapted to more general domains (in progress)
3. The numerical analysis is not completely satisfactory since the adiabatic regime requires both large time T and N (difficult to estimate).
4. The domain deformation can be replaced by a large potential simulating the domain deformation. In this way the control becomes a bilinear control. This problem has been widely studied in the literature with different strategies.

Control of the 2-d Schrödinger equation with large potentials

We consider now the Schrödinger equation with bilinear control

$$\begin{cases} i\partial_t\psi + \Delta\psi - a(t,x)\psi = 0 & x \in \Omega,\ t \geq 0, \\ \psi = 0, & x \in \partial\Omega,\ t \geq 0, \\ \psi(0) = \psi^0 \in L^2(\Omega, \mathbb{C}). \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is fixed and the control now is

$$a(t,x)$$

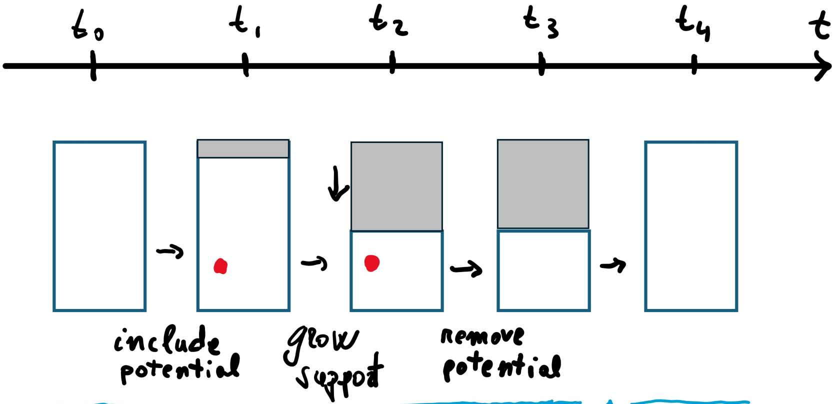
The idea now is to simulate the domain deformation with a strong potential

Related works

Most of the works consider electric fields $a(t, x) = v(t)\mu(x)$ where $v(t)$ is the intensity of the field (control) and $\mu(x)$ the dipolar moment (smooth)

- **Global approximate controllability:** Mirrahimi and Beauchard' 09, Boscain and Adami' 05, Boscain, Chittaro, Gauthier, Mason, Rossi and Sigalotti' 12, Boussaid, Caponigro and Chambrion' 22, ...
- **Local exact controllability:** Ball, Marsden and Slemrod' 85 (Negative result), Beauchard and Laurent 11', Puel' 16...
- Nonlinear models, systems, networks, etc...

The peculiarity of our result is in the explicit form of the control. It produces an adibatic regime almost any time.



* Adiabatic process
* Simple eigenvalues
* Energy is conserved at modes

* double eigenvalue
* cross energy modes