Power Series Methods for Backstepping Kernels: Theory, Practice, and Recent Developments

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Backstepping for Partial Differential Equations

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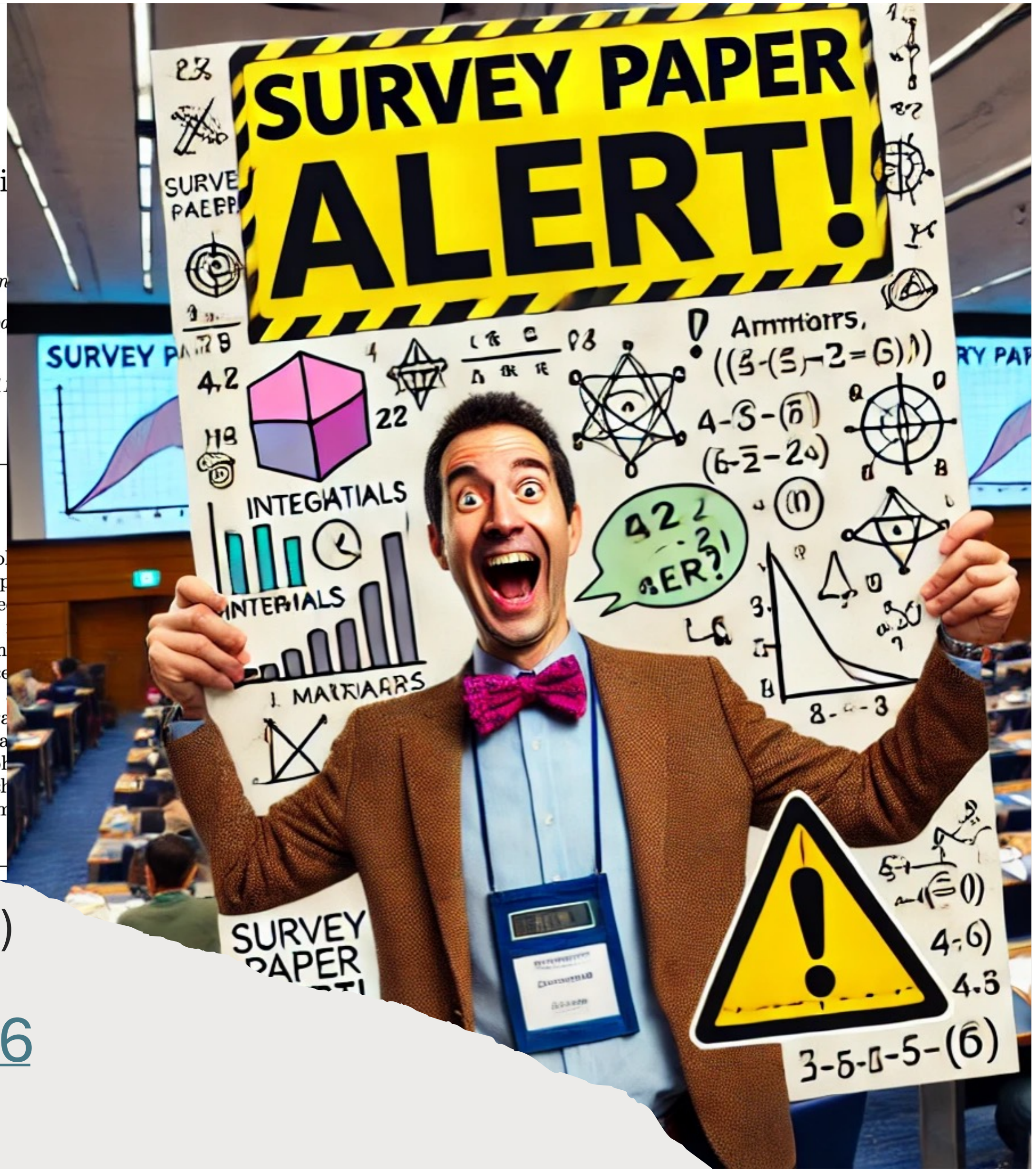
tract

ems modeled by partial differential equations (PDEs) are at least as ubiquitous as systems that are by nature
nsional and modeled by ordinary differential equations (ODEs). And yet, systematic and readily usable methodo
uch a significant portion of real systems, have been historically scarce. Around the year 2000, the backstepping app
DE control began to offer not only a less abstract alternative to PDE control techniques replicating optimal and spe
nment techniques of the 1960s, but also enabled the methodologies of adaptive and nonlinear control, matured
s and 1990s, to be extended from ODEs to PDEs, allowing feedback synthesis for physical and engineering system
ncertain, nonlinear, and infinite-dimensional. The PDE backstepping literature has grown in its nearly a quarter ce
evelopment to many hundreds of papers and nearly a dozen books. This survey aims to facilitate the entry, for
rcher, into this thriving area of overwhelming size and topical diversity. Designs of controllers and observers, for para
rbolic, and other classes of PDEs, in one and more dimensions (in box and spherical geometries), with nonlinear, ada
pled-data, and event-triggered extensions, are covered in the survey. The lifeblood of control are technology and ph
survey places a particular emphasis on applications that have motivated the development of the theory and which
fited from the theory and designs: applications involving flows, flexible structures, materials, thermal and chem
energy (from oil drilling to batteries and magnetic confinement fusions), and vehicles.



New Survey (under review, preprint available)

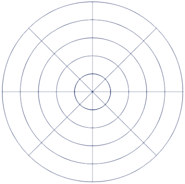
https://arxiv.org/pdf/2410.15146

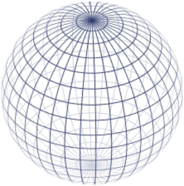
Outline

1. Introduction: Radially-Varying Reaction-Diffusion PDE on an n-dimensional Ball
    1. The challenge
    2. Power Series Solution for Kernel Equations
    3. Well-posedness and Convergence Analysis
2. Power series as a method of solution for backstepping kernel computation
3. The Timoshenko beam
4. Computational Aspects, Extensions and Challenges
5. Final remarks and conclusions

Problem

We consider the problem: stabilizing an unstable linear radially-dependent reaction-diffusion equation, evolving on an *n*-ball (disk or sphere are of most physical interest).





Main challenge: equations become singular in the radius; when applying the backstepping method, same singularity appears in kernel equations.

Solved in:

1. R. Vazquez and M. Krstic, "Boundary Control of Reaction-Diffusion PDEs on Balls in Spaces of Arbitrary Dimensions," ESAIM:Control, Optimization and Calculus of Variations, Vol. 22, No. 4, pp. 1078-1096, 2016.
2. R. Vazquez, J. Zhang, J. Qi, M. Krstic, "Kernel Well-Posedness and Computation by Power Series in Backstepping Output Feedback for Radially-Dependent Reaction-Diffusion PDEs on Multidimensional Balls," Systems & Control Letters, Vol. 177, pp. 105538, 2023

Radially-Varying Reaction-Diffusion PDE on an n-dimensional Ball

Consider the reaction-diffusion system in an n-dimensional ball of radius R $B^n(R)$:

$$\frac{\partial u}{\partial t} = \epsilon \triangle_n u + \lambda(r)u$$

where:

- $u = u(t, \vec{x})$ is the state variable
- $\vec{x} \in B^n(R) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq R\}$
- $\lambda(r)$ is the radially-varying reaction coefficient
- $\epsilon > 0$ is the diffusion coefficient
- $\epsilon \triangle_n$ the Laplacian in dimension n

Boundary conditions:

$$u(t, \vec{x})\big|_{\|\vec{x}\|=R} = U(t, \vec{x})$$

Coordinate System: n-Dimensional Spherical Coordinates

The ultraspherical coordinate system consists of:

- One radial coordinate $r = \|\vec{x}\|$
- $(n-1)$ angular coordinates $\vec{\theta} = (\theta_1, \ldots, \theta_{n-1})$ where:
  - $\theta_1, \ldots, \theta_{n-2} \in [0, \pi]$ (polar angles)
  - $\theta_{n-1} \in [0, 2\pi)$ (azimuthal angle)

Cartesian to spherical transformation:

$$
\begin{aligned}
x_1 &= r \cos\theta_1 \\
x_2 &= r \sin\theta_1 \cos\theta_2 \\
x_3 &= r \sin\theta_1 \sin\theta_2 \cos\theta_3 \\
&\vdots \\
x_n &= r \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-2} \sin\theta_{n-1}
\end{aligned}
$$

Spherical Harmonics Decomposition

We decompose the solution and control using n-dimensional ultraspherical harmonics[1]:

$$u(t, r, \vec{\theta}) = \sum_{l=0}^{\infty} \sum_{m=0}^{N(l,n)} u_l^m(r, t) Y_{lm}^n(\vec{\theta})$$

where:

- Y_{lm}^n is the m-th n-dimensional ultraspherical harmonic of degree l
- $N(l, n)$ is the number of linearly independent harmonics:
  - $N(0, n) = 1$ (mean value over n-ball)
  - For $l > 0$: $N(l, n) = \frac{2l+n-2}{l}\binom{l+n-3}{l-1}$
- Y_{lm}^n are eigenfunctions of the Laplace-Beltrami operator

[1] K.Atkinson and W. Han, Spherical Harmonics and Approximations on the Unit Sphere: An Introduction, Springer, 2012.

The Reduced 1-D Problem

From the spherical harmonics decomposition, each mode satisfies:

$$\partial_t u_l^m = \frac{\epsilon}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r u_l^m \right) - l(l+n-2) \frac{\epsilon}{r^2} u_l^m + \lambda(r) u_l^m$$

evolving in $r \in (0, R]$, with boundary conditions:

$$u_l^m(t, R) = U_l^m(t) \quad \text{(control)}$$

Note:

- The r^{-2} term appears from the angular derivatives
- Singular behavior at $r = 0$ requires careful analysis
- Control only acts at the boundary $r = R$

Natural Stability of Higher Modes

Lemma (Higher Mode Stability)

Given $\lambda(r)$ and R, there exists $L \in \mathbb{N}$ such that for all $l > L$, the equilibrium $u_l^m \equiv 0$ is open-loop exponentially stable.
Specifically, with $U_l^m = 0$, there exists $D_1 > 0$ such that:

$$\|u_l^m(t,\cdot)\|_{L^2} \leq e^{-D_1 t}\|u_l^m(0,\cdot)\|_{L^2}$$

Key idea of proof: Use the L^2 norm as Lyapunov function:

$$\|f\|_{L^2}^2 = \int_0^R |f(r)|^2 r^{n-1} dr$$

The $l(l+n-2)$ term dominates $\lambda(r)$ for large l.

The Control Problem

Main goal: Design a state feedback control law $U_l^m(t)$ that stabilizes the unstable modes ($l \leq L$).

Approach: Use the backstepping method

- Transform the system into a stable target system
- Design through Volterra integral transformation
- Key challenge: Finding the transformation kernel

Target system: We want to achieve

$$\partial_t w_l^m = \epsilon \frac{\partial_r (r^{n-1} \partial_r w_l^m)}{r^{n-1}} - \epsilon l(l+n-2) \frac{w_l^m}{r^2} - c w_l^m$$

with $c > 0$ and boundary condition $w_l^m(t, R) = 0$

The Backstepping Transform

Consider the Volterra transformation:

$$w_l^m(t,r) = u_l^m(t,r) - \int_0^r K_{lm}^n(r,\rho) u_l^m(t,\rho) d\rho$$

This leads to:

- Control law from transformation at $r = R$:

$$U_l^m(t) = \int_0^R K_{lm}^n(R,\rho) u_l^m(t,\rho) d\rho$$

- Kernel PDE for $K_{lm}^n(r,\rho)$ (after simplification, in domain $\mathcal{T} = \{(r,\rho) : 0 \leq \rho \leq r \leq R\}$):

$$\frac{1}{r^{n-1}} \partial_r \left(r^{n-1} \partial_r K_{lm}^n \right) - \partial_\rho \left(\rho^{n-1} \partial_\rho \left(\frac{K_{lm}^n}{\rho^{n-1}} \right) \right) - l(l+n-2) \left(\frac{1}{r^2} - \frac{1}{\rho^2} \right) K_{lm}^n = \frac{\lambda(\rho) + c}{\epsilon} K_{lm}^n$$

$$2\epsilon \frac{d}{dr} \left(K_{lm}^n(r,r) \right) = -(\lambda(r) + c)$$

The Kernel Equation Challenge

The kernel equation has several challenging features:

- Singular coefficients at $r = 0$ and $\rho = 0$
- Traditional approaches for kernel well-posedness fail:
  - Successive approximations lead to singular integrals: only works for $l = 0$ (mean value) and $n = 3$ (trivially reduces to 1-D case[2]), $n = 2$ (combinatorial proof based on Catalan's numbers[3])
  - Standard numerical schemes struggle with singularities
  - Explicit solutions only known for very special cases (constant λ):

$$K_{lm}^{n}(r,\rho) = -\rho \left(\frac{\rho}{r}\right)^{l+n-2} \frac{\lambda + c}{\epsilon} \frac{\mathrm{I}_1\left[\sqrt{\frac{\lambda+c}{\epsilon}(r^2-\rho^2)}\right]}{\sqrt{\frac{\lambda+c}{\epsilon}(r^2-\rho^2)}}$$

Key Insight: Try power series solution

- Similar to Frobenius method for ODEs
- Must prove existence, convergence, need to handle the singularities

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[2] R. Vazquez and M. Krstic, "Boundary control and estimation of reaction-diffusion equations on the sphere under revolution symmetry conditions," International Journal of Control, vol. 92, pp. 2-11, 2019.

[3] R. Vazquez and M. Krstic, "Boundary control of a singular reaction-diffusion equation on a disk," CPDE 2016, 2016.

Power Series Approach: Setting Up

Define a change of variables:

$$K_{lm}^n(r,\rho) = G_{lm}^n(r,\rho)\rho\left(\frac{\rho}{r}\right)^{l+n-2}$$

The G-kernel must satisfy:

$$\begin{aligned}\frac{\lambda(\rho)+c}{\epsilon}G_{lm}^n = \partial_{rr}G_{lm}^n + (3-n-2l)\frac{\partial_r G_{lm}^n}{r} \\ - \partial_{\rho\rho}G_{lm}^n + (1-n-2l)\frac{\partial_\rho G_{lm}^n}{\rho}\end{aligned}$$

with boundary condition:

$$G_{lm}^n(r,r) = -\frac{\int_0^r(\lambda(\sigma)+c)d\sigma}{2r\epsilon}$$

Power Series Solution

Assuming $\lambda(r)$ is analytic:

$$\frac{\lambda(r) + c}{\epsilon} = \sum_{i=0}^{\infty} \lambda_i r^i$$

Seek solution of the form:

$$G_{lm}^{n}(r, \rho) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} C_{ij} r^j \rho^{i-j} \right)$$

From the boundary condition:

$$\forall i, \quad \sum_{j=0}^{i} C_{ij} = -\frac{\lambda_i}{2(i+1)}$$

Even Property and Recursion

Theorem (Evenness Requirement)

If $\lambda(r)$ is not even, then there are values of $l \in \mathbb{N}$ for which there is no solution to the kernel equations in power series form.

When $\lambda(r)$ is even:

- Only even powers appear in the series, thus consider $\frac{\lambda(r)+c}{\epsilon} = \sum_{i=0}^{\infty} \lambda_i r^{2i}$, $G_{lm}^{n}(r,\rho) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} C_{ij} r^{2j} \rho^{2(i-j)}\right)$
- The coefficients satisfy a recursion:

$$(j+1)(j+1-\gamma)C_{i(j+1)} - (i-j)(j-i-\gamma)C_{ij} = \sum_{k=j}^{i-1} C_{kj}\lambda_{i-1-k}$$

where $\gamma' = \frac{\gamma}{2} = \frac{n}{2} + l - 1 \geq 0$

Main Convergence Result

Theorem (Well-posedness)

Assume $\lambda(r)$ is an even analytic function in $[0, R]$. Then:

1. *For given $n > 1$ and all $l \in \mathbb{N}$, there exists a unique power series solution $G^n_{lm}(r, \rho)$*
2. *The solution is even in both variables*
3. *The series converges in the domain $\mathcal{T} = \{(r, \rho) : 0 \leq \rho \leq r \leq R\}$*

Remark:

- $r = \|\vec{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$
- Non-even $\lambda(r)$ implies non-smooth coefficients in physical space

Sketch of Convergence Proof

Key steps to prove convergence:

1. Connection with Gauss hypergeometric functions
2. For odd dimension n:
   - Define special coefficients L_{ij} involving Gamma functions
   - Find how the coefficients grow
3. For even dimension n:
   - Solve partially up to order $\gamma - 1$ by applying Fuchs' theorem for regular singular points on a recursive set of ODEs
   - Find how the coefficients grow for higher order
4. Define $\alpha_i = \sum_{j=0}^{i} |C_{ij}|$. Need to show that $\sum_{i=0}^{\infty} \alpha_i r^{2i}$ converges.
5. Apply ratio test to prove absolute convergence. Based on: V. Leon and B. Scardua, "On singular Frobenius for second order linear partial differential equations," preprint downloaded from ArXiv, `https://arxiv.org/abs/1907.02620`, 2019.

Connection with Gauss Hypergeometric Functions

Key insight: The coefficients relate to hypergeometric series
Define:

$$\kappa(i,\gamma) = 1 + \sum_{j=0}^{i-1} \prod_{k=j}^{k=i-1} a_{ik}(\gamma)$$

where

$$a_{ij}(\gamma) = \frac{(j+1)(j+1-\gamma)}{(i-j)(i-j+\gamma)}$$

Then, for i positive and $\gamma > 0$,

$$\kappa(i,\gamma) = {}_2F_1(-i, \gamma - i; 1+\gamma; 1) = \frac{2i!}{i!} \frac{\Gamma(\gamma+1)}{\Gamma(i+\gamma+1)} > 0$$

This connects our recursion to classical special functions theory.

Solvability

One can find:

$$C_{ii} = -\frac{1}{\kappa(i,\gamma')}\left[\frac{\lambda_i}{2\epsilon(2i+1)} + H_i\right],$$

with H_i a function of previous coefficients that always exists.
Then:

$$C_{ij} \quad = \quad \left[\prod_{k=j}^{k=i-1} a_{ik}(\gamma')\right] C_{ii} + \hat{B}_{(i-1)j} + \sum_{r=j+1}^{i-1}\prod_{k=r}^{k=i-1} a_{ik}(\gamma')\hat{B}_{(i-1)r},$$

Where $\hat{B}_i$ is also defined from previous coefficients.
Thus the key is that $\kappa(i,\gamma') \neq 0$, which is proved obtaining the representation in previous page.
Thus we can always obtain the coefficients C_{ij}.

Why Dimension Matters: The γ' Split

The Critical Parameter

$$\gamma' = \frac{n}{2} + l - 1$$

- n is spatial dimension
- l is spherical harmonic degree
- This appears in coefficient denominators!

Odd n

- γ' is half-integer
- e.g., for $n = 3$:

$$\gamma' = \frac{3}{2} + l - 1 = \frac{1}{2} + l$$

- Never integer for any l!

Even n

- γ' is integer
- e.g., for $n = 2$:

$$\gamma' = 1 + l - 1 = l$$

- Always integer!

The Heart of the Matter: Coefficient Structure

Key Coefficient Formula

$$L_{ij} = \binom{i}{j} \frac{(i+\gamma')(i-1+\gamma')\cdots(i-j+\gamma'+1)}{(1-\gamma')(2-\gamma')\cdots(j-\gamma')}$$

Used to find an explicit formula for the coefficients and exploited for many properties.

Odd Case (γ' non-integer)

- Denominator never zero
- Can directly compute all L_{ij}
- Series coefficients well-defined
- Convergence follows from bounds

Even Case (γ' integer)

- Get terms like $\frac{1}{0}$
- Can't compute L_{ij} directly
- Need special treatment
- Must split solution

Odd Dimension: Direct Convergence

Strategy

Define and bound coefficient sum: $\alpha_i = \sum_{j=0}^{i} |C_{ij}|$

Key Inequality

Show α_i satisfies: $\alpha_i \leq b_i|\lambda_i| + c_i \sum_{k=0}^{i-1} \alpha_k |\lambda_{i-1-k}|$ where: b_i decreasing, $c_i \to 0$ as $i \to \infty$

Conclusion

If $\lambda(r)$ analytic in disc $|r| < R$:

$$\sum_{i=0}^{\infty} \alpha_i r^{2i} \text{ converges for } |r| < R$$

Even Dimension: The Split Solution

Step 1: Partial Solution

Up to order $\gamma' - 1$: $F(r,\rho) = \sum_{i=0}^{\gamma'-1} r^{2i}\phi_i(\rho^2)$. Each ϕ_i solves ODE:

$$4x\phi''_{\gamma'-1} + 2(2+\gamma')\phi'_{\gamma'-1} + \frac{\lambda(x)+c}{\epsilon}\phi_{\gamma'-1} = 0$$

Use Frobenius for those!

Step 2: Remainder Solution

Define $\check{G}^n_{lm} = G^n_{lm} - F$

- Starts at order $2\gamma'$
- Avoids division by zero
- Now similar to odd case
- Can prove convergence for $\check{G}^n_{lm}$

Summary of Power Series Solution

Key Achievements:

- First rigorous proof of well-posedness for backstepping kernels via power series
- Discovered necessary evenness condition for $\lambda(r)$
- Unified treatment for all dimensions $n > 1$
- Explicit recursive formulas for coefficients

Practical Implications:

- Simple numerical implementation
- Symbolic computation possible
- No need for discretization or mesh
- High precision achievable

The Full Picture: Physical Space Stability

Theorem (Complete System Stability)

Under the assumptions:

- $\lambda(r)$ *even and analytic*
- *Kernels* $K^n_{lm}(r,\rho)$ *from power series solution*

The complete physical solution:

$$u(t,r,\vec{\theta}) = \sum_{l=0}^{\infty} \sum_{m=0}^{N(l,n)} u_l^m(r,t) Y^n_{lm}(\vec{\theta})$$

with control:

$$u(t,\vec{x})\Big|_{\|\vec{x}\|=R} = \sum_{l=0}^{L} \sum_{m=0}^{N(l,n)} \left(\int_0^R K^n_{lm}(R,\rho) u_l^m(t,\rho) d\rho \right) Y^n_{lm}(\vec{\theta})$$

is exponentially stable at the origin.

Proof Strategy

1. For $l > L$: Natural stability (from angular derivatives)

$$\|u_l^m(t,\cdot)\|_{L^2} \leq e^{-D_1 t}\|u_l^m(0,\cdot)\|_{L^2}$$

2. For $l \leq L$: Controlled modes via backstepping

$$\|u_l^m(t,\cdot)\|_{L^2} \leq Ce^{-D_2 t}\|u_l^m(0,\cdot)\|_{L^2}$$

3. Combining all modes:

$$\|u(t,\cdot)\|_{L^2(B^n(R))} \leq Ce^{-Dt}\|u(0,\cdot)\|_{L^2(B^n(R))}$$

where $D = \min\{D_1, D_2\}$

Outline

1. Introduction: Radially-Varying Reaction-Diffusion PDE on an n-dimensional Ball ✓
2. Power series as a method of solution for backstepping kernel computation
   1. From complex to simple by complex numbers
   2. Examples using Mathematica
   3. Handling discontinuous kernels
3. The Timoshenko beam
4. Computational Aspects, Extensions and Challenges
5. Final remarks and conclusions

Power Series: From Complex to Simple

- We developed power series for the **radially-varying ball**:
    - Complex geometry
    - Singularities at origin
    - Required Gauss hypergeometric theory
- Key realization: Method is **simpler** for basic cases
    - More direct proofs
    - Easier implementation
    - Clear convergence conditions
- Can become a **general tool** for kernel computation, specially for beginners!

Basic Power Series Framework

Let us demonstrate with a simple example:

- Start with basic 1-D backstepping (e.g. reaction-diffusion equation)
- Show explicit power series computation
- Illustrate convergence proof using complex analysis
- Demonstrate straightforward implementation

Key Theme: What was developed for a complex case becomes a powerful general method

Backstepping Method for 1-D Reaction Diffusion Equations

Consider

$$\begin{aligned} u_t &= \epsilon u_{xx} + \lambda(x) u \\ u(t,L) &= U(t) \\ u(t,0) &= 0 \end{aligned}$$

$\epsilon > 0$, $\lambda(x)$ a function in the domain $x \in [0, L]$. Potentially unstable

The feedback $U = \int_0^L K(L,\xi)u(\xi)d\xi$ is stabilizing, by choosing $c \geq 0$ and solving

$$\begin{aligned} K_{xx}(x,\xi) - K_{\xi\xi}(x,\xi) &= \frac{\lambda(\xi) + c}{\epsilon} K(x,\xi) \\ K(x,x) &= -\frac{1}{2\epsilon} \int_0^x (\lambda(\xi) + c)\, d\xi \\ K(x,0) &= 0 \end{aligned}$$

in the *triangular* domain $\mathcal{T} = \{(x,\xi) : 0 \leq \xi \leq x \leq L\}$

At the end of the day, we only need $K(L,\xi)$

Kernel PDEs

$$K_{xx} - K_{\xi\xi} = \frac{\lambda(\xi) + c}{\epsilon} K$$
$$K(x, x) = -\frac{1}{2\epsilon} \int_0^x [\lambda(\sigma) + c] d\sigma$$
$$K(x, 0) = 0$$

This is a Classical Goursat-type problem but with integral boundary condition.

- Second-order hyperbolic PDE
- On triangular domain $0 \leq \xi \leq x \leq L$
- With non-standard boundary conditions

If $\lambda(x) = \lambda$ constant, and calling $\bar{\lambda} = \frac{\lambda + c}{\epsilon}$, then we know

$$K(x, y) = -\bar{\lambda} y \frac{\mathrm{I}_1\left(\sqrt{\bar{\lambda}\left(x^2 - y^2\right)}\right)}{\sqrt{\bar{\lambda}\left(x^2 - y^2\right)}}$$

For very specific shapes of $\lambda(x)$ other solutions exist.

There is no general explicit solution (or hope of getting one)

The Power Series Framework: Theory

Consider the kernel equations:

$$K_{xx}(x,\xi) - K_{\xi\xi}(x,\xi) = \frac{\lambda(\xi)+c}{\epsilon}K(x,\xi)$$
$$K(x,x) = -\frac{1}{2\epsilon}\int_0^x [\lambda(\sigma)+c]d\sigma$$
$$K(x,0) = 0$$

Key Insight: Extend to complex domain

- Let $\mathcal{D}_L$ be complex disk of radius L: $\{z \in \mathbb{C} : |z| < L\}$
- Consider kernel on polydisk $\mathcal{D}_{L+\delta} \times \mathcal{D}_{L+\delta}$
- Analyticity in complex domain $\Rightarrow$ power series convergence

Main Theorem

Theorem

If there exists $\delta > 0$ such that λ is analytic on $\mathcal{D}_{L+\delta}$, then:

1. *The kernel equation solution $K(x, \xi)$ extends to an analytic function on $\mathcal{D}_{L+\delta/2} \times \mathcal{D}_{L+\delta/2}$*
2. *This solution is unique*
3. *The power series converges in this domain*

Key to proof: Leverage classical successive approximation results in complex domain

R. Vazquez, G. Chen, J. Qiao, M. Krstic, "The power series method to compute backstepping kernel gains: theory and practice," CDC 2023.

Proof Strategy (Part 1)

1. Transform to integral equation via rotation and integration:

$$K(x,\xi) = G\left(\frac{x+\xi}{2}, \frac{x-\xi}{2}\right)$$

2. Write G as successive approximation series:

$$G(x,\xi) = \sum_{i=0}^{\infty} G_i(x,\xi)$$

where:

$$G_0 = -\frac{1}{4\epsilon}\int_0^x \lambda\left(\frac{s}{2}\right) + c\,ds$$

$$G_{i+1} = \frac{1}{4\epsilon}\int_\xi^x \int_0^\xi [\lambda(\frac{\tau - s}{2}) + c] G_i(\tau, s)\,ds\,d\tau$$

Proof Strategy (Part 2)

3. Consider the integrals as path integrals in the complex plane. Complex line integrals are path-independent for analytic functions
4. Show recursively that each G_i is analytic
5. Prove uniform convergence using already known bound from succesive approximations proof.
6. Apply Weierstrass M-test to get uniform convergence

Convergence $+$ analyticity $\Rightarrow$ unique power series solution

Advantages of Complex Analysis Approach

- Provides **clear conditions** for existence:
  - Analyticity of coefficients
  - Size of domain of convergence
- Gives **uniqueness** of solution
  - By identity theorem for analytic functions
  - Power series must have unique coefficients
- **Constructive proof**:
  - Shows why substitution method works
  - Guarantees convergence of numerical scheme

Implementation in Mathematica: Key Steps

1. Introduce kernel power series:

$$K(x,\xi) = \sum_{i=0}^{n}\sum_{j=0}^{i} K_{ij} x^{i-j}\xi^{j}$$

2. System coefficients expanded automatically:
   - Series[$\lambda(x)$, $\{x, 0, n\}$]
   - Handles any analytic function
   - Automatic term collection
3. Substitute into PDE and boundary conditions:
   - D[expr, $\{x, 2\}$] - D[expr, $\{\xi, 2\}$] $==$ ...
   - Coefficient[expr, $x^i\xi^j$] gives equations

Mathematica Implementation

```
In[95]:= n = 10;
         K[x_, y_] = Normal[Series[G[x*t, y*t], {t, 0, n}]] /. t→1;
         epsilon = 1;
         lambda[y_] = y^2*Cos[3*y] + y;
         c = 3;

In[100]:= LHS = epsilon*D[D[K[x, y], x], x] - epsilon*D[D[K[x, y], y], y];

In[101]:= RHS = Normal[Series[(lambda[y*t] + c)*K[x*t, y*t], {t, 0, n-2}]] /. t→1;

In[102]:= BC1 = D[K[x, y], x] /. y→0;

In[108]:= Integral[x_] = 1/(2*epsilon)*Integrate[(lambda[y] + c), {y, 0, x}];

In[109]:= BC2 = K[x, x] + Normal[Series[Integral[x*t], {t, 0, n}]] /. t→1;

In[110]:= soln = SolveAlways[Join[{LHS == RHS}, {BC2 == 0}, {BC1 == 0}], {x, y}];

In[111]:= sol[x_, y_] = K[x, y] /. soln[[1]]
```

Out[111]=

$$-\frac{3y}{2}-\frac{xy}{4}-\frac{11x^2y}{16}-\frac{x^3y}{8}+\frac{13x^4y}{768}-\frac{3x^5y}{128}-\frac{361x^6y}{92160}+\frac{103x^7y}{107520}+\frac{221x^8y}{368640}-\frac{69x^9y}{1146880}+$$
$$\frac{25y^3}{48}+\frac{145x^2y^3}{384}-\frac{x^3y^3}{64}-\frac{517x^4y^3}{18432}+\frac{283x^5y^3}{15360}+\frac{4177x^6y^3}{552960}-\frac{1033x^7y^3}{860160}+\frac{y^4}{8}+\frac{xy^4}{48}+$$
$$\frac{11x^2y^4}{192}+\frac{x^3y^4}{96}-\frac{13x^4y^4}{9216}+\frac{x^5y^4}{512}+\frac{361x^6y^4}{1105920}+\frac{133y^5}{3840}+\frac{xy^5}{128}-\frac{1201x^2y^5}{30720}+\frac{83x^3y^5}{3072}+$$
$$\frac{677x^4y^5}{46080}-\frac{337x^5y^5}{81920}-\frac{5y^6}{192}-\frac{29x^2y^6}{1536}+\frac{x^3y^6}{1280}+\frac{517x^4y^6}{368640}-\frac{12933y^7}{71680}-\frac{221xy^7}{9216}-$$
$$\frac{33127x^2y^7}{430080}-\frac{1265x^3y^7}{73728}-\frac{229y^8}{107520}-\frac{23xy^8}{53760}+\frac{283x^2y^8}{286720}+\frac{837031y^9}{7741440}+\frac{1099xy^9}{98304}+\frac{66863y^{10}}{7741440}$$

Key Features of Implementation

- **Automation:**
  - No manual derivation of recursions
  - System detects required equations
  - Handles any analytic coefficient
- **Verification:**
  - Can substitute back into PDE
  - Check boundary conditions
  - Verify convergence numerically
- **Analysis:**
  - Parameter studies
  - Order requirements
  - Convergence rates

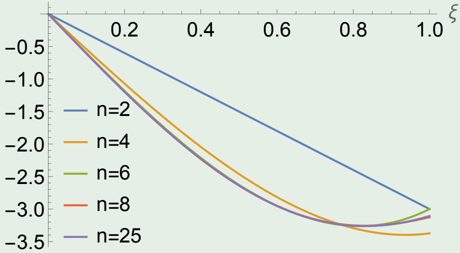
Basic Examples: Power Series Convergence

Example (Reaction-Diffusion with Smooth Coefficient)

Consider:

$$\lambda(x) = 3 + x^2 \sin(3x)$$

Analytic in $\mathbb{C}$ due to entire functions x^2 and $\sin(3x)$ $\rightarrow$ **converges**



Note rapid convergence by order 8

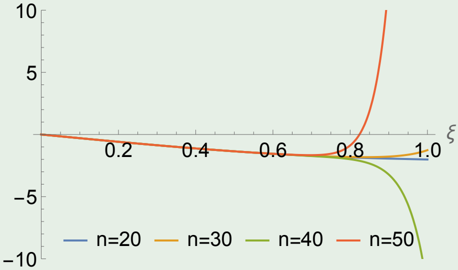
Basic Examples: Power Series Divergence

Example (Non-analytic Coefficient)

Consider:

$$\lambda(x) = \sqrt{0.5 + x^2}$$

- Branch point at $x = \pm i\sqrt{0.5}$
- Inside unit disk $\mathcal{D}_1$
- Violates analyticity requirement
- Series **diverges** as shown:



Basic Examples: Parametric Solutions

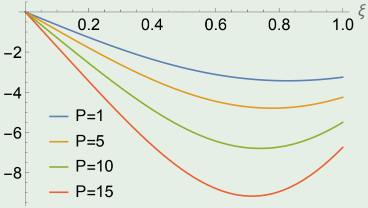
Example (Linear Parametric Coefficient)

Consider:

$$\lambda(x) = 1 + Px$$

Symbolically compute power series in both x and parameter P:

$$K(L, \xi) = \sum_{i,j} K_{ij}(P) \xi^i P^j$$



Value: Single computation for family of problems

Basic Examples: Space-Varying Diffusion

Consider:

$$\epsilon(x)K_{xx} - \epsilon(\xi)K_{\xi\xi} = [\lambda(\xi) + c]K$$

Additional requirement: $\epsilon(z) \neq 0$ in $\mathcal{D}_{L+\delta}$

Example (Convergent Case)

- $\epsilon(x) = 2 + x^2$
- Zeros at $x = \pm i\sqrt{2}$
- Outside unit disk
- Series **converges**

Example (Divergent Case)

- $\epsilon(x) = 2 + 3x^2$
- Zeros at $x = \pm i\sqrt{2/3}$
- Inside unit disk
- Series **diverges**

Space-Varying Diffusion: Extension

Thus for the more general PDE with space-varying diffusion:

$$\epsilon(x)K_{xx}(x,\xi) - \epsilon(\xi)K_{\xi\xi}(x,\xi) = [\lambda(\xi) + c]K(x,\xi)$$
$$-2\epsilon(x)\frac{d}{dx}K(x,x) = -\epsilon'(x) - \lambda(x) - c$$
$$K(x,0) = 0$$

Theorem

If $\exists\delta > 0$ *such that* λ *and* ϵ *are analytic on* $\mathcal{D}_{L+\delta}$, *and* $|\epsilon(z)| > 0$ $\forall z \in \mathcal{D}_{L+\delta}$, *then there exists a unique power series solution converging in* $\mathcal{D}_{L+\delta/2} \times \mathcal{D}_{L+\delta/2}$.

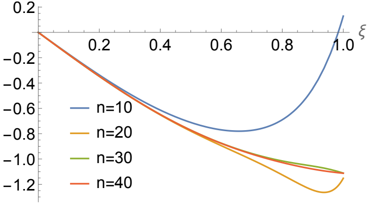
Space-Varying Diffusion: Examples

Convergent Case:

$\lambda(x) = 3 + x\sin(6x)$

$\epsilon(x) = 2 + x^2$

$L = 1,\ c = 3$

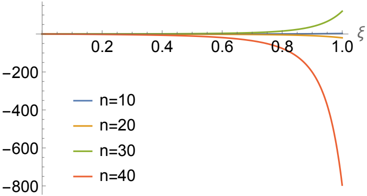


Divergent Case:

$\lambda(x) = 3 + x\sin(6x)$

$\epsilon(x) = 2 + 3x^2$

$L = 1,\ c = 3$



Key Point: Divergence due to zero of $\epsilon(x)$ at $x^* = \frac{\sqrt{2}i}{3}$ inside unit disk

Hyperbolic System Example

Consider coupled hyperbolic system:

$$\begin{aligned} v_{1t} &= -\mu_1 v_{1x} + c_1(x)v_1 + c_2(x)v_2 \\ v_{2t} &= \mu_2 v_{2x} + c_3(x)v_1 + c_4(x)v_2 \end{aligned}$$

With boundary conditions:

$$\begin{aligned} v_1(t,0) &= qv(t,0) \\ v_2(t,L) &= U(t) \end{aligned}$$

Challenge: Multiple coupled kernels required for stabilization

Hyperbolic System: Kernel Equations

Kernel system:

$$\begin{aligned}\mu(x)K^{vv}_x + \mu(\xi)K^{vv}_\xi &= -\mu'(\xi)K^{vv} + c_2(\xi)K^{vu} \\ &\quad + [c_4(x) - c_4(\xi)]K^{vu} \\ \mu(x)K^{vu}_x - \epsilon(\xi)K^{vu}_\xi &= \epsilon'(\xi)K^{vu} + c_3(\xi)K^{vv} \\ &\quad + [c_4(x) - c_1(\xi)]K^{vv}\end{aligned}$$

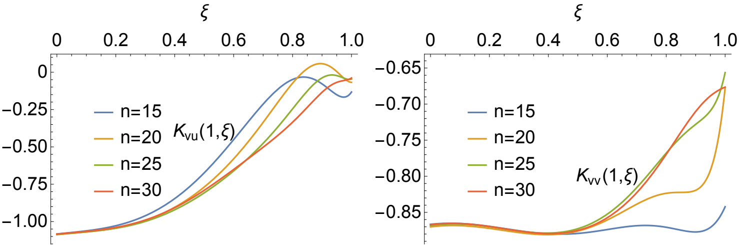
With boundary conditions:

$$\begin{aligned}K^{vv}(x,0) &= \frac{q\epsilon(0)}{\mu(0)}K^{vu}(x,0) \\ [\epsilon(x)+\mu(x)]K^{vu}(x,x) &= -c_3(x)\end{aligned}$$

Hyperbolic System: Results

Choosing coefficients:

- $\mu(x) = 1.5 + x^2$
- $\epsilon(x) = 1.2 + x^3$
- $c_1(x) = 3\cos(x)$
- $c_2(x) = \sin(2x)$
- $c_3(x) = 1 + 2e^x$
- $c_4(x) = \frac{1}{3+x^2}$



Note: Space-varying transport speeds slow convergence

Discontinuous Kernels: Motivation

- Many backstepping designs lead to **discontinuous kernels**
- Example: Motion planning for coupled transport equations
- Plant:

$$v_{1t} - \mu_1 v_{1x} = \sigma_{12}(x) v_2$$
$$v_{2t} - \mu_2 v_{2x} = \sigma_{21}(x) v_1$$

with $\mu_1 > \mu_2 > 0$ and

$$v_1(t,1) = U_1(t) \qquad v_2(t,1) = U_2$$

Design $U_1(t)$ and $U_2(t)$ so that $v_1(t,0) = \Phi_1(t)$ and $v_2(t,0) = \Phi_2(t)$ for some functions Φ_1, Φ_2 for $t \geq t_M$

The Motion Planning Control Law

Control law structure:

$$U_1 = \Phi_1(t + \frac{1}{\mu_1}) + \int_0^1 L_{11}(1,\xi)v_1(\xi)d\xi + \int_0^1 L_{12}(1,\xi)v_2(\xi)d\xi$$

$$U_2 = \Phi_2(t + \frac{1}{\mu_2}) - \int_0^1 \frac{\mu_1}{\mu_2} L_{21}(\xi,0)\Phi_1(t + \frac{1-\xi}{\mu_2})d\xi + \int_0^1 L_{21}(1,\xi)v_1(\xi)d\xi + \int_0^1 L_{22}(1,\xi)v_2(\xi)d\xi$$

Four kernels ($L_{11}, L_{12}, L_{21}, L_{22}$) needed

The Kernel Equations

$$
\begin{aligned}
\mu_1 \partial_x L_{11} + \mu_1 \partial_\xi L_{11} &= \sigma_{21}(\xi) L_{12} \\
\mu_1 \partial_x L_{12} + \mu_2 \partial_\xi L_{12} &= \sigma_{12}(\xi) L_{11} \\
\mu_2 \partial_x L_{21} + \mu_1 \partial_\xi L_{21} &= \sigma_{21}(\xi) L_{22} \\
\mu_2 \partial_x L_{22} + \mu_2 \partial_\xi L_{22} &= \sigma_{12}(\xi) L_{21}
\end{aligned}
$$

With boundary conditions:

$$
\begin{gathered}
L_{11}(x,0) = L_{12}(x,0) = L_{22}(x,0) = 0 \\
L_{12}(x,x) = \frac{\sigma_{12}(x)}{\mu_2 - \mu_1}, \quad L_{21}(x,x) = \frac{\sigma_{21}(x)}{\mu_1 - \mu_2}
\end{gathered}
$$

The Characteristic Line and Discontinuity

- L_{12} has **two** boundary conditions:
  - $L_{12}(x, 0) = 0$
  - $L_{12}(x, x) = \frac{\sigma_{12}(x)}{\mu_2 - \mu_1}$
- Characteristic line: $\xi = \frac{\mu_2}{\mu_1} x$
- Solution: Define piecewise

$$L_{12}(x, \xi) = \begin{cases} L_{12}^1(x, \xi) & \text{if } \xi < \frac{\mu_2}{\mu_1} x \\ L_{12}^2(x, \xi) & \text{if } \xi > \frac{\mu_2}{\mu_1} x \end{cases}$$

Propagation of Discontinuity

- L_{12} appears in L_{11} equation
- L_{11} must be defined piecewise:

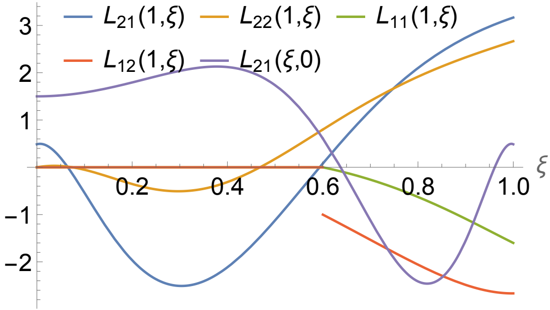
$$L_{11}(x,\xi) = \begin{cases} L_{11}^1(x,\xi) & \text{if } \xi < \frac{\mu_2}{\mu_1}x \\ L_{11}^2(x,\xi) & \text{if } \xi > \frac{\mu_2}{\mu_1}x \end{cases}$$

- But L_{11} must be continuous:

$$L_{11}^1(x, \frac{\mu_2}{\mu_1}x) = L_{11}^2(x, \frac{\mu_2}{\mu_1}x)$$

Results for Transport System

For $\mu_1 = 0.5$, $\mu_2 = 0.3$, $\sigma_{12}(x) = 0.2 + x/3$, $\sigma_{21}(x) = 0.3 + x^2/3$:



Note:

- Discontinuous $L_{12}(1, \xi)$
- Continuous but non-differentiable $L_{11}(1, \xi)$ at $\xi = 0.6$

Power Series Method: Theoretical Foundation

Theorem (Power Series Convergence)

If there exists $\delta > 0$ such that σ_{12} and σ_{21} are analytic on $\mathcal{D}_{1+\delta}$, and $\mu_1 > \mu_2 > 0$, then there exists piecewise-defined power series solutions for $L_{11}, L_{12}, L_{21}, L_{22}$ so that:

1. *Each kernel has a unique power series representation in each region*
2. *The series converge in $\mathcal{D}_{1+\delta/2} \times \mathcal{D}_{1+\delta/2}$*
3. *The kernels solve the backstepping PDEs*

Outline

Timoshenko Beam: The Model

Consider a Timoshenko beam with displacement $u(x,t)$ and rotation angle $\alpha(x,t)$:

$$\begin{aligned} \varepsilon u_{tt} &= u_{xx} - \alpha_x, \\ \mu \alpha_{tt} &= \alpha_{xx} + \frac{a}{\varepsilon}\left(u_x - \alpha\right), \end{aligned}$$

with boundary conditions:

$$\begin{aligned} u_x(0,t) &= \alpha(0,t) - \theta u_t(0,t) - \xi u(0,t), \\ u_x(1,t) &= V_1(t),\ \alpha_x(0,t) = 0,\ \alpha_x(1,t) = V_2(t) \end{aligned}$$

Where:

- $\varepsilon, \mu > 0$ are physical parameters
- θ, ξ are anti-damping and anti-stiffness coefficients
- $V_1(t), V_2(t)$ are control inputs

The Mathematical Challenge

- The system lives in function space:

$$\mathcal{H} = H^1(0,1) \times L^2(0,1) \times H^1(0,1) \times L^2(0,1)$$

- Key challenges:
  - Coupled wave equations
  - Anti-damping at uncontrolled boundary
  - Want to achieve prescribed decay rate
- Assumption: $\theta \neq \sqrt{\varepsilon}$ (non-resonance)

The Well-Posedness Result

Theorem

Consider initial conditions $(u_0, \alpha_0) \in H^1(0,1)$, $(u_{0t}, \alpha_{0t}) \in L^2(0,1)$. *If* $\theta \neq \sqrt{\varepsilon}$, *then:*

1. There exists a unique solution:

$$(u, u_t, \alpha, \alpha_t) \in C([0,\infty); \mathcal{H})$$

2. Without control $(V_1 = V_2 = 0)$, *the system is unstable*

Use backstepping to stabilize!

Key Idea: Transform Coordinates

Transform to new variables using Riemann-like transformation:

$$
\begin{aligned}
p(t,x) &= u_x(t,x) + \sqrt{\varepsilon} u_t(t,x), \\
q(t,x) &= u_x(t,x) - \sqrt{\varepsilon} u_t(t,x), \\
r(t,x) &= \alpha_x(t,x) + \sqrt{\mu} \alpha_t(t,x), \\
s(t,x) &= \alpha_x(t,x) - \sqrt{\mu} \alpha_t(t,x), \\
x_1(t) &= u(0,t), \\
x_2(t) &= \alpha(0,t)
\end{aligned}
$$

This transforms coupled wave equations into more manageable form.

The Transformed System

In new variables we get $(2+2)\times(2+2)$ system with ODEs:

$$
\begin{aligned}
p_t &= \frac{1}{\sqrt{\varepsilon}}p_x - \frac{1}{2\sqrt{\varepsilon}}(r+s),\\
q_t &= -\frac{1}{\sqrt{\varepsilon}}q_x - \frac{1}{2\sqrt{\varepsilon}}(r+s),\\
r_t &= \frac{1}{\sqrt{\mu}}r_x + \frac{a}{2\varepsilon\sqrt{\mu}}(p+q) - \frac{a}{2\varepsilon\sqrt{\mu}}\left[\int_0^x (r+s)dy + 2x_2\right],\\
s_t &= -\frac{1}{\sqrt{\mu}}s_x + \frac{a}{2\varepsilon\sqrt{\mu}}(p+q) - \frac{a}{2\varepsilon\sqrt{\mu}}\left[\int_0^x (r+s)dy + 2x_2\right],\\
\dot{x}_1 &= \frac{2}{\sqrt{\varepsilon}-\theta}\left[\xi x_1 - x_2 + p(0,t)\right],\\
\dot{x}_2 &= -\frac{1}{\sqrt{\mu}}s(0,t)
\end{aligned}
$$

Matrix Form of Transformed System

Define state vectors:

$$Z = \begin{bmatrix} p \\ r \end{bmatrix}, Y = \begin{bmatrix} q \\ s \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The system becomes:

$$\begin{aligned} Z_t &= \Sigma Z_x + \Lambda_1(Z + Y) + \Lambda_2 X + \int_0^x F[Z + Y]dy \\ Y_t &= -\Sigma Y_x + \Lambda_1(Y + Z) + \Lambda_2 X + \int_0^x F[Z + Y]dy \\ \dot{X} &= (A + B_2 D)X + (B_1 + B_2 C)Z(0, t) \end{aligned}$$

With boundary conditions:

$$Z(1, t) = V, \; Y(0, t) = CZ(0, t) + DX$$

The Backstepping Approach

Use Volterra transformation:

$$\begin{aligned}\sigma(x,t) &= Z(x,t) - \int_0^x K(x,y)Z(y,t)dy \\ &\quad - \int_0^x L(x,y)Y(y,t)dy - \Phi(x)X(t)\end{aligned}$$

Need to find:

- Kernel matrices $K(x,y)$, $L(x,y)$
- Matrix function $\Phi(x)$
- Target system that achieves stability

Key challenge: Multiple coupled kernel PDEs!

The Target System

Choose target system:

$$
\begin{aligned}
\sigma_t &= \Sigma \sigma_x + \Omega(x)\sigma \\
\psi_t &= -\Sigma \psi_x + \Lambda_1(\psi + \sigma) + \int_0^x \Xi_2 \sigma dy \\
&\quad + \int_0^x \Xi_3 \psi dy + \Xi_1 X \\
\dot{X} &= E_1 X + E_2 \sigma(0, t)
\end{aligned}
$$

Where:

- $\Omega(x)$ has special structure to decouple σ
- E_1 can be shaped via design parameters ($\Phi(0)$).
- After finite time, $\sigma \to 0$

The Kernel Equations

The kernels must satisfy:

$$\begin{aligned}
\Sigma K_x + K_y \Sigma &= (K+L)\,\Lambda_1 - \Omega K - F + \int_y^x (K+L)\,F ds, \\
\Sigma L_x - L_y \Sigma &= (K+L)\,\Lambda_1 - \Omega L - F + \int_y^x (K+L)\,F ds, \\
\Phi_x &= \Sigma^{-1}\Phi A - \Sigma^{-1}\Lambda_2 + \Sigma^{-1}\Phi B_2 D \\
&\quad -\Sigma^{-1}\Omega\Phi + \int_0^x \Sigma^{-1}(K-L)\Lambda_2 dy \\
&\quad +\Sigma^{-1} L(x,0)\Sigma D
\end{aligned}$$

With boundary conditions:

$$\begin{aligned}
\Sigma L(x,x) + L(x,x)\Sigma &= -\Lambda_1 \\
\Sigma K(x,x) - K(x,x)\Sigma &= -\Lambda_1 + \Omega(x) \\
K(x,0)\Sigma - L(x,0)\Sigma C &= \Phi B
\end{aligned}$$

and $\Phi(0)$ that can be chosen.

Structure of Kernel Solutions

Theorem (Kernel Regularity)

The kernel equations have unique solutions (K, L, Φ) where:

- K_{ij}, L_{ij} *are piecewise* C^1 *in each region*
- *Discontinuities occur along characteristics:*

$$\xi = \frac{\mu_i}{\mu_j} x$$

- *Components bounded by* Me^{Mx} *for some* $M > 0$

This justifies using different power series in each region!

The Main Stabilization Result

Theorem (Exponential Stability)

Consider initial conditions $(u_0, \alpha_0) \in H^1$, $(u_{0t}, \alpha_{0t}) \in L^2$. *For any* $C_2 > 0$, *one can choose* $\Phi(0)$ *and there exists* $C_1 > 0$ *such that the closed-loop system verifies:*

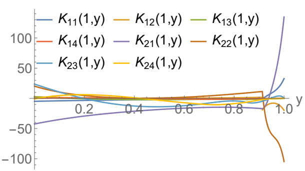
$$\|X(t)\|_{\mathcal{H}} \leq C_1 e^{-C_2 t} \|X(0)\|_{\mathcal{H}}$$

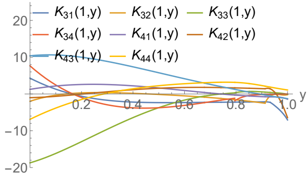
G. Chen, R. Vazquez, M. Krstic, "Rapid Stabilization of Timoshenko Beam by PDE Backstepping," IEEE Transactions on Automatic Control, vol. 69, pp. 1141-1148, 2024.

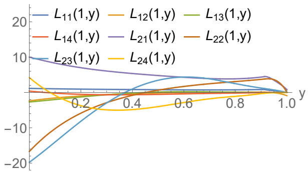
From Theory to Numerics

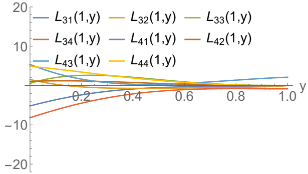
- Theory tells us:
  - Power series exist
  - Where discontinuities appear
  - What regularity to expect
- Numerical implementation:
  - 48 kernel functions
  - 7 regions from discontinuities
  - Series in each region
- Key link: Theoretical structure guides numerical method

Computed Kernel Gains









Solutions of gain kernels $K_{ij}(1, y), L_{ij}(1, y), 1 \leq i \leq 4, 1 \leq j \leq 4$

Notable features:

- Clear discontinuities in several kernels
- Smooth behavior between discontinuities
- Power series captures all features accurately

Outline

1. Introduction: Radially-Varying Reaction-Diffusion PDE on an n-dimensional Ball ✓
2. Power series as a method of solution for backstepping kernel computation✓
3. The Timoshenko beam✓
4. Computational Aspects, Extensions and Challenges
   1. Matlab numerical algebra instead of Mathematica symbolical computation
   2. Localized power series expansions
   3. Patches of power series
5. Final remarks and conclusions

MATLAB vs Mathematica: A Performance Comparison

Key Differences:

- Mathematica: Symbolic computation
  - Exact but slow for large orders
  - Memory intensive
  - Complex expressions
- MATLAB: Numerical linear algebra
  - Fast sparse matrix operations
  - Efficient memory usage
  - Optimized for large systems

Performance Gains:

- Orders of magnitude faster computation
- Higher orders achievable ($N > 100$)
- Maintained precision

The MATLAB Implementation Framework

Core Components:

- Vector-matrix formulation
- Transformation matrices for operators
- Sparse matrix handling
- Efficient linear system solver

Key Features:

- Automatic equation generation
- Built-in sparse matrix optimization
- Direct access to numerical libraries
- Easy integration with visualization tools

Sparsity Analysis

Matrix Structure:

- System matrix sparsity increases with order
- For $N = 50$: 99.2% sparsity
- Memory savings scale with problem size

Order N	Sparsity	Speed-up
25	98.2%	5x
50	99.2%	10x
100	99.6%	20x

Localized Power Series: Theory

Key Concept:

$$K(x,\xi) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} K_{ij}(x - x_0)^{i-j}(\xi - \xi_0)^j$$

Advantages:

- Choose expansion point strategically
- Avoid singularities in complex plane
- Better convergence for oscillatory solutions
- Handle previously divergent cases

Requirements:

- Analyticity in shifted domain
- Proper choice of (x_0, ξ_0)

Localized Series: Implementation

Transformation Steps:

- Change of variables: $\tilde{x} = x - x_0$, $\tilde{\xi} = \xi - \xi_0$
- Transform boundary conditions
- Adjust integral terms
- Modify system matrix

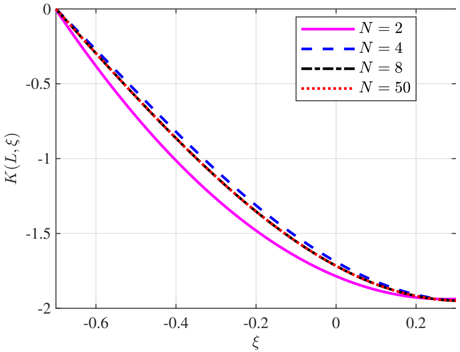
Computational Impact:

- Slightly reduced sparsity
- Moderate increase in computation time
- Balanced by improved convergence
- Enables solution of new problems

Example: From Divergent to Convergent

Consider $\lambda(x) = \sqrt{0.5 + x^2}$:

- **Original series** (at origin):
  - Divergent due to branch points
  - No solution possible
- **Localized series** ($x_0 = 0.5$, $\xi_0 = 0.7$):
  - Convergent solution
  - Clear physical interpretation



Patches of Power Series: Concept

Key Idea:

- Multiple localized expansions
- Different centers for different regions
- Smooth connections between patches
- Optimal order for each patch

Benefits:

- Better approximation of oscillatory kernels
- Lower orders needed per patch
- More flexible handling of singularities
- Improved numerical stability

Future Directions: Patches Implementation

Technical Challenges:

- Optimal patch placement
- Connection conditions between patches
- Error control at boundaries
- Automatic patch generation

Applications:

- Complex multi-kernel systems
- Systems with discontinuities
- Highly oscillatory solutions
- Neural operator training

Toolbox Development

Current Features:

- Efficient sparse matrix operations
- Localized series capability
- Automatic equation generation
- Example library

Planned Extensions:

- Patch management system
- Automatic singularity detection
- Neural operator interface
- Parameter optimization

To be presented in Dec 2024: X. Lin, R. Vazquez, M. Krstic, "Towards a MATLAB Toolbox to compute backstepping kernels using the power series method," CDC 2024.

Outline

1. Introduction: Radially-Varying Reaction-Diffusion PDE on an n-dimensional Ball ✓
2. Power series as a method of solution for backstepping kernel computation✓
3. The Timoshenko beam✓
4. Computational Aspects, Extensions and Challenges✓
5. Final remarks and conclusions
    1. Key achievements
    2. Impact on the field
    3. Future work

Final Remarks: Key Theoretical Results

Radially-Varying PDE Results:

- First rigorous proof of convergence for backstepping kernels
- Discovery of evenness condition for $\lambda(r)$
- Unified treatment for all dimensions $n > 1$
- Complete system stability proof
- Connection with Gauss hypergeometric functions

Power Series Method:

- From complex to simple back to complex
- Explicit convergence conditions
- Treatment of discontinuous kernels

Practical Implementations

Computational Framework:

- Efficient Mathematica and MATLAB implementation
- Localized power series approach
- Orders of magnitude speed improvement in Matlab
- Handling of previously unsolvable cases

Timoshenko Beam:

- Solution of 48 coupled kernel functions
- Treatment of 7 regions from discontinuities

Future Directions

Theoretical Extensions:

- Patched power series development
- Non-even reaction coefficients?
- Other methods?
- Higher-dimensional problems

Practical Development:

- Create a real toolbox
- Find a good framework for polynomial operations
- Implement the patched power series approach
- Integration with neural approaches for training
- Optimization of kernels

Thank You!

Questions?

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