# A fixed energy Born approximation for the Calderón problem

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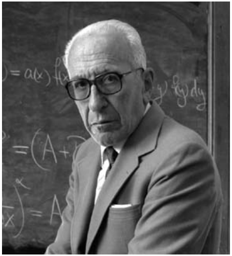
Based on joint works with C. J. Meroño (UPM), and D. Sánchez-Mendoza (UPM)

# The Calderón Problem

During his early career, as a research engineer in the geophysical division at YPF, Alberto Calderón considered the following question:

### Inverse conductivity problem

Can one reconstruct the conductivity from measurements made only at the *boundary* of a conductor?



Motivation: Knowledge of the conductivity function gives an image of the interior of the conductor. Non-invasive testing applications: *Electrical impedance tomography* (EIT). An analogous problem can be formulated in the context of *Geophysics*.

# The Calderón-Gel'fand Problem

Let $\Omega$ be a bounded domain of $\mathbb{R}^d$, $d \geq 2$ with smooth boundary and $q \in L^\infty(\Omega, \mathbb{R})$ a real potential and $\kappa \in \mathbb{R}$ the *energy*.

If $\kappa$ **is not a Dirichlet eigenvalue of** $-\Delta + q$ then, given $f \in \mathcal{C}^\infty(\partial\Omega)$, there exists a unique $u \in H^2(\Omega)$ that solves:

$$\begin{cases} (-\Delta - \kappa + q(x))u(x) = 0, & x \in \Omega, \\ u|_{\partial\Omega} = f. \end{cases}$$

The **Dirichlet-to-Neumann (DtN) map** at energy $\kappa$ defined by $q$ maps $f$ (Dirichlet datum) to the **normal derivative** of the corresponding solution on the boundary (Neumann data):

$$\Lambda_{q,\kappa} : f \longmapsto \Lambda_{q,\kappa} f := \partial_\nu u|_{\partial\Omega}.$$

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If $\Omega \subseteq \mathbb{R}^d$ is the **unit ball** then for $q =$

- If $\kappa = 0$

$$\Lambda_{0,0} = \sqrt{-\Delta_{\partial\Omega} + \left(\frac{d-2}{2}\right)^2} - \frac{d-2}{2}$$

  and $\mathrm{Sp}_{L^2(\partial\Omega)}\, \Lambda_{0,\kappa} = \mathbb{N}$.

- If $\kappa \neq 0$ then $\Lambda_{0,\kappa}$ has the same eigenfunctions as $\Lambda_{0,0}$ (spherical harmonics) but the spectrum changes:

$$\lambda_\ell[0,\kappa] = \ell - \sqrt{\kappa}\frac{J_{\ell+1+\nu_d}(\sqrt{\kappa})}{J_{\ell+\nu_d}(\sqrt{\kappa})}, \qquad \ell \in \mathbb{N},$$

  where

$$\nu_d := \frac{d-2}{2}.$$

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# The Calderón-Gel'fand Problem

The DtN map enjoys a number of interesting properties:

- $\Lambda_{0,0} = \sqrt{-\Delta_{\partial\Omega}} + B$ where $B \in \mathcal{L}(L^2(\partial\Omega))$ is a **bounded** operator on $L^2(\partial\Omega)$.
- $\Lambda_{q,\kappa} = \Lambda_{0,\kappa} + K$ where $K \in \mathcal{K}(L^2(\partial\Omega))$ is a **compact** operator on $L^2(\partial\Omega)$.

**The Calderón-Gel'fand problem ∼'55**

Is $q$ uniquely determined by the DtN map $\Lambda_{q,\kappa}$? If so, reconstruct the potential $q$ from the boundary data $\Lambda_{q,\kappa}$.

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Is $q$ uniquely determined by the DtN map $\Lambda_{q,\kappa}$? If so, reconstruct the potential $q$ from the boundary data $\Lambda_{q,\kappa}$.

# The forward map

The Calderón problem can be reformulated in terms of the non-linear map

$$\begin{array}{rccc} \Phi^\kappa : & X & \longrightarrow & \mathcal{K}(L^2(\partial\Omega)) \\ & q & \longmapsto & \Lambda_{q,\kappa} - \Lambda_{0,\kappa} \end{array}$$

(called the **forward map**) where (for instance)

$$X := \{q \in L^\infty(\Omega, \mathbb{R}) \, : \, \kappa \notin \mathrm{Sp}_{H_0^1(\Omega)}(-\Delta + q)\}.$$

- **The uniqueness aspect.** Is the map $\Phi^\kappa$ injective?
- **The stability issue.** Find a modulus of continuity for $(\Phi^\kappa)^{-1}$:

  $$\|q_1 - q_2\|_{L^\infty(\Omega)} \leq \omega(\|\Phi^\kappa(q_1) - \Phi^\kappa(q_2)\|_{\mathcal{L}(L^2)}) = \omega(\|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathcal{L}(L^2)}),$$

  at least uniformly for $q_1, q_2$ in some compact set.
- **The reconstruction aspect.** Find an effective formula to compute $q$ in terms of $\Lambda_{q,\kappa}$. Related to the **characterization of the range** $\Phi^\kappa(X)$.

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# Sketch of known results

**Uniqueness.** The map $\Phi^\kappa$ is injective:

- $d \geq 3$. Starting from the work of Sylvester and Uhlmann '87 for smooth potentials.
- $d = 2$. First complete result: Bukhgeim '08, smooth potentials.
- Proofs involve construction of particular oscillatory solutions to the elliptic problem: **Complex Geometric Optics** (**CGO**) solutions.

**Stability** The map $(\Phi^\kappa)^{-1}$ is **discontinuous** but

- $d \geq 3$. Conditional stability results ($q$ is supposed *a priori* to lie on a compact set in $L^p(\Omega)$). Starting from Alessandrini '88.
- The (conditional) modulus of continuity is **logarithmic** and this is optimal: Mandache '01.
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**Reconstruction.**

- Uniqueness proofs can be converted after some work into effective algorithms to reconstruct $q$ from $\Lambda_{q,\kappa}$. This was started by Nachmann '88, Novikov '88.
- Many other approaches, for instance: one-step linearization, Harrach, Seo '10.

**Characterization of the range.** There are at the moment no complete characterizations of $\Phi^\kappa(X)$, the set of DtN operators at.

- This is also relevant in numerical applications because the Calderón problem is ill-posed, *i.e.* $(\Phi^\kappa)^{-1}$ is discontinuous, and its conditional modulus of continuity is poorly conditioned.
- Partial characterization for $d = 2$ for some conductivities: Ingerman '00, Sharafutdinov '11.
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# Important aspects that will not be addressed

- Calderón's original approach focuses on reconstructing a conductivity matrix, a positive definite matrix $A \in L^\infty(\Omega, \mathbb{R}^{d\times d})$ from the Dirichlet-to-Neumann map of the problem:

$$\begin{cases} \operatorname{div}(A(x)\nabla u(x)) - \kappa u(x) = 0, & x \in \Omega, \\ u|_{\partial\Omega} = f. \end{cases}$$

  or a Riemannian metric on a compact manifold with boundary. This is the **anisotropic** Calderón problem.
- Some of our results have a counterpart in this setting: radial conductivities. Ongoing work with Daudé, Meroño and Nicoleau.
- The eigenvalue problem for the DtN map is known as the **Steklov problem**. Spectral theory/geometry of DtN maps is an area of strong active research.

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# Classical strategy I. Integration by parts

It allows to transfer information from the boundary $\partial\Omega$ to the interior $\Omega$.

For every $f, g \in H^{1/2}(\partial\Omega)$ the following holds:

$$\langle f, (\Lambda_{q,\kappa} - \Lambda_{0,\kappa})g\rangle_{H^{1/2}\times H^{-1/2}} = \int_\Omega q(x)u(x)v(x)\,dx,$$

where $u$ and $v$ solve:

$$\begin{cases} \Delta u + \kappa u - qu &= 0 \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= g \end{cases} \qquad \begin{cases} \Delta v + \kappa v &= 0 \quad \text{in } \Omega, \\ v|_{\partial\Omega} &= f \end{cases}$$

This follows from the weak definition of the DtN map

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# Classical strategy II. CGO solutions

**Complex Geometric Optics (CGO)** solutions are designed to extract information from $\Omega$ via Alessandrini's identity.

Let $\zeta \in \mathcal{V}_d$ where

$$\mathcal{V}_{\kappa,d} := \{\zeta \in \mathbb{C}^d : \zeta_1^2 + \ldots + \zeta_d^2 = -\kappa, \ |\zeta| = \sqrt{2}\}.$$

Given $h > 0$, we introduce the $\kappa$-harmonic linear exponential functions

$$e_{\zeta/h}(x) = e^{\frac{\zeta}{h}\cdot x}, \qquad x \in \mathbb{R}^d.$$

A CGO solution is a family of functions $\psi_\zeta^h \in H^1(\Omega)$ that solve

$$-\Delta\psi_\zeta^h + q\psi_\zeta^h - \kappa\psi_\zeta^h = 0, \quad \text{in } \Omega$$

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$$\psi_\zeta^h = e_{\zeta/h}(1 + r_{h,\zeta}), \qquad \lim_{h\to 0^+} \|r_{h,\zeta}\|_{L^2(\Omega)} = 0.$$

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Take any $\xi \in \mathbb{R}^d$ and chose $\zeta_1, \zeta_2 \in \mathcal{V}_d$ with $\zeta_1 + \zeta_2 = -ih\xi$ ( $d \geq 3$ only!).

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The approach based on Complex Geometric Optics solutions and its variants has a certain number of limitations.

- CGOs and their variants cannot be used to deal with the general anisotropic Calderón problem.
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# Approach based on the Born Approximation

**Idea.** Linearize $(\Phi^\kappa)^{-1}$. But $(\Phi^\kappa)^{-1}$ is not even continuous.

**However...** The map $\Phi^\kappa$ is Fréchet differentiable, denote by $d\Phi_0^\kappa$ its differential at $q = 0$. One could then try to use as an approximation of the potential the function:

$$q_\kappa^{\mathrm{B}} := (d\Phi_0^\kappa)^{-1}(\Lambda_{q,\kappa} - \Lambda_{0,\kappa}).$$

This is the **Born approximation** referred to in the title.

**Huge problem.** This is formal, *a priori* there is no guarantee that $\Lambda_{q,\kappa} - \Lambda_{0,\kappa}$ lies in the range of the differential $d\Phi_0^\kappa(L^\infty(\Omega))$.

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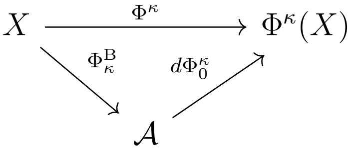


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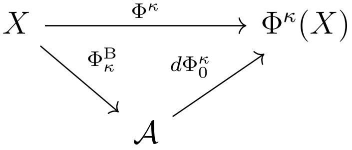


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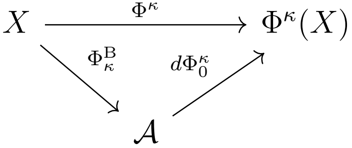


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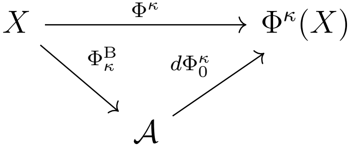


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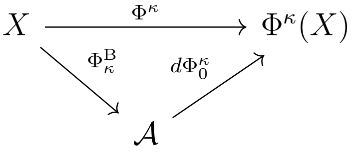


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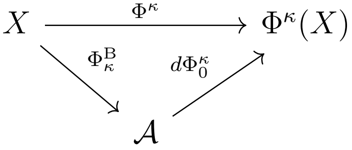


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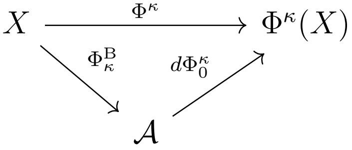


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# The radial Calderón-Gel'fand problem

We assume:

- $\Omega = \mathbb{B}^d := \{x \in \mathbb{R}^d \ : \ |x| \leq 1\}$, so that $\partial\Omega = \mathbb{S}^{d-1}$.
- $q \in X_{\text{rad}}$, where $X_{\text{rad}}$ consists of those **radial** potentials $q$ in $L^\infty(\mathbb{B}^d, \mathbb{R})$ such that $\ker_{H_0^1}(-\Delta + q - \kappa) = \{0\}$.

Suppose that $q \in X_{\text{rad}}$. **In this case, the DtN map $\Lambda_{q,\kappa}$ is completely determined by its eigenvalues.**

$\Lambda_{q,\kappa}$ is invariant by the action of $\mathrm{SO}(d)$ and commutes with $\Delta_{\mathbb{S}^{d-1}}$. Therefore **the eigenspaces** of $\Lambda_{q,\kappa}$ and $\Delta_{\mathbb{S}^{d-1}}$ **coincide** and are $\mathfrak{H}_\ell$, the **spherical harmonics** of degree $\ell$.

In other words

$$\Lambda_{q,\kappa}|_{\mathfrak{H}_\ell} = \lambda_\ell[q,\kappa]\mathrm{Id}_{\mathfrak{H}_k}.$$

# The radial Calderón-Gel'fand problem

We assume:

- $\Omega = \mathbb{B}^d := \{x \in \mathbb{R}^d \,:\, |x| \leq 1\}$, so that $\partial\Omega = \mathbb{S}^{d-1}$.
- $q \in X_{\text{rad}}$, where $X_{\text{rad}}$ consists of those **radial** potentials $q$ in $L^\infty(\mathbb{B}^d, \mathbb{R})$ such that $\ker_{H_0^1}(-\Delta + q - \kappa) = \{0\}$.

Suppose that $q \in X_{\text{rad}}$. **In this case, the DtN map $\Lambda_{q,\kappa}$ is completely determined by its eigenvalues.**

$\Lambda_{q,\kappa}$ is invariant by the action of $\mathrm{SO}(d)$ and commutes with $\Delta_{\mathbb{S}^{d-1}}$. Therefore **the eigenspaces** of $\Lambda_{q,\kappa}$ and $\Delta_{\mathbb{S}^{d-1}}$ **coincide** and are $\mathfrak{H}_\ell$, the **spherical harmonics** of degree $\ell$.

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# Direct problem: Spectrum of the DtN

**Theorem (M, Meroño, Sánchez-Mendoza '24)**

*Let $d \geq 2$ and $\kappa \neq 0$. Define the $\kappa$-**moments** of $q$ as:*

$$\sigma_\ell[q,\kappa] := \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{B}^d} q(x) \varphi_\ell(\sqrt{\kappa}|x|)^2 \, dx,$$

*where*

$$\varphi_\ell(r) := \frac{J_{\ell+\nu_d}(r)}{r^{\nu_d}}.$$

*Then, for all $\ell > \ell_q \geq 0$,*

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# Direct problem: Spectrum of the DtN

The Fréchet differential at the zero potential $d\Phi_0^\kappa$ turns out to be:

$$\begin{array}{rccc} d\Phi_0^\kappa : & L^\infty(\mathbb{B}^d) & \longrightarrow & \mathcal{K}(L^2(\mathbb{S})) \\ & q & \longmapsto & K_q \end{array}$$

where $K_q = d\Phi_0^\kappa(q)$ is the operator that has the **same eigenspaces** as $\Lambda_{q,\kappa}$ and eigenvalues

$$\operatorname{Sp} K_q = \left\{ \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{B}^d} q(x) \frac{\varphi_\ell(\sqrt{\kappa}|x|)^2}{\varphi_\ell(\sqrt{\kappa})^2} dx, \quad \ell \in \mathbb{N} \right\}.$$

In other words, $K_q$ is the radial operator whose eigenvalues are the **Hausdorff moments** of $q$:

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# A formula for the Born approximation

**Theorem (M, Meroño, Sánchez-Mendoza '24)**

*Suppose $q \in \mathcal{E}'_{\text{rad}}(\mathbb{B}^d)$ with $d \geq 2$. Then the Fourier transform of $q$ (as a distribution in $\mathcal{E}'(\mathbb{R}^d)$) is*

$$\widehat{q}(\xi) = (2\pi)^d \sum_{\ell=0}^{\infty} \sigma_\ell[q,\kappa] Z_{\ell,d}\left(1 - \frac{|\xi|^2}{2\kappa}\right),$$

*where $Z_{\ell,d}$ is the zonal harmonic of order $\ell$.*

*If $q_\kappa^{\mathrm{B}} \in \mathcal{E}'_{\text{rad}}(\mathbb{B}^d)$ exists then*

$$\widehat{q_\kappa^{\mathrm{B}}}(\xi) = (2\pi)^d \sum_{\ell=0}^{\infty} (\lambda_\ell[q,\kappa] - \lambda_\ell[0,\kappa])\varphi_\ell(\sqrt{\kappa})^2 Z_{\ell,d}\left(1 - \frac{|\xi|^2}{2\kappa}\right).$$

This formula can be used numerically regardless of existence.

There is an analogous formula in the non radial case $d = 2, 3$.

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# Numerical results in the radial case

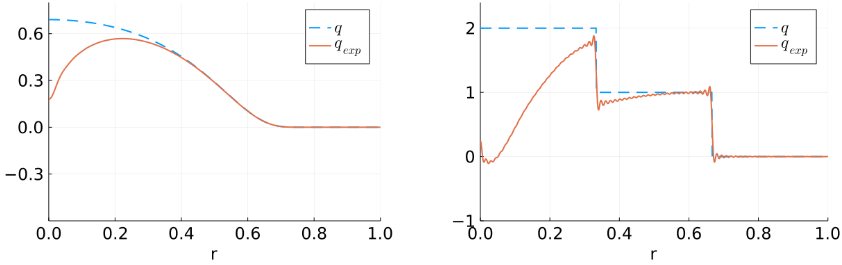


Figure: Born approximation of a smooth potential (left) and a step potential (right).

From Barceló, Castro, M, Meroño '24.

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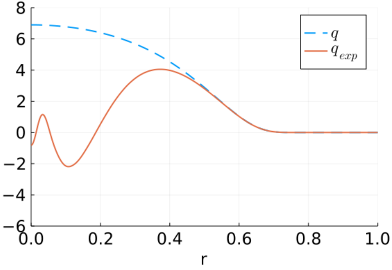
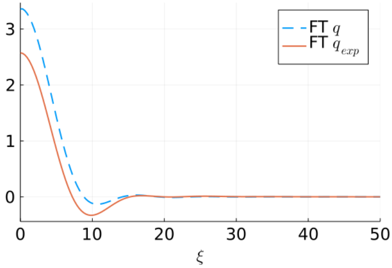



Potential




Fourier transforms

Figure: Born approximation of a smooth potential and its Fourier transform.

From Barceló, Castro, M, Meroño '24.

# Numerical results in the radial case

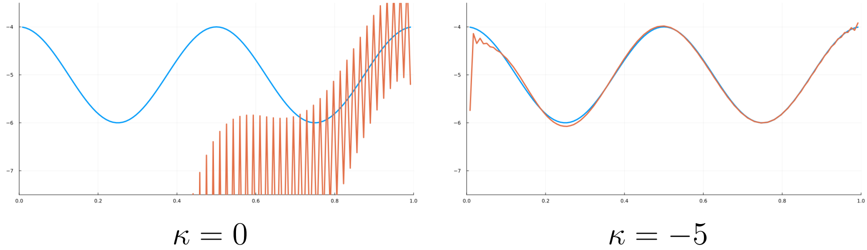



Figure: Plots of $q(x) = \cos(4\pi|x|) - 5$ (blue) and $\kappa + (q-\kappa)^{\mathrm{B}}_{\kappa}$ (orange).

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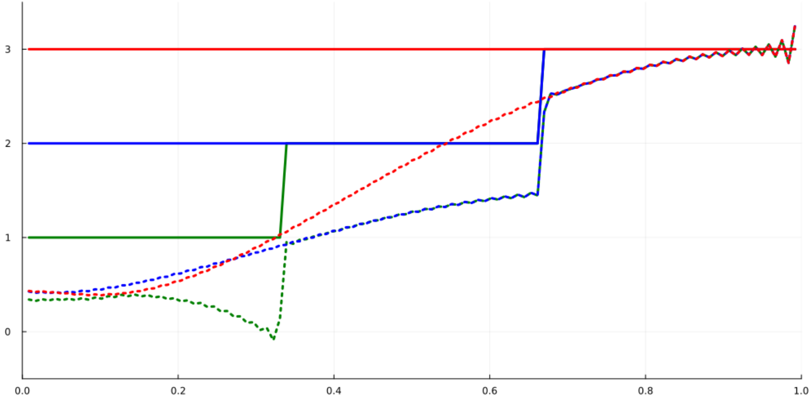


Figure: Plots of $q_i$ (solid) their respective Born approximations $q^{\mathrm{B}}_{i,\kappa}$ (dashed) at $\kappa = -1$.

$$q_3(x) := 3,$$

$$q_2(x) = q_3(x) - \chi_{(0,\frac{2}{3})}(|x|), \qquad q_1(x) = q_2(x) - \chi_{(0,\frac{1}{3})}(|x|).$$

# Existence: $(d\Phi_0^\kappa)^{-1}(\Lambda_{q,\kappa} - \Lambda_{0,\kappa})$ is well-defined

**Theorem (M, Meroño, Sánchez-Mendoza '24)**

*Assume that $q \in X_{\mathrm{rad}}$, $d \geq 2$, is radial. Then the moment problem*

$$\sigma_\ell[q_\kappa^{\mathrm{B}}, \kappa] = (\lambda_\ell[q,\kappa] - \lambda_\ell[0,\kappa])\varphi_\ell(\sqrt{\kappa})^2 \quad \textit{for all } \ell \in \mathbb{N},$$

*has a unique solution $q_\kappa^{\mathrm{B}} \in \mathcal{E}'_{\mathrm{rad}}(\mathbb{B}^d)$. This solution is of the form:*

$$q_\kappa^{\mathrm{B}}\text{"} = \text{"}a_q + s_q, \qquad a_q \in L^1_{\mathrm{loc}}(\mathbb{B}^d \setminus \{0\}), \quad \operatorname{supp} s_q = \{0\},$$

*and there exists $\ell_q \in \mathbb{N}$ such that*

$$a_q \in L^1(\mathbb{B}^d, |x|^{2\ell_q} dx), \quad s_q \textit{ is of order } < 2\ell_q.$$

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# Partial characterization of $\Phi^\kappa(X_{\mathrm{rad}})$

**Corollary** (M, Meroño, Sánchez-Mendoza '24)

*There exist $\ell_q \in \mathbb{N}$ such that*

$$\lambda_\ell[q,\kappa] = \lambda_\ell[0,\kappa] + \sigma_\ell^\kappa, \qquad \ell \geq \ell_q$$

*where $\sigma_\ell^\kappa$ is the sequence of moments:*

$$\sigma_\ell^\kappa = \int_0^1 \frac{J_{\ell+\nu_d}(\sqrt{\kappa} s)^2}{J_{\ell+\nu_d}(\sqrt{\kappa})^2} f_q(s)\, s\, ds$$

*of some function $f_q \in L^\infty((0,1))$.*

One can modify a result of Hausdorff 1922 to characterized those sequences.
We have analogous results when $q \in L^p_{\mathrm{rad}}(\mathbb{B}^d)$ with $p > d/2$.

# Uniqueness: $(\Phi_\kappa^{\mathrm{B}})^{-1}$ is well defined

The inverse problem amounts to recovering $q$ from $q_\kappa^{\mathrm{B}}$.

**Theorem (M, Meroño, Sánchez-Mendoza '24)**

*The map:*

$$\Phi_\kappa^{\mathrm{B}} : X_{\mathrm{rad}} \longrightarrow \mathcal{A} := (d\Phi_0^\kappa)^{-1}(\Phi(X_{\mathrm{rad}})) : q \longmapsto q_\kappa^{\mathrm{B}}$$

*is bijective. Moreover for every $0 < b < 1$*

$$(q_1)_\kappa^{\mathrm{B}}(x) = (q_2)_\kappa^{\mathrm{B}}(x) \quad \textit{a.e. for } b < |x| < 1$$

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$$q_1(x) = q_2(x) \ \textit{a.e. for } b < |x| < 1.$$

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# Good approximation properties

- $q_\kappa^{\mathrm{B}}$ contains the leading singularities of $q$.
- $q_\kappa^{\mathrm{B}}$ is a good approximation for $q$ close to the boundary

**Theorem (M, Meroño, Sánchez-Mendoza '24)**

*Let* $q \in X_{\mathrm{rad}}$ *such that* $\operatorname{ess\,supp} q \subset B(0;\rho)$ *for some* $0 < \rho \leq 1$. *Then* $q_\kappa^{\mathrm{B}} - q \in \mathcal{C}(\mathbb{B}^d \setminus \{0\})$ *and there exists* $\alpha_q \geq 0$ *such that:*

$$\left|(q_\kappa^{\mathrm{B}} - q)(x)\right| \leq C_q \left( \frac{(\rho^2 - |x|^2)_+}{|x|^{\alpha_q + 1}} \right)^2 .$$

*In addition, if* $q \in \mathcal{C}^m(\mathbb{B}^d)$ *with* $m \in \mathbb{N}$, *then*

$$q_\kappa^{\mathrm{B}} - q \in \mathcal{C}^{m+2}(\mathbb{B}^d \setminus \{0\}).$$

# Stability: $(\Phi_\kappa^{\mathrm{B}})^{-1}$ is Hölder continuous

**Theorem (M, Meroño, Sánchez-Mendoza '24)**

*For every $R \geq 1$ and $0 < b < 1$ there exists $C = C(b, R) > 0$ such that, for every $q_1, q_2 \in X_{\mathrm{rad}}$ of norm less or equal to $R$ and such that*

$$\int_{b<|x|<1} \left|(q_1)_\kappa^{\mathrm{B}}(x) - (q_2)_\kappa^{\mathrm{B}}(x)\right| \, dx < 1,$$

*the following holds:*

$$\int_{b<|x|<1} |q_1(x) - q_2(x)| \, dx \leq C \left( \int_{b<|x|<1} \left|(q_1)_\kappa^{\mathrm{B}}(x) - (q_2)_\kappa^{\mathrm{B}}(x)\right| \, dx \right)^{1/2}$$

# High energy limit

In the high-energy limit, the Born approximation coincides with $q$.

**Theorem (M, Meroño, Sánchez-Mendoza '24)**

$$\lim_{\kappa\to-\infty} \widehat{q_\kappa^{\mathrm{B}}}(\xi) = \widehat{q}(\xi), \qquad \forall \xi \in \mathbb{R}^d.$$

This holds even when $q$ is not radial, since $\widehat{q_\kappa^{\mathrm{B}}}(\xi)$ can always be defined (inversion of the Fourier transform is not proven unless $q$ is radial).

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Some comments on these results.

- Some of the proofs rely on the approach to Inverse Spectral Theory for operators

$$-\partial_x^2 + Q(x), \qquad \text{on } L^2(\mathbb{R}_+)$$

developed initially by Simon '99 and his notion of $A$-amplitude for the Weyl-Titchmarsh function.

- This approach allows to establish uniqueness for the Calderón problem, at least in the radial case, without relying on CGO solutions.
- We also have an algorithm form computing $q$ in terms of $q_\kappa^{\mathrm{B}}$ involving solving a non linear integro-differential equation.

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